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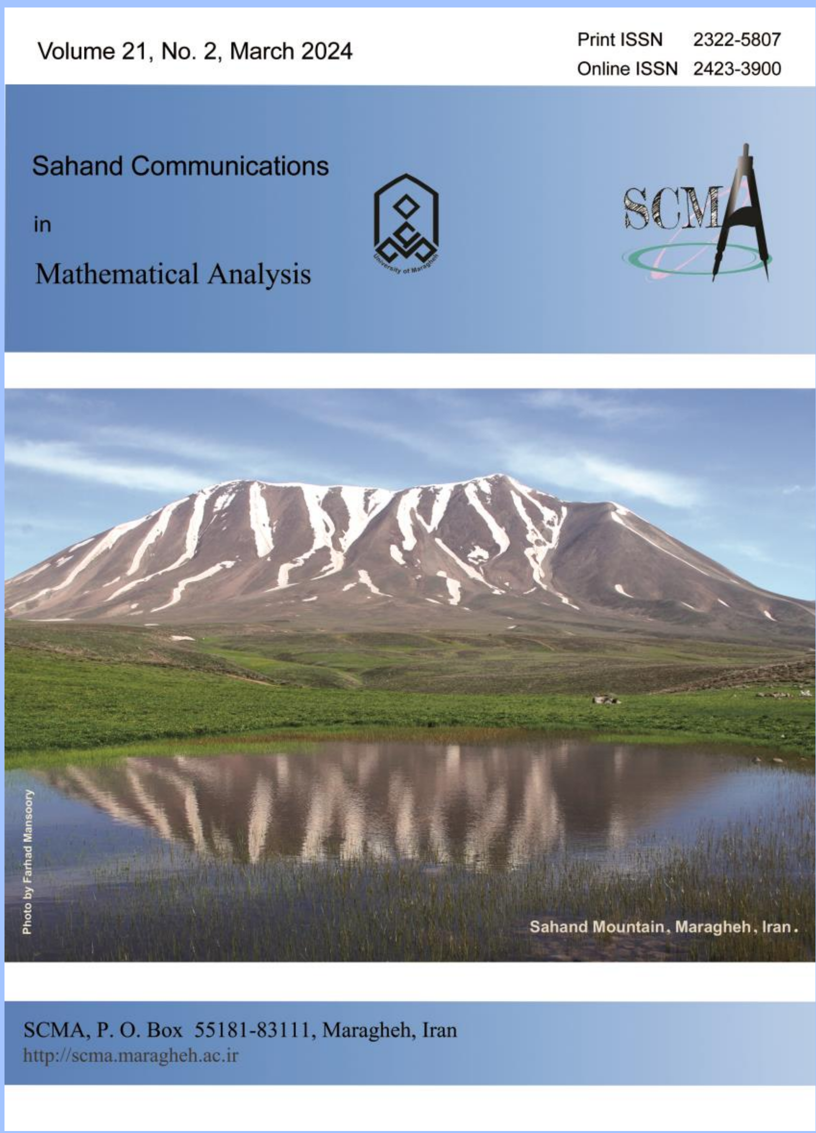

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# On Some New Classes of $\mathcal{I}-$Convergent Sequences in GNLS

Vakeel A. Khan[1], Zahid Rahman[2], Ayhan Esi[3*] and Amit Kumar[4]

Abstract. This paper is devoted to study $\mathcal{I}$-convergent,$\mathcal{I}-$null, $\mathcal{I}-$bounded and bounded sequence spaces in gradual normed linear spaces, denoted by $c^{\mathcal{I}}_{\|\cdot\|_G}, c^{\mathcal{I}}_{0\|\cdot\|_G}, \ell^{\mathcal{I}}_{\infty\|\cdot\|_G}, \ell_{\infty\|\cdot\|_G}, m^{\mathcal{I}}_{\|\cdot\|_G}$ and $m^{\mathcal{I}}_{0\|\cdot\|_G}$ respectively. We discussed some algebraic and topological properties of these classes. Also, we studied some inclusions involving these spaces.

## 1. Introduction

The term fuzzy was first introduced by Zadeh [35] in 1965 to expand the idea of classical set theory. Subsequently, the notion of fuzzy sets was fulfilled by many scientists and researchers. At the moment, it has spacious applications in science, engineering and technology. The expression "Fuzzy number" performs a vital role in the theory of fuzzy sets. Essentially, the fuzzy number is not a number, but it is the generalization of intervals. Furthermore, fuzzy numbers inherit algebraic properties of intervals, not of classical numbers. Therefore, the term "fuzzy numbers" is arguable for many researchers and authors due to their diverse behaviours. Many authors used the term "fuzzy intervals" in lieu of fuzzy numbers. Considering this confusion, Forine et al. [11] introduced a new notion in fuzzy set theory, "Gradual real numbers" as elements (singleton) of fuzzy intervals the year 2006. They showed that fuzzy interval can be represented as a set of gradual numbers that lie between two gradual numbers end-points, the same as the representation of an interval of reals. Gradual numbers represents only fuzziness with precision. Particularly, a fuzzy number can be denoted a crisp interval

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of gradual numbers. Generally, a gradual number can not be considered as a set of fuzzy numbers because the mapping from $(0, 1]$ to $\mathbb{R}$ is not necessarily one-to-one.

In the brief time, a gradual number have been applied as tools for computations on fuzzy intervals, with applications to combinatorial fuzzy optimization and evaluation of monotonic functions (see, [1, 3, 4, 7, 10–12, 14, 24, 36–38]). Kasperski et al. [14] used gradual numbers to solve combinatorial optimization problems. Afterwards, Fortin et al. [12] applied gradual numbers to present some methods for the evaluation of optimality. Also, Stock [33] used gradual numbers for the evaluation of fuzzy optimization. Recently, Sadeqi and Azari [28] in year 2011 have introduced the notion of gradual normed linear spaces (shortly GNLS) and studied some algebraic and topological properties of these spaces. Quiet recently in [8], Ettefagh et al. studied some properties of sequences in GNLS. In 2021, Chaudhury and Debnath [5] introduced the concept of $\mathcal{I}$ and $\mathcal{I}^*$-convergence of sequence in GNLS.

The concept of statistical convergence was first introduced by Fast [9] and Steinhaus [32] separately in 1951. Efficient research on statistical convergence started after the studies of Fridy [13], and thereafter, an immense quantity of literature has come into view. Afterwards, it was explored from a sequence space point of view and connected to the summability theory by fridy [13], Salat [29] and some other researchers. Convergence is also applied in mathematical analysis and number theory; for details, we refer the readers to [6, 13, 20, 25, 26, 29]. In the year 2000, Kostyrko et al. [21] used the notion of ideal $\mathcal{I}$ ($\subseteq \mathbb{N}$) to introduce an improved generalization of statistical convergence, called ideal convergence ($\mathcal{I}$-convergence). Later, the concept of $\mathcal{I}-$-convergence was also explored from the viewpoint of sequence spaces and linked with summability theory by Salat et al. [30, 31], Khan et al. [15, 16, 18] and many others for extensive detail about $\mathcal{I}$-convergence we refer [2, 15–23, 34].

This paper will define some new classes of $\mathcal{I}$-convergent sequences in gradual normed linear space $(\mathbb{V}, \|\cdot\|_G)$. In addition, we study some topological and algebraic properties, and we also present some inclusion relations for these sequence spaces.

## 2. Preliminaries

It is necessary to allude to some essential concepts related to main object of this research. Let $\omega$ represents the space of all real-valued sequences, then every subspace of $\omega$ is called a sequence space of real numbers. Let the space of all bounded, convergent and null sequences of real numbers be denoted by $\ell_\infty$, $c$ and $c_0$, respectively. Let $I = [a_1, a_2]$

and $J = [b_1, b_2]$ be intervals of numbers in $\mathbb{R}$, by $\mathcal{S}$; we denote the set of closed and bounded intervals on $\mathbb{R}$, i.e. $\mathcal{S} = \{I \subset \mathbb{R} : I = [a_1, a_2]\}$. For $I, J \in \mathcal{S}$, we define $I \leq J$ iff $a_1 \leq b_1$ and $a_2 \leq b_2$ and $d(I, J) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$. It is evident that $d$ defines a Housdroff metric on $\mathcal{S}$ and $(\mathcal{S}, d)$ is a complete metric space. Also, the relation $\leq$ is a partial order on $\mathcal{S}$.

