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Generalized Niezgoda's Inequality with Refinements and Applications

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Abstract. Motivated by the results of Niezgoda, corresponding to the generalization of Mercer's inequality for positive weights, in this paper, we consider real weights, for which we establish related results. To be more specific, Niezgoda's results are derived under Jensen Steffensen conditions. In addition, we construct some functionals enabling us to refine Niezgoda's results. Lastly, we discuss some applications.

1. Introduction

The well-known Jensen's inequality for convex functions is among the most important inequalities in mathematics and statistics. Jensen's inequality asserts a remarkable relation between the mean and the mean of function values. Any generalization or refinements of Jensen's inequality is a source of enrichment of the monotone property of mixed means. Applications of Jensen's inequality in statistics and probability related to the expectation of a convex function of a random variable are of great significance. Moreover, many other essential inequalities may be obtained from it, such as Hölder's and Minkowski's inequalities.

In 2003, A. McD. Mercer [3] has proven a variant of Jensen inequality. This variant furnished a new field for scholars. Notably, in 2009, M. Niezgoda in [21] provided a generalization of Mercer's results and pointed out the relationship between majorization ordering and Mercer's result. Furthermore, in the same article, Niezgoda extended Mercer's result to a pair of similarly separable vectors for convex functions. In 2012, Khan et al. beautified Niezgoda's result [21] by proposing refinement of Jensen−Mercer inequality in [16] (see also [20, 27, 17, 4, 5, 6, 19, 24, 15]). In the present article, we would like to give a generalization of Niezgoda's inequality and

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its refinements with applications. Specifically, we would give some relationships between generalized arithmetic, geometric and harmonic means. We would also get Ky Fan type [8, pp. 25-28], Popoviciu type [28] and Rado type [25] inequalities in our application section.

In all over the article we assume that $(\mu_1, \nu_1) \subset \mathbb{R}$ and $\mu_1 < \nu_1$. Considering positive $m-$tuple $\boldsymbol{\rho} = (\rho_1, \dots, \rho_m)$, we define inner product on $\mathbb{R}^m$ by

$$\langle \boldsymbol{\sigma}, \boldsymbol{\beta} \rangle = \sum_{j=1}^{m} \rho_j \sigma_j \beta_j \tag{1.1}$$

for $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_m)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$. Also for the positive weights $\rho_1, \dots, \rho_m$, we define the notations

$\mathrm{P}_j = \sum\limits_{i=1}^{j} \rho_i, \forall\, j \in \{1, \dots, m\}$ and of course, $P_m = \sum\limits_{j=1}^{m} \rho_j$.

Furthermore, for the real n$-$tuple $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$, we define the notations

$\mathrm{W}_j = \sum\limits_{i=1}^{j} \omega_i, \forall\, j \in \{1, \dots, n\}$ and of course, $W_n = \sum\limits_{j=1}^{n} \omega_j$.

Jensen's inequality [11, p. 43]) (see also [13] and [14]) is one of the well-known result.

Theorem 1.1. *Let* $\boldsymbol{\varsigma} = (\varsigma_1, \dots, \varsigma_n)$ *be* $n-$*tuple in* $(\mu_1, \nu_1)^n$*, and* $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n)$ *be a positive* $n-$*tuple. If* Υ *is convex function on* (μ_1, ν_1)*, then*

$$\Upsilon\left(\frac{1}{\mathrm{P}_n} \sum_{i=1}^{n} \rho_i \varsigma_i\right) \leq \frac{1}{\mathrm{P}_n} \sum_{i=1}^{n} \rho_i \Upsilon(\varsigma_i) \tag{1.2}$$

holds.

The supposition "$\boldsymbol{\rho}$ is positive $n-$tuple" in Theorem 1.1 can be compensated by "$\boldsymbol{\rho}$ is a non$-$negative $n-$tuple" with $P_n > 0$. That is acceptable to question whether the supposition "$\boldsymbol{\rho}$ is a non$-$negative $n-$tuple" can be reduced at the surcharge of tightening $\boldsymbol{\varsigma}$ more strictly. Steffensen (see [12]) was the pioneer to address this issue in Theorem 1.2 (see also [11, p. 57]).

Theorem 1.2. *Let* $\boldsymbol{\varsigma} = (\varsigma_1, \dots, \varsigma_n)$ *be a monotonic* $n-$*tuple in* $(\mu_1, \nu_1)^n$ *and* $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n)$ *is a real* $n-$*tuple such that* $\frac{1}{\mathrm{P}_n} \sum_{i=1}^{n} \rho_i \varsigma_i \in (\mu_1, \nu_1)$ *and*

$$0 \leq \mathrm{P}_i \leq \mathrm{P}_n, \qquad \mathrm{P}_n > 0, \quad for\, i \in \{1, \dots, n\}. \tag{1.3}$$

If Υ *is convex function on* (μ_1, ν_1)*, then* (1.2) *still holds.*

(1.2) *under conditions of* (1.3) *is called Jensen Steffensen inequality.*

Mercer [3] furnished a variant of (1.2) which is named as "Jensen$-$Mercer inequality".

Theorem 1.3. *Following the supposition of Theorem* 1.1*, the inequality* (1.4) *holds.*

$$\Upsilon\left(\theta+\eta-\frac{1}{\mathrm{P}_n}\sum_{i=1}^{n}\rho_i\varsigma_i\right)\leq\Upsilon(\theta)+\Upsilon(\eta)-\frac{1}{\mathrm{P}_n}\sum_{i=1}^{n}\rho_i\Upsilon(\varsigma_i),\tag{1.4}$$

where

$$\theta=\min_{\forall\varsigma_i\in(\mu_1,\nu_1)}\{\varsigma_i\}\quad \textit{and}\quad \eta=\max_{\forall\varsigma_i\in(\mu_1,\nu_1)}\{\varsigma_i\}.$$

The following generalization of (1.4) is given in [26].

Theorem 1.4. *Following the supposition of Theorem* 1.2*, inequality* (1.4) *holds.*

In paper [18], Bakula et al. proposed a paramount result which enables us to obtain (1.2) under conditions of (1.3). In this place and in all over the article, we take into consideration a convex function $\Upsilon:(\mu_1,\nu_1)\to\mathbb{R}$, where $-\infty\leq\mu_1<\nu_1\leq+\infty$, for $\Upsilon'(\varsigma)$, where $\varsigma\in(\mu_1,\nu_1)$, we may take an element of $[\Upsilon'_-(\varsigma),\Upsilon'_+(\varsigma)]$; however, without any generality loss we can set $\Upsilon'(\varsigma)=\Upsilon'(\varsigma)$ (indeed, if Υ is differentiable then $\Upsilon'(\varsigma)=\Upsilon'_+(\varsigma)=\Upsilon'_-(\varsigma)$).

Theorem 1.5. *Following the supposition of Theorem* 1.2*, we have*

$$\Upsilon(\mathrm{c})+\Upsilon'(\mathrm{c})(\bar{\varsigma}-\mathrm{c})\leq\frac{1}{\mathrm{P}_n}\sum_{i=1}^{n}\rho_i\Upsilon(\varsigma_i)\leq\Upsilon(\mathrm{d})+\frac{1}{\mathrm{P}_n}\sum_{i=1}^{n}\rho_i\Upsilon'(\varsigma_i)(\varsigma_i-\mathrm{d})$$

hold $\forall\ \mathrm{c},\mathrm{d}\in(\mu_1,\nu_1)$*, where*

$$\bar{\varsigma}:=\frac{1}{\mathrm{P}_n}\sum_{i=1}^{n}\rho_i\varsigma_i.$$

Now we state the definition of majorization from [21] as follows. Let two m-tuples $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_m)$ and $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_m)$ be such that $\sigma_{[1]}\geq\cdots\geq\sigma_{[m]}$, $\beta_{[1]}\geq\cdots\geq\beta_{[m]}$ be their ordered components.

Definition 1.6. For $\boldsymbol{\sigma}$, $\boldsymbol{\beta}\in\mathbb{R}^m$

$$\sigma\prec\beta\text{if}\begin{cases}\sum\limits_{j=1}^{\kappa}\sigma_{[j]}\leq\sum\limits_{j=1}^{\kappa}\beta_{[j]}, & \kappa\in\{1,\ldots,m-1\}\\ \sum\limits_{j=1}^{m}\sigma_{[j]}=\sum\limits_{j=1}^{m}\beta_{[j]}\end{cases}$$

When $\boldsymbol{\sigma}\prec\boldsymbol{\beta}$, we say "$\boldsymbol{\beta}$ majorizes $\boldsymbol{\sigma}$" or "$\boldsymbol{\sigma}$ majorized by $\boldsymbol{\beta}$".