**Definition 2.1** ([33])**.** Let $X$ denote a set. Then, a fuzzy set $\tilde{A}$ on $X$ is a set of ordered pairs.

$$\tilde{A} = \left\{ \left(x, \mu_{\tilde{A}}(x)\right) : x \in X \right\}$$

where the membership function $\mu_{\tilde{A}} : X \to [0, 1]$ is a map from $X$ into the set of possible degrees of memberships, $[0, 1]$. Where $\mu_{\tilde{A}}(x) = 1$ and $\mu_{\tilde{A}}(x) = 0$ expressing full membership and no-membership, respectively, and values between 0 and 1 showing partial membership.

**Definition 2.2** ([22])**.** A fuzzy number is a function $\mathcal{X}$ from $\mathbb{R}$ to $[0, 1]$, which satisfying the following conditions

(i) $\mathcal{X}$ is normal, i.e., there exists an $c \in \mathbb{R}$ such that $\mathcal{X}(c) = 1$;
(ii) $\mathcal{X}$ is fuzzy convex, i.e., for any $c_1, c_2 \in \mathbb{R}$ and $\lambda \in [0, 1]$, $\mathcal{X}(\lambda c_1 + (1 - \lambda)c_2) \geq \min\{\mathcal{X}(c_1), \mathcal{X}(c_2)\}$;
(iii) $\mathcal{X}$ is upper semi continuous;
(iv) The closure of $\mathcal{X} = \{c \in \mathbb{R} : \mathcal{X}(c) \geq 0\}$, denoted by $\mathcal{X}^\circ$ is compact.

The above properties shows that for each $\alpha \in (0, 1]$ the $\alpha-$level set,

$$\begin{aligned} \mathcal{X}^\alpha &= \{c \in \mathbb{R} : \mathcal{X}(c) \geq \alpha\} \\ &= [\underline{\mathcal{X}}^\alpha, \overline{\mathcal{X}}^\alpha] \\ &\neq \emptyset \end{aligned}$$

is compact and convex subset of $\mathbb{R}$. Let $\mathcal{S}(\mathbb{R})$ be the set of all real valued fuzzy numbers. Define $\overline{d} : \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \to \mathbb{R}$ by

$$\overline{d}(\mathcal{X}, \mathcal{Y}) = \sup_{\alpha \in [0,1]} d(\mathcal{X}^\alpha, \mathcal{Y}^\alpha)$$

In [27] Puri and Ralescu proved that $(\mathcal{S}(\mathbb{R}), \overline{d})$ is a complete metric space . For $\mathcal{X}, \mathcal{Y} \in \mathcal{S}(\mathbb{R})$ we define $\mathcal{X} \leq \mathcal{Y}$ if and only if $\underline{\mathcal{X}}^\alpha \leq \underline{\mathcal{Y}}^\alpha$ and $\overline{\mathcal{X}}^\alpha \leq \overline{\mathcal{Y}}^\alpha$ for each $\alpha \in [0, 1]$. We say that $\mathcal{X} < \mathcal{Y}$ if $\mathcal{X} \leq \mathcal{Y}$ and there exist $\alpha_0 \in [0, 1]$ such that $\underline{\mathcal{X}}^{\alpha_0} < \underline{\mathcal{Y}}^{\alpha_0}$ or $\overline{\mathcal{X}}^{\alpha_0} < \overline{\mathcal{Y}}^{\alpha_0}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be any two fuzzy numbers then $\mathcal{X}$ and $\mathcal{Y}$ are incomparable if neither$\mathcal{X} \leq \mathcal{Y}$ nor $\mathcal{Y} \leq \mathcal{X}$. For $\mathcal{X}, \mathcal{Y} \in \mathcal{S}(\mathbb{R}), c \in \mathbb{R}$ and for the $\alpha-$level sets $[\mathcal{X}]^\alpha = [a_1^\alpha, a_2^\alpha]$ and $[\mathcal{Y}]^\alpha = [b_1^\alpha, b_2^\alpha]$ the linear structure of $\mathcal{S}(\mathbb{R})$ i.e. addition $\mathcal{X} + \mathcal{Y}$ and scalar multiplication $c\mathcal{X}$, are defined as

- $[\mathcal{X} + \mathcal{Y}]^\alpha = [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha]$, for $\alpha \in [0, 1]$

- $[c\mathcal{X}]^\alpha = c[\mathcal{X}]^\alpha$, for $\alpha \in [0,1]$.

Let $\mathcal{B}$ be a subset of $\mathcal{S}(\mathbb{R})$, then $\mathcal{B}$ is called bounded above if there is a fuzzy number $\beta$ such that $\mathcal{X} \leq \beta$, for each $\mathcal{X} \in \mathcal{S}$ and $\beta$ is called the upper bound of $\mathcal{S}$. Also, the upper bound $\beta$ of $\mathcal{S}$ is said to be the least upper bound (Sup) of $\mathcal{S}$ if $\beta \leq \beta'$ for all upper bounds $\beta'$ of $\mathcal{S}$. The lower bound and greatest lower bound (inf) of set $\mathcal{S}$ can be defined similarly. If the set $\mathcal{S}$ has both bounds (above and below), $\mathcal{S}$ is a bounded set.

**Definition 2.3** ([11])**.** A gradual number, $\tilde{r}$, is defined by a function called the assignment function, $\mathcal{A}_{\tilde{r}} : (0,1] \longmapsto \mathbb{R}$ . It can be understood as a real value parametrized by $\alpha \in (0,1]$. Then, for each $\alpha$, a real value $r_\alpha$ is given by $\mathcal{A}_{\tilde{r}}\,(\alpha)$.

A gradual real number $\tilde{r}$ is said to be non-negative if for every $\alpha \in (0,1]$, we have $\mathcal{A}_{\tilde{r}}\,(\alpha) \geq 0$. Also by $G(\mathbb{R})$ and $G^*(\mathbb{R})$ we represents the set of all gradual real numbers and non-negative gradual real numbers respectively.

Let $\circ$ be any operation in $\mathbb{R}$ and let $\tilde{r_1}, \tilde{r_2} \in G(\mathbb{R})$ be gradual numbers with assignment functions $\mathcal{A}_{\tilde{r_1}}$ and $\mathcal{A}_{\tilde{r_2}}$ respectively. Then $\tilde{r_1} \circ \tilde{r_2} \in G(\mathbb{R})$ with assignment function $\mathcal{A}_{\tilde{r_1}\circ\tilde{r_2}}$ defined as

$$\mathcal{A}_{\tilde{r_1}\circ\tilde{r_2}}(\alpha) = \mathcal{A}_{\tilde{r_1}}(\alpha) \circ \mathcal{A}_{\tilde{r_2}}(\alpha), \quad \alpha \in (0,1]\,.$$

Then for any $\tilde{r_1}, \tilde{r_2} \in G(\mathbb{R})$ and every scalar $c \in \mathbb{R}$ the gradual addition, subtraction, multiplication and division can be present as follow

$$\begin{aligned}
&\mathcal{A}_{\tilde{r_1}+\tilde{r_2}}(\alpha) = \mathcal{A}_{\tilde{r_1}}(\alpha) + \mathcal{A}_{\tilde{r_2}}(\alpha), \quad \alpha \in (0,1]\\
&\mathcal{A}_{\tilde{r_1}-\tilde{r_2}}(\alpha) = \mathcal{A}_{\tilde{r_1}}(\alpha) + \mathcal{A}_{-\tilde{r_2}}(\alpha) = \mathcal{A}_{\tilde{r_1}}(\alpha) - \mathcal{A}_{\tilde{r_2}}(\alpha), \quad \alpha \in (0,1]\\
&\mathcal{A}_{c\cdot\tilde{r}}(\alpha) = c \cdot \mathcal{A}_{\tilde{r}}(\alpha), \quad \alpha \in (0,1]\\
&\mathcal{A}_{\tilde{r}_1\cdot\tilde{r}_2}(\alpha) = \mathcal{A}_{\tilde{r}_1}(\alpha) \cdot \mathcal{A}_{\tilde{r}_2}(\alpha), \quad \alpha \in (0,1]\\
&\mathcal{A}_{\frac{\tilde{r}_1}{\tilde{r}_2}}(\alpha) = \frac{\mathcal{A}_{\tilde{r}_1}(\alpha)}{\mathcal{A}_{\tilde{r}_2}(\alpha)}, \text{ if } \quad \mathcal{A}_{\tilde{r}_2}(\alpha) \neq 0 \quad \text{for all } \alpha \in (0,1]\,.
\end{aligned}$$