The majorization approach was first brought in by Hardy et al. In their book "Inequalities", [10], we can identify the famous majorization theorem. Using the definition of majorization stated above, we are ready to state an extension of inequality (1.4) presented by Niezgoda in [21]. We would call it "Niezgoda's inequality".

Theorem 1.7. *Assume Υ is continuous convex function on (μ_1, ν_1). Suppose $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m) \in (\mu_1, \nu_1)^m$ and $\mathbf{X} = (\varsigma_{ij})$ is an $n \times m$ matrix such that $\varsigma_{ij} \in (\mu_1, \nu_1)$ $\forall$ $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. If $\boldsymbol{\sigma}$ majorizes each row of $\mathbf{X}$, i.e.,*

$$\boldsymbol{\varsigma_{i.}} = (\varsigma_{i1}, \ldots, \varsigma_{im}) \prec (\sigma_1, \ldots, \sigma_m) = \boldsymbol{\sigma} \text{ for each } i \in \{1, \ldots, n\},$$

then the following inequality holds:

$$\Upsilon\left(\sum_{j=1}^{m} \sigma_j - \sum_{j=1}^{m-1} \sum_{i=1}^{n} \omega_i \varsigma_{ij}\right) \leq \sum_{j=1}^{m} \Upsilon(\sigma_j) - \sum_{j=1}^{m-1} \sum_{i=1}^{n} \omega_i \Upsilon(\varsigma_{ij}) \tag{1.5}$$

where $\sum_{i=1}^{n} \omega_i = 1$ with $\omega_i \geq 0$ $\forall i \in \{1, \ldots, n\}$.

The present article is divided into the following sections: The 1st section contains preliminaries and an introduction. In 2nd section, we generalize Niezgoda's result [21] by considering real weights satisfying the Jensen–Steffensen condition. In 3rd section, we construct functionals to establish refinements of our results proved in 2nd section. The 4th section contains applications of our main results and the last section concludes the article.

2. Generalization of Niezgoda's Inequality

In this section for a given $n \times m$ matrix $\mathbf{X} = (\varsigma_{ij})$ such that $\varsigma_{ij} \in (\mu_1, \nu_1)$ $\forall i, j$, we define a matrix $\Upsilon(\mathbf{X}) = \Upsilon(\varsigma_{ij})$. The ith row and jth column of $\mathbf{X}$ are described by $\boldsymbol{\varsigma_{i.}}$ and $\boldsymbol{\varsigma_{.j}}$, respectively. *e.g.*, $\Upsilon(\boldsymbol{\varsigma_{i.}}) = (\Upsilon(\varsigma_{i1}), \ldots, \Upsilon(\varsigma_{im}))^T$. Now we give the generalization of Theorem 1.7.

Theorem 2.1. *Let $\Upsilon : (\mu_1, \nu_1) \longrightarrow \mathbb{R}$ be a continuous convex function. Let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m) \in (\mu_1, \nu_1)^m$ and $\mathbf{X} = (\varsigma_{ij})$ is a real $n \times m$ matrix with $\varsigma_{ij} \in (\mu_1, \nu_1)$ $\forall$ $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ such that*

$$\varsigma_{1j} \geq \varsigma_{2j} \geq \cdots \geq \varsigma_{nj} \quad \text{or} \quad \varsigma_{1j} \leq \varsigma_{2j} \leq \cdots \leq \varsigma_{nj}$$

Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ be a real $n-$tuple such that $\frac{1}{\mathrm{W}_n} \sum_{i=1}^{n} \omega_i \varsigma_{ij} \in (\mu_1, \nu_1)$ for each $j \in \{1, \ldots, m\}$ and the conditions on weights given in (1.3) *hold. If for each $i \in \{1, \ldots, n\}$ we have*

$$\sum_{j=1}^{m} \varsigma_{ij} = \sum_{j=1}^{m} \sigma_j \tag{2.1}$$

$$\sum_{j=1}^{m} \varsigma_{ij} \Upsilon'(\varsigma_{ij}) = \sum_{j=1}^{m} \sigma_j \Upsilon'(\varsigma_{ij}), \tag{2.2}$$

then we have the following inequality

$$\Upsilon\left(\sum_{j=1}^{m} \sigma_j - \frac{1}{\mathrm{W}_n} \sum_{i=1}^{n} \sum_{j=1}^{\kappa-1} \omega_i \varsigma_{ij} - \frac{1}{\mathrm{W}_n} \sum_{i=1}^{n} \sum_{j=\kappa+1}^{m} \omega_i \varsigma_{ij}\right) \tag{2.3}$$

$$\leq \sum_{j=1}^{m} \Upsilon(\sigma_j) - \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=1}^{\kappa-1}\omega_i \Upsilon(\varsigma_{ij}) - \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=\kappa+1}^{m}\omega_i \Upsilon(\varsigma_{ij})$$

where $\kappa \in \{1, \ldots, m\}$.

Proof. Fix $\kappa \in \{1, \ldots, m\}$, using first (2.1) and then using Jensen−Steffensen inequality we get,

$$\begin{aligned}(2.4)\qquad & \Upsilon\left(\sum_{j=1}^{m}\sigma_j - \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=1}^{\kappa-1}\omega_i\varsigma_{ij} - \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=\kappa+1}^{m}\omega_i\varsigma_{ij}\right) \\ &= \Upsilon\left(\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\left(\sum_{j=1}^{m}\sigma_j - \sum_{j=1}^{\kappa-1}\varsigma_{ij} - \sum_{j=\kappa+1}^{m}\varsigma_{ij}\right)\right) \\ &= \Upsilon\left(\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\varsigma_{i\kappa}\right) \leq \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{i\kappa}).\end{aligned}$$

Now from Theorem 1.5, we have

$$\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_i) \leq \Upsilon(\mathrm{d}) + \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\Upsilon'(\varsigma_i)(\varsigma_i - \mathrm{d}).$$

Replace first d by σ_j and ς_i by ς_{ij} $\forall\ i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, we have

$$\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij}) \leq \Upsilon(\sigma_j) + \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\Upsilon'(\varsigma_{ij})(\varsigma_{ij} - \sigma_j) \quad \forall\, j \in \{1, \ldots, m\}.$$

By taking sum from 1 to m over j we have

$$\begin{aligned}(2.5)\qquad & \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=1}^{m}\omega_i\Upsilon(\varsigma_{ij}) \\ &\leq \sum_{j=1}^{m}\Upsilon(\sigma_j) + \frac{1}{\mathrm{W}_n}\sum_{j=1}^{m}\sum_{i=1}^{n}\omega_i\Upsilon'(\varsigma_{ij})(\varsigma_{ij} - \sigma_j), \\ &= \sum_{j=1}^{m}\Upsilon(\sigma_j) + \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\left(\sum_{j=1}^{m}\Upsilon'(\varsigma_{ij})\varsigma_{ij} - \sum_{j=1}^{m}\Upsilon'(\varsigma_{ij})\sigma_j\right).\end{aligned}$$

By using (2.2) in (2.5), the second term in right hand side vanishes, we have

$$\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=1}^{m}\omega_i\Upsilon(\varsigma_{ij}) \leq \sum_{j=1}^{m}\Upsilon(\sigma_j)$$

and finally

$$(2.6)\qquad \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{i\kappa})$$

$$\leq \sum_{j=1}^{m} \Upsilon(\sigma_j) - \frac{1}{\mathrm{W}_n} \sum_{i=1}^{n} \sum_{j=1}^{\kappa-1} \omega_i \Upsilon(\varsigma_{ij}) - \frac{1}{\mathrm{W}_n} \sum_{i=1}^{n} \sum_{j=\kappa+1}^{m} \omega_i \Upsilon(\varsigma_{ij}).$$

Using transitive property on (2.4) and (2.6) we get (2.3). □

Remark 2.2. It is essential to highlight that at the expense of (2.1) and (2.2) in Theorem 2.1, we relax the condition of Theorem 1.7 that $\boldsymbol{\sigma}$ majorizes each row of $\mathbf{X}$.

Remark 2.3. If in inequality (2.3) we set $\kappa = m = 2$, $\sigma_1 = \theta$, $\sigma_2 = \eta$ with $\sigma_1 \leq \sigma_2$, $\varsigma_{i1} = \varsigma_i$ and $\varsigma_{i2} = \sigma_1 + \sigma_2 - \varsigma_i$ for $i \in \{1, \dots, n\}$, then inequality (2.3) reduces to inequality (1.4). Hence, Theorem 2.1 is the generalized extension of Jensen–Mercer's inequality.

Remark 2.4. Note that the result [5, Theorem 1] still valid if we replace inequality (2) of [5] by equation (2.2) of this article.

Theorem 2.5. *Let $\Upsilon : (\mu_1, \nu_1) \longrightarrow \mathbb{R}$ be a continuous convex function. Suppose that $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_m) \in (\mu_1, \nu_1)^m$ and $\mathbf{X} = (\varsigma_{ij})$ is a real $n \times m$ matrix with $\varsigma_{ij} \in (\mu_1, \nu_1)\, \forall\, i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ such that*

$$\varsigma_{1j} \geq \varsigma_{2j} \geq \cdots \geq \varsigma_{nj} \quad \text{or} \quad \varsigma_{1j} \leq \varsigma_{2j} \leq \cdots \leq \varsigma_{nj}.$$

Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ be a real $n-$tuple such that $\frac{1}{\mathrm{W}_n} \sum_{i=1}^{n} \omega_i \varsigma_{ij} \in (\mu_1, \nu_1)$ for each $j \in \{1, \dots, m\}$ and the conditions on weights given in (1.3) holds, and the vector $\boldsymbol{\upsilon} \in \mathbb{R}^m$ with $\upsilon_\kappa \neq 0$, $\forall \kappa \in \{1, \dots, m\}$. If for each $i \in \{1, \dots, n\}$ we have

(i) $\langle \boldsymbol{\sigma} - \boldsymbol{\varsigma_{i.}} , \boldsymbol{\upsilon} \rangle = 0$ *and*
(ii) $\langle \boldsymbol{\sigma} - \boldsymbol{\varsigma_{i.}} , \Upsilon'(\boldsymbol{\varsigma_{i.}}) \rangle = 0$,

then we have the following inequality

$$\rho_\kappa \Upsilon \left(\sum_{j=1}^{m} \sigma_j \epsilon \rho_j \upsilon_j - \sum_{j=1}^{\kappa-1} \epsilon \rho_j \upsilon_j \sum_{i=1}^{n} \omega_i \varsigma_{ij} - \sum_{j=\kappa+1}^{m} \epsilon \rho_j \upsilon_j \sum_{i=1}^{n} \omega_i \varsigma_{ij} \right) \tag{2.7}$$
$$\leq \sum_{j=1}^{m} \rho_j \Upsilon(\sigma_j) - \sum_{j=1}^{\kappa-1} \rho_j \sum_{i=1}^{n} \omega_i \Upsilon(\varsigma_{ij}) - \sum_{j=\kappa+1}^{m} \rho_j \sum_{i=1}^{n} \omega_i \Upsilon(\varsigma_{ij}),$$

where $\kappa \in \{1, \dots, m\}$ and $\epsilon = \dfrac{1}{\rho_\kappa \upsilon_\kappa}$ with $\rho_\kappa > 0$.

Proof. Fix $\kappa \in \{1, \dots, m\}$. Under the assumption of the theorem, it follows from Proposition 1.5 that

$$\frac{1}{\mathrm{W}_n} \sum_{i=1}^{n} \omega_i \Upsilon(\varsigma_i) \leq \Upsilon(\mathrm{d}) + \frac{1}{\mathrm{W}_n} \sum_{i=1}^{n} \omega_i \Upsilon'(\varsigma_i)(\varsigma_i - \mathrm{d}).$$

Replace d by σ_j and ς_i by ς_{ij}, we have

$$\frac{1}{\mathrm{W}_n} \sum_{i=1}^{n} \omega_i \Upsilon(\varsigma_{ij}) \leq \Upsilon(\sigma_j) + \frac{1}{\mathrm{W}_n} \sum_{i=1}^{n} \omega_i \Upsilon'(\varsigma_{ij})(\varsigma_{ij} - \sigma_j), \quad \forall\, j \in \{1, \dots, m\}.$$

By multiplying ρ_j and taking sum from 1 to m over j, we get

$$\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\sum_{j=1}^{m}\rho_j\Upsilon(\varsigma_{ij}) \leq \sum_{j=1}^{m}\rho_j\Upsilon(\sigma_j) + \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\sum_{j=1}^{m}\rho_j\Upsilon'(\varsigma_{ij})(\varsigma_{ij}-\sigma_j),$$

or we can write

(2.8)

$$\begin{aligned}\sum_{j=1}^{m}\rho_j\Upsilon(\sigma_j) - \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\sum_{j=1}^{m}\rho_j\Upsilon(\varsigma_{ij}) &\geq \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\sum_{j=1}^{m}\rho_j\Upsilon'(\varsigma_{ij})(\sigma_j-\varsigma_{ij}) \\ &= \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\langle \boldsymbol{\sigma} - \boldsymbol{\varsigma_{i.}}, \Upsilon'(\boldsymbol{\varsigma_{i.}})\rangle \\ &= 0.\end{aligned}$$

The last inequality is due to assumptions (ii). Given that, $\langle \boldsymbol{\sigma} - \boldsymbol{\varsigma_{i.}}, \boldsymbol{\upsilon}\rangle = 0$ for each $i \in \{1, \ldots, n\}$, by (1.1) we have

$$\sum_{j=1}^{m}\sigma_j\epsilon\rho_i\upsilon_j - \sum_{j=1}^{\kappa-1}\epsilon\rho_j\upsilon_i\varsigma_{ij} - \sum_{j=\kappa+1}^{m}\epsilon\rho_j\upsilon_j\varsigma_{ij} = \varsigma_{i\kappa}, \tag{2.9}$$

where $\epsilon = \dfrac{1}{\rho_\kappa\upsilon_\kappa}$, $\forall\kappa \in \{1, \ldots, m\}$. Consider L.H.S of (2.7), using first (2.9) and then applying Jensen Steffensen inequality we get,

(2.10)

$$\begin{aligned}&\rho_\kappa\Upsilon\left(\sum_{j=1}^{m}\sigma_j\epsilon\rho_j\upsilon_j - \frac{1}{\mathrm{W}_n}\sum_{j=1}^{\kappa-1}\epsilon\rho_j\upsilon_j\sum_{i=1}^{n}\omega_i\varsigma_{ij} - \frac{1}{\mathrm{W}_n}\sum_{j=\kappa+1}^{m}\epsilon\rho_j\upsilon_j\sum_{i=1}^{n}\omega_i\varsigma_{ij}\right) \\ &\quad= \rho_\kappa\Upsilon\left(\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\left(\sum_{j=1}^{m}\sigma_j\epsilon\rho_j\upsilon_j - \sum_{j=1}^{\kappa-1}\epsilon\rho_j\upsilon_j\omega_i\varsigma_{ij} - \sum_{j=\kappa+1}^{m}\epsilon\rho_j\upsilon_j\varsigma_{ij}\right)\right) \\ &\quad= \rho_\kappa\Upsilon\left(\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\varsigma_{i\kappa}\right) \\ &\quad\leq \rho_\kappa\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{i\kappa})\end{aligned}$$

from (2.8)

(2.11)

$$\begin{aligned}&\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\rho_\kappa\Upsilon(\varsigma_{i\kappa}) \\ &\quad\leq \sum_{j=1}^{m}\rho_j\Upsilon(\sigma_j) - \frac{1}{\mathrm{W}_n}\sum_{j=1}^{\kappa-1}\rho_j\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij}) - \frac{1}{\mathrm{W}_n}\sum_{j=\kappa+1}^{m}\rho_j\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij}).\end{aligned}$$

Using transitive property on (2.10) and (2.11) we get (2.7). □

Corollary 2.6. *Let all the assumptions of Theorem* 2.1 *be valid and let* $\boldsymbol{\upsilon} = (1, \dots, 1)$. *Then we have the following inequality*

$$\Upsilon\left(\sum_{j=1}^{m}\sigma_j\widetilde{\rho}_j - \sum_{j=1}^{\kappa-1}\widetilde{\rho}_j\sum_{i=1}^{n}\omega_i\varsigma_{ij} - \sum_{j=\kappa+1}^{m}\widetilde{\rho}_j\sum_{i=1}^{n}\omega_i\varsigma_{ij}\right) \leq \sum_{j=1}^{m}\widetilde{\rho}_j\Upsilon(\sigma_j) - \sum_{j=1}^{\kappa-1}\widetilde{\rho}_j\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij}) - \sum_{j=\kappa+1}^{m}\widetilde{\rho}_j\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij}), \tag{2.12}$$

where $\kappa \in \{1, \dots, m\}$ *and* $\widetilde{\rho}_j = \frac{\rho_j}{\rho_\kappa}$ *with* $\rho_\kappa > 0$.