Let $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ be gradual numbers with assignment functions $\mathcal{A}_{\tilde{r}_1}(\alpha)$ and $\mathcal{A}_{\tilde{r}_2}(\alpha)$ respectively and $\alpha \in (0,1]$, then;

(i) $\tilde{r}_1 = \tilde{r}_2 \quad \Leftrightarrow \quad \mathcal{A}_{\tilde{r}_1}(\alpha) = \mathcal{A}_{\tilde{r}_2}(\alpha)$.
(ii) $\tilde{r}_1 \geq \tilde{r}_2 \quad \Leftrightarrow \quad \mathcal{A}_{\tilde{r}_1}(\alpha) \geq \mathcal{A}_{\tilde{r}_2}(\alpha)$.
(ii) $\tilde{r}_1 \leq \tilde{r}_2 \quad \Leftrightarrow \quad \mathcal{A}_{\tilde{r}_1}(\alpha) \leq \mathcal{A}_{\tilde{r}_2}(\alpha)$.

**Definition 2.4** ([38])**.** Let $\tilde{r} \in G(\mathbb{R})$ be a number with assignment functions $\mathcal{A}_{\tilde{r}}(\alpha)$, and $\alpha \in (0,1]$, then the absolute value of $\tilde{r}$ is given by the mapping $|\tilde{r}| : (0,1] \to G^*(\mathbb{R})$ defined by $\mathcal{A}_{|\tilde{r}|}(\alpha) = |\mathcal{A}_{\tilde{r}}(\alpha)|$.

**Definition 2.5** ([28])**.** Let $\mathbb{V}$ be a vector space of real numbers and $\|\cdot\|_G : \mathbb{V} \to G^*(\mathbb{R})$ be a mapping. Then $(\mathbb{V}, \|\cdot\|_G)$ is called gradual normed linear space (GNLS) with gradual norm $\|\cdot\|_G$ iff

($\mathcal{G}_1$) $\|u\|_G = 0 \quad \Leftrightarrow \quad u = 0,\ \forall u \in \mathbb{V};$
($\mathcal{G}_2$) $\|cu\|_G = |c|\|u\|_G, \quad \forall c \in \mathbb{R};$
($\mathcal{G}_3$) $\|u+v\|_G \leq \|u\|_G + \|v\|_G, \quad \forall u, v \in \mathbb{V}.$

The above definition can be written in the form assignment of function as follow:
For any $u, v \in \mathbb{V}, c \in \mathbb{R}$ and $\alpha \in (0,1]$, the pair $(\mathbb{V}, \|\cdot\|_G)$ is called gradual normed linear space (GNLS) with gradual norm $\|\cdot\|_G$ iff

($\mathcal{G}_1$) $\mathcal{A}_{\|u\|_G}(\alpha) = \mathcal{A}_{\tilde{0}}(\alpha) \quad \Leftrightarrow \quad u = 0,\ \forall u \in \mathbb{V};$
($\mathcal{G}_2$) $\mathcal{A}_{\|cu\|_G}(\alpha) = |c|\mathcal{A}_{\|u\|_G}(\alpha), \quad \forall c \in \mathbb{R};$
($\mathcal{G}_3$) $\mathcal{A}_{\|u+v\|_G}(\alpha) \leq \mathcal{A}_{\|u\|_G}(\alpha) + \mathcal{A}_{\|v\|_G}(\alpha), \quad \forall u, v \in \mathbb{V}.$

**Example 2.6.** Let $\mathbb{V} = \mathbb{R}^n$ be a real vector space and $v = (v_1, v_2, \ldots, v_n) \in \mathbb{V},\ \alpha \in (0,1]$, then the norm $\|\cdot\|_G$ defined by $\mathcal{A}_{\|v\|_G}(\alpha) = e^{\alpha} \sum_{i=1}^{n} |v_i|$ is a gradual norm on $\mathbb{V} = \mathbb{R}^n$ and the pair $(\mathbb{V}, \|\cdot\|_G)$ is GNLS.

For more examples about GNLS we refer the readers to [28].

**Definition 2.7** ([21])**.** A family of sets $\mathcal{I} \subseteq \mathbb{N}$ is said to be ideal if and only if

(i) $\emptyset \in \mathcal{I}$
(ii) $\mathcal{A}, \mathcal{B} \in \mathcal{I} \quad \Rightarrow \quad \mathcal{A} \cup \mathcal{B} \in \mathcal{I},$
(iii) $\mathcal{A} \in \mathcal{I}$ and $\mathcal{B} \subseteq \mathcal{A} \quad \Rightarrow \quad \mathcal{B} \in \mathcal{I}.$

An ideal $\mathcal{I} \neq \emptyset$ is called proper (non-trivial) if $\mathcal{I} \neq 2^{\mathbb{N}}$.

Let $\mathcal{I}$ be a non-trivial ideal then $\mathcal{I}$ is called admissible if $\mathcal{I} \supseteq \{\{n\} : n \in \mathbb{N}\}$.
A non-trivial ideal $\mathcal{I}$ is called maximal if there cannot exists any non-trivial ideal $\mathcal{B} \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset.

**Definition 2.8** ([20])**.** A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called filter on $\mathbb{N}$ if and only if;

(i) $\emptyset \notin \mathcal{F}$,
(ii) for each $\mathcal{A}, \mathcal{B} \in \mathcal{F}$ we have $\mathcal{A} \cap \mathcal{B} \in \mathcal{F}$,
(iii) for every $\mathcal{A} \in \mathcal{F}$ and $\mathcal{B} \supseteq \mathcal{A}$, we have $\mathcal{B} \in \mathcal{F}$.

If $\mathcal{I}$ is a non-trivial ideal of $\mathbb{N}$, then the family of sets $\mathcal{F}(\mathcal{I}) = \{\mathcal{C} \subset \mathbb{N} : \exists \quad \mathcal{A} \in \mathcal{I} : \mathcal{C} = \mathbb{N} \backslash \mathcal{A}\}$ is a filter of $\mathbb{N}$ and known as filter associated with $\mathcal{I}$.

**Definition 2.9** ([20])**.** Let $\mathcal{I}$ be a proper ideal of $\mathbb{N}$, then a sequence $u = (u_i) \in \omega$ is said to be $\mathcal{I}-$convergent to a real number $u_0$ if for every $\varepsilon > 0$ we have $\mathcal{A}(\varepsilon) = \{i \in \mathbb{N} : |u_i - u_0| \geq \varepsilon\} \in \mathcal{I}$. Here $u_0$ is called $\mathcal{I}-$limit of the sequence $(u_i) \in \omega$ and denoted as $\mathcal{I} - \lim_{i \to \infty} u_i = u_0$.

**Definition 2.10** ([19])**.** A sequence $u = (u_i) \in \omega$ is said to be $\mathcal{I}-$null if for every $\varepsilon > 0$ we have $\{i \in \mathbb{N} : |u_i - 0| \geq \varepsilon\} \in \mathcal{I}$, denoted by $\mathcal{I} - \lim_{i \to \infty} u_i = 0$.

**Definition 2.11** ([18])**.** Let $\mathcal{I}$ be a admissible ideal of $\mathbb{N}$, then a sequence $(u_i) \in \omega$ is said to be $\mathcal{I}$–Cauchy if for each $\varepsilon > 0$ there exists a number $\mu = \mu(\varepsilon) \in \mathbb{N}$ such that $\{i \in \mathbb{N} : |u_i - u_\mu| \geq \varepsilon\} \in \mathcal{I}$.