For instance, if $\widetilde{\rho}_j = 1$, $(\widetilde{\rho}_1 = \cdots = \widetilde{\rho}_m)$, *then* (2.12) *reduces to*

$$\Upsilon\left(\sum_{j=1}^{m}\sigma_j - \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=1}^{\kappa-1}\omega_i\varsigma_{ij} - \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=\kappa+1}^{m}\omega_i\varsigma_{ij}\right) \leq \sum_{j=1}^{m}\Upsilon(\sigma_j) - \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=1}^{\kappa-1}\omega_i\Upsilon(\varsigma_{ij}) - \frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=\kappa+1}^{m}\omega_i\Upsilon(\varsigma_{ij}) \tag{2.13}$$

and, in particular, for $\kappa = m$, (2.13) *reduces to*

$$\Upsilon\left(\sum_{j=1}^{m}\sigma_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n}\omega_i\varsigma_{ij}\right) \leq \sum_{j=1}^{m}\Upsilon(\sigma_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij}). \tag{2.14}$$

Furthermore, to be more specific, for $m = 2$, (2.14) *reduces to* (1.4).

Remark 2.7. It is important to highlighted that in our Theorem 2.5 we relax the condition of similarly separable vectors as stated in Theorem 3.1 of [21].

Corollary 2.8. *Let all the assumptions of Theorem* 2.1 *be valid and let* $\boldsymbol{\upsilon} = (1, 2 \dots, m)$. *Then we have the following inequality*

$$\Upsilon\left(\sum_{j=1}^{m}\sigma_j\widetilde{\rho}_j\widetilde{\upsilon}_j - \sum_{j=1}^{\kappa-1}\widetilde{\rho}_j\widetilde{\upsilon}_j\sum_{i=1}^{n}\omega_i\varsigma_{ij} - \sum_{j=\kappa+1}^{m}\widetilde{\rho}_j\sum_{i=1}^{n}\omega_i\varsigma_{ij}\right) \leq \sum_{j=1}^{m}\widetilde{\rho}_j\Upsilon(\sigma_j) - \sum_{j=1}^{\kappa-1}\widetilde{\rho}_j\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij}) - \sum_{j=\kappa+1}^{m}\widetilde{\rho}_j\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij}), \tag{2.15}$$

where $\kappa \in \{1, \dots, m\}$, $\widetilde{\rho}_j = \frac{\rho_j}{\rho_\kappa}$ *with* $\rho_\kappa > 0$ *and* $\widetilde{\upsilon}_j = \frac{j}{\kappa}$.

For instance, if $\widetilde{\rho}_j = 1$, $(\widetilde{\rho}_1 = \cdots = \widetilde{\rho}_m)$, *then* (2.15) *reduces to*

$$\Upsilon\left(\sum_{j=1}^{m}\frac{j}{\kappa}\sigma_j - \sum_{j=1}^{\kappa-1}\frac{j}{\kappa}\sum_{i=1}^{n}\omega_i\varsigma_{ij} - \sum_{j=\kappa+1}^{m}\frac{j}{\kappa}\sum_{i=1}^{n}\omega_i\varsigma_{ij}\right) \leq \sum_{j=1}^{m}\Upsilon(\sigma_j) - \sum_{j=1}^{\kappa-1}\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij}) - \sum_{j=\kappa+1}^{m}\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij}), \tag{2.16}$$

and, in particular, for $\kappa = m$, (2.16) *reduces to*

$$\Upsilon\left(\sum_{j=1}^{m}\frac{j}{m}\sigma_j - \sum_{j=1}^{m-1}\frac{j}{m}\sum_{i=1}^{n}\omega_i\varsigma_{ij}\right) \leq \sum_{j=1}^{m}\Upsilon(\sigma_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij}), \tag{2.17}$$

Furthermore, to be more specific, for $m = 2$, (2.17) *reduces to*

$$\Upsilon\left(\frac{1}{2}\sigma_1 + \sigma_2 - \frac{1}{2}\sum_{i=1}^{n}\omega_i\varsigma_{i1}\right) \leq \Upsilon(\sigma_1) + \Upsilon(\sigma_2) - \sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{i1}). \tag{2.18}$$

3. Refinements

3.1. **Refinements of Niezgoda's Inequality for Index Set Functions.**

Let I be a finite non−empty set of positive integers. Let $\boldsymbol{\omega} = (\omega_i), i \in \{1, \ldots, \mathrm{n}\}$ be a real sequence and let $\mathbf{X} = (\varsigma_{ij})$ be an $n \times m$ matrix such that the entries $\varsigma_{ij} \in (\mu_1, \nu_1)\, \forall\, i, j$.

If we define the index set function $\digamma_1$ as

(3.1)

$$\digamma_1(\mathrm{I}) = W_{\mathrm{I}}\left[\sum_{j=1}^{m}\Upsilon(\sigma_j) - \frac{1}{W_{\mathrm{I}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}}\omega_i\Upsilon(\varsigma_{ij}) - \frac{1}{W_{\mathrm{I}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}}\omega_i\Upsilon(\varsigma_{ij})\right.$$
$$\left. -\Upsilon\left(\sum_{j=1}^{m}\sigma_j - \frac{1}{W_{\mathrm{I}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij} - \frac{1}{W_{\mathrm{I}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}\right)\right]$$

where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m) \in (\mu_1, \nu_1)^m$ and $W_{\mathrm{I}} = \sum_{i\in\mathrm{I}}\omega_i$, then the following result is true.

Theorem 3.1. *Let* I *and* $\bar{\mathrm{I}}$ *be two finite non−empty sets of positive integers such that* $\mathrm{I} \cap \bar{\mathrm{I}} = \emptyset$ *and* $\mathrm{I} \cup \bar{\mathrm{I}} = \{1, \ldots, n\}$. *Let* $\mathbf{X} = (\varsigma_{ij})$ *be an* $n \times m$ *matrix such that the entries* $\varsigma_{ij} \in (\mu_1, \nu_1)\, \forall\, i \in \mathrm{I}, j \in \{1, \ldots, m\}$ *and let* $\boldsymbol{\omega} = (\omega_i), i \in \mathrm{I} \cup \bar{\mathrm{I}}$ *be a real sequence such that* $\frac{1}{\mathrm{W}_S}\sum_{i\in S}\omega_i\varsigma_{ij} \in (\mu_1, \nu_1)(S = \mathrm{I}, \bar{\mathrm{I}}, \mathrm{I} \cup \bar{\mathrm{I}})$.

For $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m) \in (\mu_1, \nu_1)^m$ *and for a convex function* Υ *on an interval* (μ_1, ν_1), *if* $0 < \mathrm{W_S} < \mathrm{W}_{\mathrm{I}\cup\bar{\mathrm{I}}}$, *then we get the following inequality under the assumptions of Theorem* 2.1

$$\digamma_1(\mathrm{I} \cup \bar{\mathrm{I}}) \geq \digamma_1(\mathrm{I}) + \digamma_1(\bar{\mathrm{I}}) \tag{3.2}$$

Proof. Fix $\kappa \in \{1, \ldots, m\}$.

(3.3)

$$\digamma_1(\mathrm{I} \cup \bar{\mathrm{I}})$$
$$= W_{\mathrm{I}\cup\bar{\mathrm{I}}}\left[\sum_{j=1}^{m}\Upsilon(\sigma_j) - \frac{1}{W_{\mathrm{I}\cup\bar{\mathrm{I}}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}\cup\bar{\mathrm{I}}}\omega_i\Upsilon(\varsigma_{ij}) - \frac{1}{W_{\mathrm{I}\cup\bar{\mathrm{I}}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}\cup\bar{\mathrm{I}}}\omega_i\Upsilon(\varsigma_{ij})\right.$$

$$-\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{W_{\mathrm{I}\cup\bar{\mathrm{I}}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}\cup\bar{\mathrm{I}}}\omega_i\varsigma_{ij}-\frac{1}{W_{\mathrm{I}\cup\bar{\mathrm{I}}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}\cup\bar{\mathrm{I}}}\omega_i\varsigma_{ij}\right)\Bigg],$$

while convexity of Υ and Jensen−Steffensen inequality yields

(3.4)

$$\begin{aligned}
&\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{W_{\mathrm{I}\cup\bar{\mathrm{I}}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}\cup\bar{\mathrm{I}}}\omega_i\varsigma_{ij}-\frac{1}{W_{\mathrm{I}\cup\bar{\mathrm{I}}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}\cup\bar{\mathrm{I}}}\omega_i\varsigma_{ij}\right)\\
&\quad=\Upsilon\left[\frac{1}{W_{\mathrm{I}\cup\bar{\mathrm{I}}}}\left(W_{\mathrm{I}\cup\bar{\mathrm{I}}}\sum_{j=1}^{m}\sigma_j-\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}\cup\bar{\mathrm{I}}}\omega_i\varsigma_{ij}-\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}\cup\bar{\mathrm{I}}}\omega_i\varsigma_{ij}\right)\right]\\
&\quad=\Upsilon\left[\frac{1}{W_{\mathrm{I}\cup\bar{\mathrm{I}}}}\left\{\left(\sum_{i\in\mathrm{I}}\omega_i+\sum_{i\in\bar{\mathrm{I}}}\omega_i\right)\sum_{j=1}^{m}\sigma_j-\sum_{j=1}^{\kappa-1}\left(\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}+\sum_{i\in\bar{\mathrm{I}}}\omega_i\varsigma_{ij}\right)\right.\right.\\
&\qquad\left.\left.-\sum_{j=\kappa+1}^{m}\left(\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}+\sum_{i\in\bar{\mathrm{I}}}\omega_i\varsigma_{ij}\right)\right\}\right]\\
&\quad\leq\frac{W_{\mathrm{I}}}{W_{\mathrm{I}\cup\bar{\mathrm{I}}}}\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{W_{\mathrm{I}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}-\frac{1}{W_{\mathrm{I}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}\right)\\
&\qquad+\frac{W_{\bar{\mathrm{I}}}}{W_{\mathrm{I}\cup\bar{\mathrm{I}}}}\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\bar{\mathrm{I}}}\omega_i\varsigma_{ij}-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\bar{\mathrm{I}}}\omega_i\varsigma_{ij}\right).
\end{aligned}$$

Finally combining (3.3) and inequality (3.4) we get

$$\begin{aligned}
&F_1(\mathrm{I}\cup\bar{\mathrm{I}})\\
&\quad\geq W_{\mathrm{I}}\left(\sum_{j=1}^{m}\Upsilon(\sigma_j)-\frac{1}{W_{\mathrm{I}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}}\omega_i\Upsilon(\varsigma_{ij})-\frac{1}{W_{\mathrm{I}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}}\omega_i\Upsilon(\varsigma_{ij})\right.\\
&\qquad\left.-\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{W_{\mathrm{I}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}-\frac{1}{W_{\mathrm{I}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}\right)\right)\\
&\qquad+W_{\bar{\mathrm{I}}}\left(\sum_{j=1}^{m}\Upsilon(\sigma_j)-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\bar{\mathrm{I}}}\omega_i\Upsilon(\varsigma_{ij})-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\bar{\mathrm{I}}}\omega_i\Upsilon(\varsigma_{ij})\right.\\
&\qquad\left.-\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\bar{\mathrm{I}}}\omega_i\varsigma_{ij}-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\bar{\mathrm{I}}}\omega_i\varsigma_{ij}\right)\right).\\
&\quad=F_1(\mathrm{I})+F_1(\bar{\mathrm{I}}).
\end{aligned}$$

□

The following corollaries give certain refinements in connection with the index set function.

Corollary 3.2. *Let* $\mathrm{I}_{\mathrm{l}} = \{1, \ldots, \mathrm{l}\}$ *where* $\mathrm{l} \in \{1, \ldots, n\}$. *Suppose that* $\mathbf{X} = (\varsigma_{ij})$ *be an* $n \times m$ *matrix such that the entries* $\varsigma_{ij} \in (\mu_1, \nu_1)\ \forall\ i \in \mathrm{I}_n,\ j \in \{1, \ldots, m\}$ *and let* $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m) \in (\mu_1, \nu_1)^m$.
For a convex function Υ *on an interval* (μ_1, ν_1), *if* $0 < W_S < W_{\mathrm{I}_n}$ *and* $S \in \{\mathrm{I}_1, \ldots, \mathrm{I}_n\}$, *(where equality holds for* $S = \mathrm{I}_n$*), then under the assumptions of Theorem* 2.1 *we have*

$$\digamma_1(\mathrm{I}_{n-1}) \le \digamma_1(\mathrm{I}_n), \tag{3.5}$$

Proof. Set $\kappa \in \{1, \ldots, m\}$. Since Υ is convex function hence by a property of convex function we have

$$\Upsilon(\varsigma_n) \le \Upsilon(\mathrm{d}) + \Upsilon'(\varsigma_n)(\varsigma_n - \mathrm{d})$$

Replace $\varsigma_n = \varsigma_{nj}$ and $\mathrm{d} = \sigma_j$, $\forall j \in \{1, \ldots, m\}$, then we have

$$\sum_{j=1}^{m} \Upsilon(\varsigma_{nj}) \le \sum_{j=1}^{m} \Upsilon(\sigma_j) + \sum_{j=1}^{m} \Upsilon'(\varsigma_{nj})(\varsigma_{nj} - \sigma_j). \tag{3.6}$$

By using (2.2) we have 2nd term in R.H.S of (3.6) vanishes and then by using (2.1)we have
(3.7)

$$\Upsilon\left(\sum_{j=1}^{m} \sigma_j - \sum_{j=1}^{\kappa-1} \varsigma_{nj} - \sum_{j=\kappa+1}^{m} \varsigma_{nj}\right) \le \sum_{j=1}^{m} \Upsilon(\sigma_j) - \sum_{j=1}^{\kappa-1} \Upsilon(\varsigma_{nj}) - \sum_{j=\kappa+1}^{m} \Upsilon(\varsigma_{nj}).$$

From given condition we have $0 < W_{\mathrm{I}_{n-1}} < W_{\mathrm{I}_n}$ which implies $0 < \omega_n = W_{\mathrm{I}_n} - W_{\mathrm{I}_{n-1}}$. So by applying (3.7) we have

$$\begin{aligned} \digamma_1(\{n\}) = \omega_n \Bigg(&\sum_{j=1}^{m} \Upsilon(\sigma_j) - \sum_{j=1}^{\kappa-1} \Upsilon(\varsigma_{nj}) - \sum_{j=\kappa+1}^{m} \Upsilon(\varsigma_{nj}) \\ &-\Upsilon\left(\sum_{j=1}^{m} \sigma_j - \sum_{j=1}^{\kappa-1} \varsigma_{nj} - \sum_{j=\kappa+1}^{m} \varsigma_{nj}\right)\Bigg) \ge 0. \end{aligned} \tag{3.8}$$

As

$$\digamma_1(\mathrm{I}_n) = \digamma_1(\mathrm{I}_{n-1} \cup \{n\}).$$

Since, $\mathrm{I}_{n-1} \cap \{n\} = \emptyset$, hence by Theorem 2.1 and then by using inequality (3.8) we get

$$\digamma_1(\mathrm{I}_n) = \digamma_1(\mathrm{I}_{n-1} \cup \{n\}) \ge \digamma_1(\mathrm{I}_{n-1}) + \digamma_1(\{n\}) \ge \digamma_1(\mathrm{I}_{n-1}). \qquad \square$$

Remark 3.3. Theorem 3.1 and Corollary 3.2 are also valid under the assumptions of Theorem 2.5 and for the index set functional defined as

$$\digamma_1(\mathrm{I}) = W_{\mathrm{I}}\left[\sum_{j=1}^{m} \rho_j \Upsilon(\sigma_j) - \frac{1}{W_{\mathrm{I}}} \sum_{j=1}^{\kappa-1} \rho_j \sum_{i \in \mathrm{I}} \omega_i \Upsilon(\varsigma_{ij}) - \frac{1}{W_{\mathrm{I}}} \sum_{j=\kappa+1}^{m} \rho_j \sum_{i \in \mathrm{I}} \omega_i \Upsilon(\varsigma_{ij})\right.$$

$$-\rho_\kappa \Upsilon\left(\sum_{j=1}^{m}\sigma_j\epsilon\rho_j\upsilon_j-\frac{1}{W_{\mathrm{I}}}\sum_{j=1}^{\kappa-1}\epsilon\rho_j\upsilon_j\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}-\frac{1}{W_{\mathrm{I}}}\sum_{j=\kappa+1}^{m}\epsilon\rho_j\upsilon_j\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}\right)\Bigg].$$

Remark 3.4. Theorem 3.1 and Corollary 3.2 results are the generalized extension of corresponding results in [2] and [9].