**Definition 2.12** ([31])**.** A sequence $u = (u_i) \in \omega$ is called $\mathcal{I}$–bounded if there exists a real number $\mu > 0$, such that the set $\{i \in \mathbb{N} : |u_i| > \mu\} \in \mathcal{I}$.

**Definition 2.13** ([31])**.** Let $\mathbb{V}$ be the sequence spaces of real numbers, then $\mathbb{V}$ is said to be solid (or normal) if for $(u_i) \in \mathbb{V},\ (\lambda_i) \in G(\mathbb{R})$ with $|\lambda_i| \leq 1$ we have $(\lambda_i\ u_i) \in \mathbb{V}$ for all $i \in \mathbb{N}$.

**Definition 2.14** ([28])**.** Let $(\mathbb{V}, \|\cdot\|_G)$ be a GNLS. Then the sequence $(u_i) \in \mathbb{V}$ is said to be gradually convergent to $u \in \mathbb{V}$, iff for all $\alpha \in (0,1]$, we have $\lim\limits_{i\to\infty} \mathcal{A}_{\|u_i - u\|_G}(\alpha) = \mathcal{A}_{\tilde{0}}(\alpha)$. Also $(u_i) \in \mathbb{V}$ is said to be gradually Cauchy iff for all $\alpha \in (0,1]$ and $\mathcal{N} = \mathcal{N}_\varepsilon(\alpha) \in \mathbb{N}$, we have $\lim\limits_{j\to\infty} \mathcal{A}_{\|u_i - u_\mathcal{N}\|_G}(\alpha) = \mathcal{A}_{\tilde{0}}(\alpha)$.

The GNLS $(\mathbb{V}, \|\cdot\|_G)$ is called complete if every Cauchy sequence in $(\mathbb{V}, \|\cdot\|_G)$ is convergent.

**Theorem 2.15** ([28])**.** *Let $(\mathbb{V}, \|\cdot\|_G)$ be a GNLS, then every convergent sequence in $\mathbb{V}$ is a Cauchy sequence.*

**Definition 2.16** ([5])**.** Let $(u_i) \in (\mathbb{V}, \|\cdot\|_G)$, be a sequence and $\alpha \in (0,1]$. Then $(u_i)$ is said to be gradually $\mathcal{I}-$convergent to $u \in \mathbb{V}$, if for every $\varepsilon > 0$; we have $\left\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - u\|_G}(\alpha) \geq \varepsilon\right\} \in \mathcal{I}$; and denoted by $u_i \xrightarrow{\mathcal{I}-\|\cdot\|_G} u$.

**Definition 2.17** ([5])**.** Let $(u_i) \in (\mathbb{V}, \|\cdot\|_G)$ be a sequence and $\alpha \in (0,1]$. Then $(u_i)$ is said to be gradually $\mathcal{I}-$Cauchy if for every $\varepsilon > 0$, there exists a number $\mathcal{N} = \mathcal{N}_\varepsilon(\alpha) \in \mathbb{N}$, such that $\left\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - u_\mathcal{N}\|_G}(\alpha) \geq \varepsilon\right\}$ $\in \mathcal{I}$.

**Definition 2.18** ([31])**.** Let $\mathbb{V}$ be a sequence space and $\mathcal{K} = \{k_i \in \mathbb{N} : k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$. A $\mathcal{K}-$ step space of $\mathbb{V}$ is a sequence space

$$\lambda_\mathcal{K}^\mathbb{V} = \{(u_{k_i}) \in \omega : (u_k) \in \mathbb{V}\}.$$

A canonical pre–image of a sequence $(u_{k_i}) \in \lambda_\mathcal{K}^\mathbb{V}$ is a sequence $(v_k) \in \omega$ defined as follows:

$$v_k = \begin{cases} u_k, & \text{if } k \in \mathcal{K} \\ 0, & \text{otherwise.} \end{cases}$$

A canonical pre–image of step space $\lambda_\mathcal{K}^\mathbb{V}$ is a set of canonical pre–images of all elements in $\lambda_\mathcal{K}^\mathbb{V}$. i.e. $v$ is in canonical pre–image of $\lambda_\mathcal{K}^\mathbb{V}$ if and only if $v$ is canonical pre–image of some element $u \in \lambda_\mathcal{K}^\mathbb{V}$.

**Definition 2.19** ([31])**.** A sequence space $\mathbb{V}$ is said to be monotone, if it have the canonical pre-images of its step spaces. i.e. for all $\mathcal{K} \subseteq \mathbb{N}$ and $(u_k) \in \mathbb{V}$, we have $(\lambda_k u_k) \in \mathbb{V}$, where

$$\lambda_k = \begin{cases} 1 & \text{, if } k \in \mathcal{K} \\ 0 & \text{, otherwise.} \end{cases}$$

**Definition 2.20** ([18])**.** A sequence space $\mathbb{V}$ is said to be convergence free, if $(u_i) \in \mathbb{V}$, whenever $(u_i) \in \mathbb{V}$ and $(v_i) = 0 \;\Rightarrow\; (u_i) = 0,\ \forall\ i \in \mathbb{N}$.

**Definition 2.21** ([31])**.** Let $u = (u_k)$ and $v = (v_k)$ be two sequences, then we say that $u_k = v_k$ for almost all $k$ relative to $\mathcal{I}$ (in short $a.a.k.r.\mathcal{I}$) if the set $\{k \in \mathbb{N} : u_k \neq v_k\} \in \mathcal{I}$.

**Lemma 2.22** ([31])**.** *Every solid space is monotone.*

**Lemma 2.23** ([30])**.** *Let $\mathcal{I} \in 2^{\mathbb{N}}$ be a maximal ideal and $\mathcal{C} \in \mathcal{F}(\mathcal{I})$ then for each $\mathcal{M} \subset \mathbb{N}$ if $\mathcal{M} \notin \mathcal{I}$, then $\mathcal{M} \cap \mathcal{C} \notin \mathcal{I}$.*

## 3. Main Results

Throughout in this paper we assume that $\mathcal{I}$ is an admissible ideal of subsets of $\mathbb{N}$ and $\mathbb{V}$ be a GNLS also $u = (u_i),\ v = (v_i)$, be sequences in $\mathbb{V}$. Now let $\varepsilon > 0$ and $\alpha \in (0, 1]$, then we define some new sequence spaces as

$$c^{\mathcal{I}}_{\|\cdot\|_G} := \Big\{u = (u_i) \in \mathbb{V} : \\ \big\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - r\|_G}(\alpha) \geq \varepsilon\big\} \in \mathcal{I} \text{ for some } r \in \mathbb{V}(\mathcal{I})\Big\}. \tag{3.1}$$

$$c^{\mathcal{I}}_{0\ \|\cdot\|_G} := \Big\{u = (u_i) \in \mathbb{V} : \\ \big\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - 0\|_G}(\alpha) \geq \varepsilon\big\} \in \mathcal{I} \text{ for some } 0 \in \mathbb{V}(\mathcal{I})\Big\}. \tag{3.2}$$

$$\ell^{\mathcal{I}}_{\infty\ \|\cdot\|_G} := \Big\{u = (u_i) \in \mathbb{V} : \\ \exists\ \mu > 0 \text{ s.t } \big\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - 0\|_G}(\alpha) \geq \mu\big\} \in \mathcal{I}\Big\}. \tag{3.3}$$

$$\ell_{\infty\ \|\cdot\|_G} := \{u = (u_i) \in \mathbb{V} : \sup_n \mathcal{A}_{\|u_i - 0\|_G}(\alpha) < \infty\}. \tag{3.4}$$

For convenience of our work we represent

$$m^{\mathcal{I}}_{0\ \|\cdot\|_G} := c^{\mathcal{I}}_{0\ \|\cdot\|_G} \cap \ell_{\infty\ \|\cdot\|_G}, \qquad m^{\mathcal{I}}_{\|\cdot\|_G} := c^{\mathcal{I}}_{\|\cdot\|_G} \cap \ell_{\infty\ \|\cdot\|_G}.$$

The inclusions $c^{\mathcal{I}}_{0\ \|\cdot\|_G} \subset \ell^{\mathcal{I}}_{\infty\ \|\cdot\|_G} \subset \ell_{\infty\ \|\cdot\|_G}$ is apparent from the definitions of $c^{\mathcal{I}}_{0\ \|\cdot\|_G}$, $\ell^{\mathcal{I}}_{\infty\ \|\cdot\|_G}$ and $\ell_{\infty\ \|\cdot\|_G}$.