3.2. **Refinements of Niezgoda's Inequality for D Functional.** Let a function Υ define on an interval (μ_1,ν_1) and suppose that $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_m)\in(\mu_1,\nu_1)^m$ and $\mathbf{X}=(\varsigma_{ij})$ is a real $n\times m$ matrix such that $\varsigma_{ij}\in(\mu_1,\nu_1)\,\forall\, i\in\{1,\ldots,n\}$ and $j\in\{1,\ldots,m\}$. Then for any non−empty subset I of $\{1,\ldots,n\}$ we take $\bar{\mathrm{I}}:=\{1,\ldots,n\}\backslash\mathrm{I}\neq\emptyset$ and $\boldsymbol{\omega}=(\omega_1,\ldots,\omega_n)$ be a real n−tuple and we define $W_{\mathrm{I}}=\sum_{i\in\mathrm{I}}\omega_i$ and $W_{\bar{\mathrm{I}}}=W_n-\sum_{i\in\mathrm{I}}\omega_i$ such that $0<W_S<W_n$ and $\frac{1}{W_S}\sum_{i\in S}\omega_i\varsigma_{ij}\in(\mu_1,\nu_1)$ where $S\in\{\mathrm{I},\bar{\mathrm{I}},\{1,\ldots,n\}\}$. If we define a D functional as

$$D(\boldsymbol{\omega},\mathbf{X},\Upsilon;\mathrm{I}):=\frac{1}{\mathrm{W}_n}W_{\mathrm{I}}\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{W_{\mathrm{I}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}-\frac{1}{W_{\mathrm{I}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}\right)$$
$$+\frac{1}{\mathrm{W}_n}W_{\bar{\mathrm{I}}}\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\bar{\mathrm{I}}}\omega_j\varsigma_{ij}-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\bar{\mathrm{I}}}\omega_j\varsigma_{ij}\right),$$

then the following theorem is valid.

Theorem 3.5. *Under the assumptions of Theorem* 2.1*, for any non−empty subset* I *of* $\{1,\ldots,n\}$ *we have*

$$\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=1}^{\kappa-1}\omega_i\varsigma_{ij}-\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=\kappa+1}^{m}\omega_i\varsigma_{ij}\right)\tag{3.9}$$
$$\leq D(\boldsymbol{\omega},\mathbf{X},\Upsilon;\mathrm{I})$$
$$\leq\sum_{j=1}^{m}\Upsilon(\sigma_j)-\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=1}^{\kappa-1}\omega_i\Upsilon(\varsigma_{ij})-\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=\kappa+1}^{m}\omega_i\Upsilon(\varsigma_{ij}).$$

Proof. By the property of convex function we have

$$\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=1}^{\kappa-1}\omega_i\varsigma_{ij}-\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\sum_{j=\kappa+1}^{m}\omega_i\varsigma_{ij}\right)$$
$$=\Upsilon\left[\frac{1}{\mathrm{W}_n}\sum_{i=1}^{n}\omega_i\left(\sum_{j=1}^{m}\sigma_j-\sum_{j=1}^{\kappa-1}\varsigma_{ij}-\sum_{j=\kappa+1}^{m}\omega_i\varsigma_{ij}\right)\right]$$
$$=\Upsilon\left[\frac{1}{\mathrm{W}_n}W_{\mathrm{I}}\left(\frac{1}{W_{\mathrm{I}}}\sum_{i\in\mathrm{I}}\omega_i\left(\sum_{j=1}^{m}\sigma_j-\sum_{j=1}^{\kappa-1}\varsigma_{ij}-\sum_{j=\kappa+1}^{m}\varsigma_{ij}\right)\right)\right.$$

$$+\frac{1}{\mathrm{W}_n}W_{\bar{\mathrm{I}}}\left(\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{i\in\bar{\mathrm{I}}}\omega_i\left(\sum_{j=1}^{m}\sigma_j-\sum_{j=1}^{\kappa-1}\varsigma_{ij}-\sum_{j=1}^{\kappa-1}\varsigma_{ij}\right)\right)\Bigg]$$

$$\leq\frac{1}{\mathrm{W}_n}W_{\mathrm{I}}\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{W_{\mathrm{I}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}-\frac{1}{W_{\mathrm{I}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}\right)$$

$$+\frac{1}{\mathrm{W}_n}W_{\bar{\mathrm{I}}}\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\bar{\mathrm{I}}}\omega_i\varsigma_{ij}-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\bar{\mathrm{I}}}\omega_i\varsigma_{ij}\right)$$

$$=D(\boldsymbol{\omega},\mathbf{X},\Upsilon\,;\mathrm{I}).$$

Now using generalized Niezgoda inequality (2.3) in the following functional

$$D(\boldsymbol{\omega},\mathbf{X},\Upsilon\,;\mathrm{I})$$

$$:=\frac{1}{\mathrm{W}_n}W_{\mathrm{I}}\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{W_{\mathrm{I}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}-\frac{1}{W_{\mathrm{I}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}\right)$$

$$+\frac{1}{\mathrm{W}_n}W_{\bar{\mathrm{I}}}\Upsilon\left(\sum_{j=1}^{m}\sigma_j-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\bar{\mathrm{I}}}\omega_j\varsigma_{ij}-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\bar{\mathrm{I}}}\omega_j\varsigma_{ij}\right),$$

$$\leq\frac{1}{\mathrm{W}_n}W_{\mathrm{I}}\left(\sum_{j=1}^{m}\Upsilon(\sigma_j)-\frac{1}{W_{\mathrm{I}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}}\omega_i\Upsilon(\varsigma_{ij})-\frac{1}{W_{\mathrm{I}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}}\omega_i\Upsilon(\varsigma_{ij})\right)$$

$$+\frac{1}{\mathrm{W}_n}W_{\bar{\mathrm{I}}}\left(\sum_{j=1}^{m}\Upsilon(\sigma_j)-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\bar{\mathrm{I}}}\omega_j\Upsilon(\varsigma_{ij})-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\bar{\mathrm{I}}}\omega_j\Upsilon(\varsigma_{ij})\right),$$

$$=\sum_{j=1}^{m}\Upsilon(\sigma_j)-\frac{1}{\mathrm{W}_n}\sum_{j=1}^{\kappa-1}\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij})-\frac{1}{\mathrm{W}_n}\sum_{j=\kappa+1}^{m}\sum_{i=1}^{n}\omega_i\Upsilon(\varsigma_{ij}),$$

for any I, which validate the 2nd inequality in (3.9). □

Remark 3.6. Theorem 3.5 is also valid under the assumptions of Theorem 2.5, for D functional defined as

$$D(\boldsymbol{\rho},\boldsymbol{\omega},\mathbf{X},\Upsilon\,;\mathrm{I})$$

$$:=\frac{1}{\mathrm{W}_n}W_{\mathrm{I}}\rho_\kappa\Upsilon\left(\sum_{j=1}^{m}\sigma_j\epsilon\rho_j\upsilon_j-\frac{1}{W_{\mathrm{I}}}\sum_{j=1}^{\kappa-1}\epsilon\rho_j\upsilon_j\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}-\frac{1}{W_{\mathrm{I}}}\sum_{j=\kappa+1}^{m}\epsilon\rho_j\upsilon_j\sum_{i\in\mathrm{I}}\omega_i\varsigma_{ij}\right)$$

$$+\frac{1}{\mathrm{W}_n}W_{\bar{\mathrm{I}}}\rho_\kappa\Upsilon\left(\sum_{j=1}^{m}\sigma_j\epsilon\rho_j\upsilon_j-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=1}^{\kappa-1}\epsilon\rho_j\upsilon_j\sum_{i\in\bar{\mathrm{I}}}\omega_i\varsigma_{ij}-\frac{1}{W_{\bar{\mathrm{I}}}}\sum_{j=\kappa+1}^{m}\epsilon\rho_j\upsilon_j\sum_{i\in\bar{\mathrm{I}}}\omega_i\varsigma_{ij}\right).$$

4. Applications

($\hbar$): For $\emptyset \neq \mathrm{I} \subseteq \{1,\dots,n\}$, the arithmetic, geometric, harmonic and power means of order $\mathrm{r} \in \mathbb{R}$ are defined as $\hat{\mathrm{A}}_{\mathrm{I}}, \hat{\mathrm{G}}_{\mathrm{I}}, \hat{\mathrm{H}}_{\mathrm{I}}$ and $\hat{\mathrm{M}}_{\mathrm{I}}^{[\mathrm{r}]}$ respectively with $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_m)$ $\boldsymbol{\varsigma_{.j}} = (\varsigma_{1j},\dots,\varsigma_{nj})$ such that $\sigma_j, \varsigma_{ij} \in (\mu_1,\nu_1)^+ \subseteq \mathbb{R}^+$ $\forall$ $i \in \{1,\dots,n\}$, $j \in \{1,\dots,m\}$. Let $\omega_i, where\, i \in \mathrm{I}$, are the positive weights in $\mathbb{R}^+$. While for $\mathrm{I} = \{1,\dots,n\}$, the generalized arithmetic, generalized geometric, generalized harmonic and generalized power means are denoted by $\hat{\mathrm{A}}_n, \hat{\mathrm{G}}_n, \hat{\mathrm{H}}_n$ and $\hat{\mathrm{M}}_n^{[\mathrm{r}]}$ respectively.

All over the section we suppose that ln and exp have the natural domain. If we describe

Generalized Arithmetic Mean

$$\mathrm{A}_\sigma = \sum_{j=1}^{m} \sigma_j$$

$$\mathrm{A}_{\mathrm{I}} = \frac{1}{W_{\mathrm{I}}} \sum_{j=1}^{\kappa-1} \sum_{i \in \mathrm{I}} \omega_i \varsigma_{ij} + \frac{1}{W_{\mathrm{I}}} \sum_{j=\kappa+1}^{m} \sum_{i \in \mathrm{I}} \omega_i \varsigma_{ij}$$

$$\hat{\mathrm{A}}_{\mathrm{I}} = \mathrm{A}_\sigma - \mathrm{A}_{\mathrm{I}}$$

Generalized Geometric Mean

$$\mathrm{G}_\sigma = \exp\left(\sum_{j=1}^{m} \ln(\sigma_j) \right)$$

$$\mathrm{G}_{\mathrm{I}} = \exp\left(\frac{1}{W_{\mathrm{I}}} \sum_{j=1}^{\kappa-1} \sum_{i \in \mathrm{I}} \omega_i \ln(\varsigma_{ij}) + \frac{1}{W_{\mathrm{I}}} \sum_{j=\kappa+1}^{m} \sum_{i \in \mathrm{I}} \omega_i \ln(\varsigma_{ij}) \right)$$

$$\hat{\mathrm{G}}_{\mathrm{I}} = \frac{\mathrm{G}_\sigma}{\mathrm{G}_{\mathrm{I}}}$$

Generalized Harmonic Mean

$$\mathrm{H}_\sigma = \left(\sum_{j=1}^{m} \frac{1}{\sigma_j} \right)^{-1}$$

$$\mathrm{H}_{\mathrm{I}} = \left(\frac{1}{W_{\mathrm{I}}} \sum_{j=1}^{\kappa-1} \sum_{i \in \mathrm{I}} \omega_i \frac{1}{\varsigma_{ij}} + \frac{1}{W_{\mathrm{I}}} \sum_{j=\kappa+1}^{m} \sum_{i \in \mathrm{I}} \omega_i \frac{1}{\varsigma_{ij}} \right)^{-1}$$

$$\frac{1}{\hat{\mathrm{H}}_{\mathrm{I}}} = \frac{1}{\mathrm{H}_\sigma} - \frac{1}{\mathrm{H}_{\mathrm{I}}}$$

Generalized Power Mean

$$\mathrm{M}_\sigma^{[\mathrm{r}]} = \sum_{j=1}^{m} \sigma_j^{\mathrm{r}}$$

$$\mathrm{M}_{\mathrm{I}}^{[\mathrm{r}]}=\frac{1}{W_{\mathrm{I}}}\sum_{j=1}^{\kappa-1}\sum_{i\in\mathrm{I}}\omega_i(\varsigma_{ij})^{\mathrm{r}}+\frac{1}{W_{\mathrm{I}}}\sum_{j=\kappa+1}^{m}\sum_{i\in\mathrm{I}}\omega_i(\varsigma_{ij})^{\mathrm{r}}$$

$$\hat{\mathrm{M}}_{\mathrm{I}}^{[\mathrm{r}]}=(\mathrm{M}_{\sigma}^{[\mathrm{r}]}-\mathrm{M}_{\mathrm{I}}^{[\mathrm{r}]})^{\frac{1}{\mathrm{r}}}.$$

Theorem 4.1.

(4.1) $\quad (i)\ \hat{\mathrm{A}}_n \geq \hat{\mathrm{G}}_n$

(4.2) $\quad (ii)\ \dfrac{\hat{\mathrm{A}}_n(\varsigma)}{\hat{\mathrm{A}}_n(1-\varsigma)} \geq \dfrac{\hat{\mathrm{G}}_n(\varsigma)}{\hat{\mathrm{G}}_n(1-\varsigma)}$ *provided that* $0<\varsigma_{ij}\leq \dfrac{1}{2}$ *for all* i, j.

Proof. (i) Applying (2.3) to the convex function $\Upsilon(\varsigma)=-\ln\varsigma$, we obtain (4.1).

(ii) Applying (2.3) to the convex function $\Upsilon(\varsigma)=\ln\left(\frac{1-\varsigma}{\varsigma}\right)$ for $0<\varsigma\leq\frac{1}{2}$ we obtain required inequality (4.2). □

Remark 4.2. The inequality (4.2) is a generalized variant of weighted Ky Fan's inequality (see, for example, [8, pp. 25-28]).

Theorem 4.3.

$$\text{(4.3)}\qquad (i)\ \left(\frac{\hat{\mathrm{A}}_n}{\hat{\mathrm{G}}_n}\right)^{W_n} \geq \left(\frac{\hat{\mathrm{A}}_{n-1}}{\hat{\mathrm{G}}_{n-1}}\right)^{W_{n-1}}$$

$$\text{(4.4)}\qquad (ii)\ W_n\left(\hat{\mathrm{A}}_n-\hat{\mathrm{G}}_n\right)\geq W_{n-1}\left(\hat{\mathrm{A}}_{n-1}-\hat{\mathrm{G}}_{n-1}\right)$$

Proof.
- Applying (3.5) to the convex function $\Upsilon(\varsigma)=-\ln\varsigma$, we obtain (4.3).
- Applying (3.5) to the convex function $\Upsilon(\varsigma)=\exp(\varsigma)$ and replacing σ_j with $\ln(\sigma_j)$ and ς_{ij} with $\ln(\varsigma_{ij})$, for all $i\in\mathrm{I}$ and $j\in\{1,\ldots,m\}$ we obtain (4.4). □

Remark 4.4. If in Theorem 4.3 we put $\omega_i=i\ \forall i\in\mathrm{I}$, then we get the following results, which are of Popoviciu-[28] and Rado- [25] types, respectively, (see also [7, p. 194]).

Corollary 4.5.

$$(i)\ \left(\frac{\hat{\mathrm{A}}_n}{\hat{\mathrm{G}}_n}\right)^{n} \geq \left(\frac{\hat{\mathrm{A}}_{n-1}}{\hat{\mathrm{G}}_{n-1}}\right)^{n-1}$$

$$(ii)\ n\left(\hat{\mathrm{A}}_n-\hat{\mathrm{G}}_n\right)\geq (n-1)\left(\hat{\mathrm{A}}_{n-1}-\hat{\mathrm{G}}_{n-1}\right)$$

Proof. Follows directly from Theorem 4.3 for $\omega_i=1$ for all $i\in\{1,\ldots,n\}$. □

Corollary 4.6.

$$(i)\ \left(\frac{\hat{\mathrm{G}}_n}{\hat{\mathrm{H}}_n}\right)^{W_n} \geq \left(\frac{\hat{\mathrm{G}}_{n-1}}{\hat{\mathrm{H}}_{n-1}}\right)^{W_{n-1}}$$

$$(ii)\ W_n\left(\frac{1}{\hat{\mathrm{H}}_n}-\frac{1}{\hat{\mathrm{G}}_n}\right)\geq W_{n-1}\left(\frac{1}{\hat{\mathrm{H}}_{n-1}}-\frac{1}{\hat{\mathrm{G}}_{n-1}}\right)$$