**Definition 3.1.** A sequence $(u_i) \in \mathbb{V}$ is said to be gradually $\mathcal{I}-$null if there exists a real number $u_0$, such that for every $\varepsilon > 0$, and $\alpha \in (0,1]$, we have $\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - 0\|}(\alpha) \geq \varepsilon\} \in \mathcal{I}$.

**Definition 3.2.** A sequence $u = (u_i) \in \mathbb{V}$ is said to be gradually $\mathcal{I}-$bounded if there exists a real number $r > 0$, such that, the set $\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - 0\|_G}(\alpha) > r\} \in \mathcal{I}$.

**Remark 3.3.** If $\mathcal{I}$ is proper ideal and $\mathcal{I} \supseteq \{\{i\} : i \in \mathbb{N}\}$, then the sequence spaces $c^{\mathcal{I}}_{\|\cdot\|_G}$, $c^{\mathcal{I}}_{0 \quad \|\cdot\|_G}$, $\ell^{\mathcal{I}}_{\infty \ \|\cdot\|_G}$ are coincide with $c_{\ \|\cdot\|_G}$, $c_{0 \quad \|\cdot\|_G}$, $\ell_{\infty \ \|\cdot\|_G}$, respectively.

**Theorem 3.4.** *The classes* $c^{\mathcal{I}}_{\|\cdot\|_G}$, $c^{\mathcal{I}}_{0 \quad \|\cdot\|_G}$, $\ell^{\mathcal{I}}_{\infty \quad \|\cdot\|_G}$, $m^{\mathcal{I}}_{\|\cdot\|_G}$ *and* $m^{\mathcal{I}}_{0 \quad \|\cdot\|_G}$ *are linear over* $\mathbb{R}$.

*Proof.* (i) Let $u = (u_i)$ and $v = (v_i)$ be two arbitrary sequences in $c^{\mathcal{I}}_{\|\cdot\|_G}$ and $c_1, c_2$ be scalars, then for every $\varepsilon > 0$ and $\alpha\,(0,1]$, there exists $r_1, r_2 \in \mathbb{V}$ such that

$$\left\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - r_1\|_G}(\alpha) \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}$$
$$\left\{i \in \mathbb{N} : \mathcal{A}_{\|v_i - r_2\|_G}(\alpha) \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}.$$

Let us define

$$\mathcal{A}_u = \left\{n \in \mathbb{N} : \mathcal{A}_{\|u_i - r_1\|_G}(\alpha) < \frac{\varepsilon}{2|c_1|}\right\} \in \mathcal{F}(\mathcal{I})$$
$$\mathcal{A}_v = \left\{i \in \mathbb{N} : \mathcal{A}_{\|v_i - r_2\|_G}(\alpha) < \frac{\varepsilon}{2|c_2|}\right\} \in \mathcal{F}(\mathcal{I}),$$

such that $\mathcal{A}^c_u, \mathcal{A}^c_v \in \mathcal{I}$ then

$$\begin{aligned} \mathcal{A}_{uv} &= \left\{i \in \mathbb{N} : \mathcal{A}_{\|(c_1u_i + c_2v_i) - (c_1r_1 + c_2r_2\|_G}(\alpha) < \varepsilon\right\} \\ &\supseteq \left\{\left\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - r_1\|_G}(\alpha) < \frac{\varepsilon}{2|c_1|}\right\}\right. \\ &\quad \left.\cap \left\{i \in \mathbb{N} : \mathcal{A}_{\|v_i - r_2\|_G}(\alpha) < \frac{\varepsilon}{2|c_2|}\right\}\right\} \in \mathcal{F}(\mathcal{I}) \end{aligned} \tag{3.5}$$

Since the sets $\mathcal{A}_u$ and $\mathcal{A}_v$ on the RHS of (3.5) belong to $\mathcal{F}(\mathcal{I})$, by definition of $\mathcal{F}(\mathcal{I})$, the complement of LHS of (3.5) belongs to $\mathcal{I}$. That is $\mathcal{A}^c_{uv} \subseteq \mathcal{I}$, this gives that $(c_1u_i + c_2v_i) \in c^{\mathcal{I}}_{\|\cdot\|_G}$. Hence $c^{\mathcal{I}}_{\|\cdot\|_G}$ is linear over $\mathbb{R}$.

(ii) Suppose $u = (u_i), v = (v_i)$ be two arbitrary sequences in $c^{\mathcal{I}}_{0 \ \|\cdot\|_G}$ and $c_1, c_2$ be scalars, then for every $\varepsilon > 0$ and any $\alpha \in (0,1]$

$$\mathcal{B}_u = \left\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - 0\|_G}(\alpha) \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}$$

$$\mathcal{B}_v = \left\{ i \in \mathbb{N} : \mathcal{A}_{\|v_i - 0\|_G}(\alpha) \geq \frac{\varepsilon}{2} \right\} \in \mathcal{I}$$

$$\mathcal{A}_{\|(c_1 u_i + c_2 v_i) - 0\|_G}(\alpha) \leq |c_1| \mathcal{A}_{\|u_i - 0\|_G}(\alpha) + |c_2| \mathcal{A}_{\|v_i - 0\|_G}(\alpha)$$

Now

$$\begin{aligned} \mathcal{B}_u v &= \left\{ i \in \mathbb{N} : \mathcal{A}_{\|}(c_1 u_i + c_2 v_i) - 0\|_G(\alpha) \geq \varepsilon \right\} \\ &\subseteq \left\{ i \in \mathbb{N} : |c_1| \mathcal{A}_{\|} u_i - 0\|_G(\alpha) \geq \frac{\varepsilon}{2} \right\} \\ &\quad \cup \left\{ i \in \mathbb{N} : |c_2| \mathcal{A}_{\|} v_i - 0\|_G(\alpha) \geq \frac{\varepsilon}{2} \right\} \\ &= \left\{ i \in \mathbb{N} : \mathcal{A}_{\|} u_i - 0\|_G(\alpha) \geq \frac{\varepsilon}{2|c_1|} \right\} \\ &\quad \cup \left\{ i \in \mathbb{N} : \mathcal{A}_{\|} v_i - 0\|_G(\alpha) \geq \frac{\varepsilon}{2|c_2|} \right\} \\ &\subseteq \mathcal{B}_u \cup \mathcal{B}_v \in \mathcal{I} \end{aligned} \tag{3.6}$$

Hence $c_{0\ \|\cdot\|_G}^{\mathcal{I}}$ is linear over $\mathbb{R}$.

The proof of remaining parts have same procedure. □

**Theorem 3.5.** *Let $c_{\|\cdot\|_G}^{\mathcal{I}}$, $c_{0\ \|\cdot\|_G}^{\mathcal{I}}$ and $\ell_{\infty\ \|\cdot\|_G}^{\mathcal{I}}$ be classes of all gradually $I-$convergent, gradually $I-$null and gradually $I-$bounded sequences in $\mathbb{V}$, then the following inclusions are proper.*

$$c_{0\ \|\cdot\|_G}^{\mathcal{I}} \subset c_{\|\cdot\|_G}^{\mathcal{I}} \subset \ell_{\infty\ \|\cdot\|_G}^{\mathcal{I}}$$

*Proof.* The inclusion $c_{0\ \|\cdot\|_G}^{\mathcal{I}} \subset c_{\|\cdot\|_G}^{\mathcal{I}}$ is evident.

Let us take $u = (u_i) \in c_{\|\cdot\|_G}^{\mathcal{I}}$ then there exists $r \in \mathbb{V}$ such that $u_i \xrightarrow{\mathcal{I}-\|\cdot\|_G} r$. That is

$$\left\{ i \in \mathbb{N} : \mathcal{A}_{\|u_i - r\|_G}(\alpha) \geq \varepsilon \right\} \in \mathcal{I}.$$

We can write

$$\begin{aligned} \mathcal{A}_{\|u_i - 0\|_G}(\alpha) &= \mathcal{A}_{\|u_i - (r - r)\|_G}(\alpha) \\ &\leq \mathcal{A}_{\|u_i - r\|_G}(\alpha) + \mathcal{A}_{\|u_i - (-r)\|_G}(\alpha). \end{aligned}$$

In view of the definition of $\ell_{\infty\|\cdot\|_G}^{\mathcal{I}}$ it follows that $(u_i) \in \ell_{\infty\|\cdot\|_G}^{\mathcal{I}}$.
Hence $c_{\|\cdot\|_G}^{\mathcal{I}} \subset \ell_{\infty\|\cdot\|_G}^{\mathcal{I}}$. Finally

$$c_{0\|\cdot\|_G}^{\mathcal{I}} \subset c_{\|\cdot\|_G}^{\mathcal{I}} \subset \ell_{\infty\|\cdot\|_G}^{\mathcal{I}}.$$ □

**Theorem 3.6.** *Let $\mathscr{X} = \left\{ c_{0\ \|\cdot\|_G}^{\mathcal{I}}, \ell_{\infty\ \|\cdot\|_G}^{\mathcal{I}}, m_{0\ \|\cdot\|_G}^{\mathcal{I}} \right\}$. Then the class of spaces $\mathscr{X}$ is solid and monotone.*

*Proof.* We shall prove the theorem for $\mathscr{X} = c^{\mathcal{I}}_{0\|\cdot\|_G}$, the proof for remaining classes have same procedure. Let $u = (u_i) \in \mathscr{X}$ and $\alpha \in (0,1]$, then for every $\varepsilon > 0$

$$\mathcal{B}_u = \left\{ i \in \mathbb{N} : \mathcal{A}_{\|u_i - 0\|_G}(\alpha) \geq \frac{\varepsilon}{2} \right\} \in \mathcal{I}. \tag{3.7}$$

Chose $\beta = (\beta_i)$ with $|\beta| \leq 1,\ i \in \mathbb{N}$, then

$$\mathcal{A}_{\|(\beta_i u_i) - 0\|_G}(\alpha) \leq |\beta_i| \mathcal{A}_{\|u_i - 0\|_G}(\alpha) \leq \mathcal{A}_{\|u_i - 0\|_G}(\alpha), \quad \text{for all } n \in \mathbb{N}$$

looking to (3.7), we have

$$\left\{ i \in \mathbb{N} : \mathcal{A}_{\|\beta_i u_i - 0\|_G}(\alpha) \geq \varepsilon \right\} \subseteq \left\{ i \in \mathbb{N} : \mathcal{A}_{\|u_i - 0\|_G}(\alpha) \geq \varepsilon \right\} \in \mathcal{I}.$$
$$\Rightarrow \quad \left\{ i \in \mathbb{N} : \mathcal{A}_{\|\beta_i u_i - 0\|_G}(\alpha) \geq \varepsilon \right\} \in \mathcal{I}.$$
$$\Rightarrow \quad (\beta_i u_i) \in \mathscr{X} = c^{\mathcal{I}}_{0\|\cdot\|_G}.$$

Hence $c^{\mathcal{I}}_{0\|\cdot\|_G}$ is solid. Finally by lemma (2.22), $c^{\mathcal{I}}_{0\|\cdot\|_G}$ is monotone. □

**Theorem 3.7.** *The spaces* $c^{\mathcal{I}}_{\|\cdot\|_G}$, $c^{\mathcal{I}}_{0\ \|\cdot\|_G}$ *and* $\ell^{\mathcal{I}}_{\infty\ \|\cdot\|_G}$ *are sequence algebra.*

*Proof.* Let $(u_i)$ and $(v_i)$ be two sequences in $c^{\mathcal{I}}_{0\|\cdot\|_G}$ and $\alpha \in (0,1]$, then for every given $\varepsilon_1, \varepsilon_2 > 0$ we have,

$$\left\{ i \in \mathbb{N} : \mathcal{A}_{\|u_i - 0\|_G}(\alpha) \geq \varepsilon_1 \right\} \in \mathcal{I}, \qquad \left\{ i \in \mathbb{N} : \mathcal{A}_{\|v_i - 0\|_G}(\alpha) \geq \varepsilon_2 \right\} \in \mathcal{I}.$$

In other word

$$\mathcal{I}\text{-} \lim_{i \to \infty} \mathcal{A}_{\|u_i - 0\|_G}(\alpha) = \mathcal{A}_0(\alpha) = \tilde{0}$$
$$\mathcal{I}\text{-} \lim_{i \to \infty} \mathcal{A}_{\|v_i - 0\|_G}(\alpha) = \mathcal{A}_0(\alpha) = \tilde{0}.$$

Therefore we have

$$\mathcal{I}\text{-} \lim_{i \to \infty} \mathcal{A}_{\|u_i v_i - 0\|_G}(\alpha) = \mathcal{I}\text{-} \lim_{i \to \infty} \mathcal{A}_{\|v_i u_i - 0\|_G}(\alpha) = \mathcal{A}_0(\alpha)$$
$$\Rightarrow \quad \left\{ i \in \mathbb{N} : \mathcal{A}_{\|u_i v_i - 0\|_G}(\alpha) \geq \varepsilon \right\} \in \mathcal{I}.$$

Thus $(u_i) \cdot (v_i) \in c^{\mathcal{I}}_{0\|\cdot\|_G}$ and hence $c^{\mathcal{I}}_{0\|\cdot\|_G}$ is sequence algebra. □

**Theorem 3.8.** *Let* $u = (u_i) \in \mathbb{V}$ *be a sequence, then* $u = (u_i)$ *is gradually* $\mathcal{I}-$*convergent* $\Leftrightarrow$ *for every* $\varepsilon > 0$ *and each* $\alpha \in (0,1]$ *there exists* $\mathcal{N} = \mathcal{N}_\varepsilon(\alpha) \in \mathbb{N}$ *such that*

$$\left\{ i \in \mathbb{N} : \mathcal{A}_{\|u_i - u_{\mathcal{N}}\|_G}(\alpha) < \varepsilon \right\} \in \mathcal{F}(\mathcal{I}) \tag{3.8}$$

*Proof.* Assume that $u_i \xrightarrow{\mathcal{I}-\|\cdot\|_G} r$, $r \in \mathbb{V}$. Then for every $\varepsilon > 0$ and each $\alpha \in (0,1]$ we have

$$\mathcal{S}_\varepsilon = \left\{ i \in \mathbb{N} : \mathcal{A}_{\|u_i - r\|_G}(\alpha) < \frac{\varepsilon}{2} \right\} \in \mathcal{F}(\mathcal{I})$$

chose $\mathcal{N} = \mathcal{N}_\varepsilon(\alpha) \in \mathcal{S}_\varepsilon$ then

$$\begin{aligned}\mathcal{A}_{\|u_i - u_\mathcal{N}\|_G}(\alpha) &\leq \mathcal{A}_{\|u_i-(-r)\|_G}(\alpha) + \mathcal{A}_{\|r-u_\mathcal{N}\|_G}(\alpha)\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\\ &= \varepsilon.\end{aligned}$$

Hence

$$\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - u_\mathcal{N}\|_G}(\alpha) < \varepsilon\} \in \mathcal{F}(\mathcal{I}). \tag{3.9}$$

Conversely let (3.9) holds, then;

$$\mathcal{S}'_\varepsilon = \{i \in \mathbb{N} : \mathcal{A}_{\|u_i - 0\|_G}(\alpha) \in \mathcal{J}_\varepsilon\} \in \mathcal{F}(\mathcal{I}), \quad \forall\, \varepsilon > 0$$

Where $\mathcal{J}_\varepsilon = [u_i - \varepsilon, u_i + \varepsilon]$. Then we have $\mathcal{S}'_\varepsilon \in \mathcal{F}(\mathcal{I})$ and $\mathcal{S}'_{\varepsilon/2} \in \mathcal{F}(\mathcal{I})$. Hence $\mathcal{S}'_\varepsilon \cap \mathcal{S}'_{\varepsilon/2} \in \mathcal{F}(\mathcal{I})$. This implies that,

$$\mathcal{J} = \mathcal{J}_\varepsilon \cap \mathcal{J}_{\varepsilon/2} \neq \emptyset\ .$$

That is $\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - u_\mathcal{N}\|_G}(\alpha) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$ $diam(\mathcal{J}) \leq \frac{1}{2} diam(\mathcal{J}_\varepsilon)$. Where *diam* means length of interval. Take up the same procedure, by induction we get a sequence of closed intervals.

$$\mathcal{J}_\varepsilon = \mathcal{J}_0 \supseteq \mathcal{J}_1 \supseteq \mathcal{J}_2 \ \cdots \supseteq \mathcal{J}_i \ \cdots$$

with property

$$diam(\mathcal{J}_n) \leq \frac{1}{2} diam(\mathcal{J}_{i-1}), \quad \text{for } i = 2, 3, \cdots$$

and

$$\{i \in \mathbb{N} : \mathcal{A}_{\|u_i - u_\mathcal{N}\|_G}(\alpha) < \varepsilon\} \in \mathcal{F}(\mathcal{I}).$$

Hence there exists $r \in \bigcap_{i\in\mathbb{N}} \mathcal{J}_i$, such that $u_i \xrightarrow{\mathcal{I}-\|\cdot\|_G} r$. Observing that $u = (u_i) \in \mathbb{V}$ is gradually $\mathcal{I}-$convergent. □

**Theorem 3.9.** *The classes $c^{\mathcal{I}}_{\|\cdot\|_G}$, $m^{\mathcal{I}}_{\|\cdot\|_G}$ and $m^{\mathcal{I}}_{0\|\cdot\|_G}$ are closed sub-spaces of $\ell_{\infty\|\cdot\|_G}$.*

*Proof.* Suppose $\left(u_i^{(j)}\right)$ is a Cauchy sequences in $c^{\mathcal{I}}_{\|\cdot\|_G}$, then $\left(u_i^{(j)}\right)$ is convergent in $\ell_{\infty\|\cdot\|_G}$ and $\lim_{j\to\infty} u_i^{(j)} = u_i$.

Let $u_i \xrightarrow{\mathcal{I}-\|\cdot\|_G} \beta_j$ for $j \in \mathbb{N}$. Then we have to show that;

(i) $(\beta_j) \to \beta$, where $\beta \in G(\mathbb{R})$.

(ii) $u_i \xrightarrow{\mathcal{I}-\|\cdot\|_G} \beta$.

(i) As $\left(u_i^{(j)}\right)$ is a Cauchy sequence then for every $\varepsilon > 0$, there is $\mathcal{N} = \mathcal{N}_\varepsilon(\alpha) \in \mathbb{N}$ such that

$$\mathcal{B}_{jk} = \left\{ i \in \mathbb{N} : \ \mathcal{A}_{\left\| u_i^{(j)} - u_i^{(k)} \right\|_G}(\alpha) < \frac{\varepsilon}{3}, \ \forall \ j,k \geq \mathcal{N} \right\} \tag{3.10}$$

$$\mathcal{B}_{j} = \left\{ i \in \mathbb{N} : \ \mathcal{A}_{\left\| u_i^{(j)} - \beta_j \right\|_G}(\alpha) < \frac{\varepsilon}{3}, \ \forall \ j,k \geq \mathcal{N} \right\} \tag{3.11}$$

$$\mathcal{B}_{k} = \left\{ i \in \mathbb{N} : \ \mathcal{A}_{\left\| u_i^{(j)} - \beta_k \right\|_G}(\alpha) < \frac{\varepsilon}{3}, \ \forall \ j,k \geq \mathcal{N} \right\} \tag{3.12}$$

Then $\mathcal{B}^c_{jk}, \mathcal{B}^c_j, \mathcal{B}^c_k \in \mathcal{I}$. Let $\mathcal{B}^c = \mathcal{B}^c_{jk} \cup \mathcal{B}^c_j \cup \mathcal{B}^c_k \in \mathcal{I}$. Where $\mathcal{B} = \left\{ i \in \mathbb{N} : \ \mathcal{A}_{\|\beta_j - \beta_k\|_G}(\alpha) < \varepsilon \right\} \in \mathcal{I}$.

Let $j,k \geq \mathcal{N}$ and $i \notin \mathcal{B}_j \cap \mathcal{B}_k$, then by using,(3.10),(3.11),(3.12) we have;

$$\begin{aligned} \mathcal{A}_{\|\beta_j - \beta_k\|_G}(\alpha) &\leq \mathcal{A}_{\left\| u_i^{(j)} - \beta_j \right\|_G}(\alpha) + \mathcal{A}_{\left\| u_i^{(j)} - \beta_k \right\|_G}(\alpha) + \mathcal{A}_{\left\| u_i^{(j)} - u_i^{(k)} \right\|_G}(\alpha) \\ &< \varepsilon \end{aligned}$$

Thus $(\beta_j)$ is a Cauchy sequence in $G(\mathbb{R})$ and hence $\lim_{j \to \infty} \beta_j = \beta$.

(ii) Let us take the numbers $\delta > 0$ and $\gamma$ as

$$\mathcal{A}_{\|\beta_j - \beta\|_G}(\alpha) < \frac{\delta}{3}, \quad \text{for each } j > \gamma \ . \tag{3.13}$$

Since $(u_i^{(j)}) \longrightarrow u_i$ as $j \longrightarrow \infty$, thus

$$\mathcal{A}_{\left\| u_i^{(j)} - u_i^{(k)} \right\|_G}(\alpha) < \frac{\delta}{3}, \quad \forall \ j > \gamma \ . \tag{3.14}$$

As $(u_i^{(j)}) \xrightarrow{\mathcal{I} - \|\cdot\|_G} \beta_k$, there exist $\mathcal{S} \in \mathcal{I}$, such that for each $i \notin \mathcal{S}$, we have

$$\mathcal{A}_{\left\| u_i^{(j)} - \beta_k \right\|_G}(\alpha) < \frac{\delta}{3}, \quad \forall \ j > \gamma \tag{3.15}$$

by using (3.13),(3.14),(3.15), for $k > \gamma$, we have

$$\begin{aligned} \mathcal{A}_{\|u_i - \beta\|_G}(\alpha) &\leq \mathcal{A}_{\|u_i - u_i^{(k)}\|_G}(\alpha) + \mathcal{A}_{\|u_i^{(k)} - \beta_k\|_G}(\alpha) \\ &\quad + \mathcal{A}_{\|\beta_k - \beta\|_G}(\alpha) < \delta, \ \forall \ i \notin \mathcal{S} \in \mathcal{I} \quad \Rightarrow \quad (u_i) \xrightarrow{\mathcal{I} - \|\cdot\|_G} \beta. \end{aligned}$$

Thus $c^{\mathcal{I}}_{\|\cdot\|_G}$ is closed subspace of $\ell_{\infty\|\cdot\|_G}$.

Similarly we can show that $m^{\mathcal{I}}_{0\|\cdot\|_G}$ and $m^{\mathcal{I}}_{\|\cdot\|_G}$ are closed subspace of $\ell_{\infty\|\cdot\|_G}$. □

**Theorem 3.10.** *Suppose $u=(u_k)\in\mathbb{V}$ be a sequence and let $\mathcal{I}\subseteq\mathbb{N}$ be an admissible ideal. If $v=(v_k)\in c^{\mathcal{I}}_{\|\cdot\|_G}$ is a sequence, such that $u_k=v_k$ for a.a.k.r.$\mathcal{I}$, then $u=(u_k)\in c^{\mathcal{I}}_{\|\cdot\|_G}$.*

*Proof.* Let $u_k=v_k$ for $a.a.k.r.\mathcal{I}$, i.e. $\mathcal{S}=\{k\in\mathbb{N}:u_k\neq v_k\}\in\mathcal{I}$. Let $(v_k)$ be a sequence such that $v_k\xrightarrow{\mathcal{I}-\|\cdot\|_G}\ell,\ \ell\in G(\mathbb{R})$, then for any given $\varepsilon>0$ and $\alpha\in(0,1]$ we have $\mathcal{S}_v=\left\{k\in\mathbb{N}:\mathcal{A}_{\|v_k-\ell\|_G}(\alpha)\geq\varepsilon\right\}\in\mathcal{I}$. As $\mathcal{I}$ is admissible, then the favorable result can be obtained from the following inclusions

$$\begin{aligned}\mathcal{S}_u&=\left\{k\in\mathbb{N}:\mathcal{A}_{\|u_k-\ell\|_G}(\alpha)\geq\varepsilon\right\}\\&\subseteq\{k\in\mathbb{N}:u_k\neq v_k\}\cup\left\{k\in\mathbb{N}:\mathcal{A}_{\|v_k-\ell\|_G}(\alpha)\geq\varepsilon\right\}.\end{aligned}$$

then $\mathcal{S}_u\subseteq\mathcal{S}\cup\mathcal{S}_v$. Since $\mathcal{S}\in\mathcal{I}$ and $\mathcal{S}_v\in\mathcal{I}$, then $\mathcal{S}_u\in\mathcal{I}$. Hence $(u_k)\in c^{\mathcal{I}}_{\|\cdot\|_G}$. □

**Theorem 3.11.** *Let $u=(u_i)$ be a sequence in one of the classes $c^{\mathcal{I}}_{0\|\cdot\|_G}$, $c^{\mathcal{I}}_{\|\cdot\|_G}$, $\ell^{\mathcal{I}}_{\|\cdot\|_G}$. If every sub sequence $(v_j)$ of $(u_i)$ is gradually $\mathcal{I}$-convergent to $u$, then $u_i\xrightarrow{\mathcal{I}-\|\cdot\|_G}u$.*

*Proof.* We shall prove the result by contradiction.

Let us assume that $u_i\xrightarrow{\mathcal{I}-\|\cdot\|_G}u$ is impossible. Then for any given $\varepsilon>0$ and every $\alpha\in(0,1]$, we have

$$\mathcal{S}_u=\left\{i\in\mathbb{N}:\mathcal{A}_{\|u_i-u\|_G}(\alpha)\geq\varepsilon\right\}\notin\mathcal{I}.$$

This implies that $\mathcal{S}_u$ is infinite. Now take a set $I$ as $I=\{i_1<i_2<\cdots<i_j<\cdots\}$ and let $(v_i)$ be a sub sequence of $(u_i)$, i.e. $(v_i)=u_{ij}$. Then $(v_j)$ is not gradually $\mathcal{I}-$convergent to $u$, which is a contradiction to the statement of theorem. Hence $u_i\xrightarrow{\mathcal{I}-\|\cdot\|_G}u$ if $v_j\xrightarrow{\mathcal{I}-\|\cdot\|_G}u$ □

**Theorem 3.12.** *Let $u=(u_i)$ be an arbitrary sequence in $\mathscr{X}=\left\{c^{\mathcal{I}}_{0\|\cdot\|_G},\ c^{\mathcal{I}}_{\|\cdot\|_G},\ \ell^{\mathcal{I}}_{\|\cdot\|_G}\right\}$. If $u_i\xrightarrow{\mathcal{I}-\|\cdot\|_G}u$, for $u\in\mathscr{X}$. Then $(u_i)$ is gradually $\mathcal{I}-$Cauchy.*

*Proof.* Let $(u_i)\in\mathscr{X}=c^{\mathcal{I}}_{\|\cdot\|_G}$ be gradually $\mathcal{I}-$convergent to some $u\in\mathscr{X}$, then for any given $\varepsilon>0$ and every $\alpha\in(0,1]$, we have

$$\begin{aligned}\mathcal{S}_u&=\left\{i\in\mathbb{N}:\mathcal{A}_{\|u_i-u\|_G}(\alpha)\geq\varepsilon\right\}\in\mathcal{I}\\\mathcal{F}_u&=\left\{i\in\mathbb{N}:\mathcal{A}_{\|u_i-u\|_G}(\alpha)<\varepsilon\right\}\in\mathcal{F}(\mathcal{I}).\end{aligned}$$

Let $\mathcal{N}=\mathcal{N}_\varepsilon(\alpha)\in\mathcal{F}(\mathcal{I})$ be a number, then $\mathcal{A}_{\|u_i-u_{\mathcal{N}}\|_G}(\alpha)<\varepsilon$. Now chose

$$\mathcal{S}_{u_{\mathcal{N}}}=\left\{i\in\mathbb{N}:\mathcal{A}_{\|u_i-u_{\mathcal{N}}\|_G}(\alpha)\geq 2\varepsilon\right\},$$

then we have to show that $\mathcal{S}_{u_{\mathcal{N}}} \subseteq \mathcal{S}_u$.

Let $c \in \mathcal{S}_{u_{\mathcal{N}}}$, be a number then

$$\begin{aligned} 2\varepsilon &\leq \mathcal{A}_{\|u_c - u_{\mathcal{N}}\|_G}(\alpha) \\ &\leq \mathcal{A}_{\|u_c - u\|_G}(\alpha) + \mathcal{A}_{\|u - u_{\mathcal{N}}\|_G}(\alpha) \\ &< \mathcal{A}_{\|u_c - u\|_G}(\alpha) + \varepsilon. \\ &\Rightarrow \quad \mathcal{A}_{\|u_c - u\|_G}(\alpha) \geq \varepsilon \\ &\Rightarrow \quad c \in \mathcal{S}_u \\ &\Rightarrow \quad \mathcal{S}_{u_{\mathcal{N}}} \in \mathcal{I}. \end{aligned}$$

This implies that $u_i$ is gradually $\mathcal{I}-$ Cauchy sequence.

Similarly we can show the result for $\mathscr{X} = c^{\mathcal{I}}_{0\|\cdot\|_G}$ and $\mathscr{X} = \ell^{\mathcal{I}}_{\infty\|\cdot\|_G}$. □

## Conclusion

The concept of gradual numbers reckons a previously missing gist of completeness of fuzzy set theory. Gradual numbers to measure the length of fuzzy intervals or the size of finite fuzzy set. The term gradual numbers were first introduced by Fortin, Dubois and Fargier in 2006 to solve the confusion related the algebraic behaviour of fuzzy numbers. After that, several authors and researchers extended the concept of gradual numbers and studied several algebraic and topological properties of these preceding numbers. Also, many researchers applied gradual numbers in various areas of mathematics, operations research, physics, computer science and engineering.

In this paper, we studied the $\mathcal{I}-$convergence of some classes of bounded sequences $\ell_\infty$, convergence sequences $c$ and null sequences $c_0$ in gradual normed linear spaces $(\mathbb{V}, \|\cdot\|_G)$. We defined some classes of $\mathcal{I}-$bounded sequences $\ell^{\mathcal{I}}_\infty$, $\mathcal{I}$-convergent sequences $c^{\mathcal{I}}$,and $\mathcal{I}-$null sequences $c^{\mathcal{I}}_0$ in gradual normed linear spaces, presented by (3.1),(3.2),(3.3), (3.4). Also, we discovered some algebraic and topological properties for these forgoing spaces, , such as linearity, solidity, monotonicity, closedness and the relation between gradual $\mathcal{I}-$convergence and gradual $\mathcal{I}-$Cauchy. We also showed that the inclusions $c^{\mathcal{I}}_{0\|\cdot\|_G} \subseteq c^{\mathcal{I}}_{\|\cdot\|_G} \subseteq \ell^{\mathcal{I}}_{\infty\|\cdot\|_G}$ are hold. Finally, we we would like to indicate that gradual $\mathcal{I}-$convergence is a fresh,, interesting and useful tool in fuzzy mathematics, operations research and engineering. The results discussed here yield novel tools to sort and solve some problems of sequence spaces of gradual $\mathcal{I}-$convergence in several fields of mathematics, science and engineering. These new results and methodology will help to researchers to boost the study on $\mathcal{I}-$convergence in GNLS.

## Authors Contributions

All authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript.

## Competing Interests

The authors declare that they have no competing interests.

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