Proof. Follows directly from Theorem 4.11 by the substitutions $\sigma_j \to \frac{1}{\sigma_j}$ and $\varsigma_{ij} \to \frac{1}{\varsigma_{ij}}$ for all $i \in \mathrm{I}$ and $j \in \{1,\ldots,m\}$. □

Theorem 4.7. (i) *For* $\mathrm{r} \leq 1$, *we have the following inequalities.*

$$W_n\left(\hat{\mathrm{A}_n}-\hat{\mathrm{M}}_n{}^{[\mathrm{r}]}\right)\geq W_{n-1}\left(\hat{\mathrm{A}}_{n-1}-\hat{\mathrm{M}}_{n-1}{}^{[\mathrm{r}]}\right) \tag{4.5}$$

(ii) *For* $\mathrm{r} \leq 1$, *we have the inequalities in* (4.5) *are reversed.*

Proof. For $\mathrm{r} \leq 1$, $\mathrm{r} \neq 0$, use (3.5) for the convex function $\Upsilon(\varsigma) = \varsigma^{\frac{1}{\mathrm{r}}}$, replacing σ_j with σ_j^{r} and ς_{ij} with $(\varsigma_{ij})^{\mathrm{r}}$ for all $i \in \mathrm{I}$ and $j \in \{1,\ldots,m\}$ and for $\mathrm{r} = 0$ use and (3.5) for the convex function $\Upsilon(\varsigma) = \exp\varsigma$ replacing σ_j with $\ln\sigma$ and ς_{ij} with $\ln\varsigma_{ij}$ for all $i \in \mathrm{I}$ and $j \in \{1,\ldots,m\}$, we obtain (4.9).

If $\mathrm{r} \geq 1$, ,then (4.5) reversed because $\Upsilon(\varsigma) = \varsigma^{\frac{1}{\mathrm{r}}}$ is concave. □

Corollary 4.8.

$$W_n\left(\hat{\mathrm{A}_n}-\hat{\mathrm{H}}_n\right)\geq W_{n-1}\left(\hat{\mathrm{A}}_{n-1}-\hat{\mathrm{H}}_{n-1}\right)$$

Remark 4.9. Obviously, part *(ii)* of Theorem 4.3 directly follows from Theorem 4.7.

Theorem 4.10. *Let* $\mathrm{r},\mathrm{t} \in \mathbb{R}; \mathrm{r} \leq \mathrm{t}$.

(i) *If* $\mathrm{t} \geq 0$, *then, we have the following inequalities.*

$$W_n\left(\left(\hat{\mathrm{M}}_n{}^{[s]}\right)^s-\left(\hat{\mathrm{M}}_n{}^{[\mathrm{r}]}\right)^s\right)\geq W_{n-1}\left(\left(\hat{\mathrm{M}}_{n-1}{}^{[s]}\right)^s-\left(\hat{\mathrm{M}}_{n-1}{}^{[\mathrm{r}]}\right)^s\right) \tag{4.6}$$

(ii) *For* $\mathrm{r} \leq 1$, *we have the inequalities in* (4.6) *are reversed.*

Proof. Let $\mathrm{t} \geq 0$. Applying (3.5) to the convex function $\Upsilon(\varsigma) = \varsigma^{\frac{\mathrm{t}}{\mathrm{r}}}$ and replace σ_j with σ_j^{r} and ς_{ij} with $(\varsigma_{ij})^{\mathrm{r}}$ for all $i \in \mathrm{I}$ and $j \in \{1,\ldots,m\}$, we obtain (4.6).

If $\mathrm{t} < 0$, then (4.6) reversed since $\Upsilon(\varsigma) = \varsigma^{\frac{\mathrm{t}}{\mathrm{r}}}$ is concave. □

Theorem 4.11.

$$(i)\ \hat{\mathrm{G}_n} \leq \hat{\mathrm{A}}_{\mathrm{I}}^{W_{\mathrm{I}}}.\ \hat{\mathrm{A}}_{\bar{\mathrm{I}}}^{W_{\bar{\mathrm{I}}}} \leq \hat{\mathrm{A}_n} \tag{4.7}$$

$$(ii)\ \hat{\mathrm{G}}_n \leq W_{\mathrm{I}}\hat{\mathrm{G}}_{\mathrm{I}} + W_{\bar{\mathrm{I}}}\hat{\mathrm{G}}_{\bar{\mathrm{I}}} \leq \hat{\mathrm{A}}_n \tag{4.8}$$

Proof. (i) Applying Theorem 3.5 to the convex function $\Upsilon(\varsigma) = -\ln(\varsigma)$, we obtain

$$\ln\hat{\mathrm{A}}_n \geq \left(\ln\hat{\mathrm{A}}_{\mathrm{I}}^{W_{\mathrm{I}}} + \ln\hat{\mathrm{A}}_{\bar{\mathrm{I}}}^{W_{\bar{\mathrm{I}}}}\right) \geq \ln\hat{\mathrm{G}}_n$$

from which (4.7) follows.

(ii) Applying Theorem 3.5 to the convex function $\Upsilon(\varsigma) = \exp(\varsigma)$ and replacing σ_j with $\ln(\sigma_j)$ and ς_{ij} with $\ln(\varsigma_{ij})$, for all $i \in \mathrm{I}$ and $j \in \{1,\ldots,m\}$ we obtain (4.8). □

Corollary 4.12.

$$(i)\ \frac{1}{\hat{\mathrm{G}}_n} \leq \frac{1}{\hat{\mathrm{H}}_{\mathrm{I}}^{W_{\mathrm{I}}}\ \hat{\mathrm{H}}_{\bar{\mathrm{I}}}^{W_{\bar{\mathrm{I}}}}} \leq \frac{1}{\hat{\mathrm{H}}_n}$$

$$(ii)\ \frac{1}{\hat{\mathrm{G}}_n} \leq \frac{W_{\mathrm{I}}}{\hat{\mathrm{G}}_{\mathrm{I}}} + \frac{W_{\bar{\mathrm{I}}}}{\hat{\mathrm{G}}_{\bar{\mathrm{I}}}} \leq \frac{1}{\hat{\mathrm{H}}_n}$$

Proof. Directly from Theorem 4.11 by the substitutions $\sigma_j \to \frac{1}{\sigma_j}$ and $\varsigma_{ij} \to \frac{1}{\varsigma_{ij}}$ for all $i \in \mathrm{I}$ and $j \in \{1, \ldots, m\}$. □

Theorem 4.13. (i) *For* $\mathrm{r} \leq 1$, *we have*

$$\hat{\mathrm{M}}_n^{[\mathrm{r}]} \leq W_{\mathrm{I}} \hat{\mathrm{M}}_{\mathrm{I}}^{[\mathrm{r}]} + W_{\bar{\mathrm{I}}} \hat{\mathrm{M}}_{\bar{\mathrm{I}}}^{[\mathrm{r}]} \leq \hat{\mathrm{A}}_n \tag{4.9}$$

(ii) *For* $\mathrm{r} \geq 1$, (4.9) *reversed.*

Proof. For $\mathrm{r} \leq 1$, $\mathrm{r} \neq 0$, use Theorem 3.5 for the convex function $\Upsilon(\varsigma) = \varsigma^{\frac{1}{\mathrm{r}}}$, replacing σ_j with σ_j^{r} and ς_{ij} with $(\varsigma_{ij})^{\mathrm{r}}$ for all $i \in \mathrm{I}$ and $j \in \{1, \ldots, m\}$ and for $\mathrm{r} = 0$ use and Theorem 3.5 for the convex function $\Upsilon(\varsigma) = \exp \varsigma$ replacing σ_j with $\ln \sigma_j$ and ς_{ij} with $\ln \varsigma_{ij}$ for all $i \in \mathrm{I}$ and $j \in \{1, \ldots, m\}$, we obtain (4.9).

If $\mathrm{r} \geq 1$, ,then (4.9) reversed because $\Upsilon(\varsigma) = \varsigma^{\frac{1}{\mathrm{r}}}$ is concave. □

Corollary 4.14.

$$\hat{\mathrm{H}}_n \leq W_{\mathrm{I}} \hat{\mathrm{H}}_{\mathrm{I}} + W_{\bar{\mathrm{I}}} \hat{\mathrm{H}}_{\bar{\mathrm{I}}} \leq \hat{\mathrm{A}}_n$$

Proof. Directly from Theorem 4.13 for $\mathrm{r} = -1$. □

Remark 4.15. Obviously, part (*ii*) of Theorem 4.11 directly follows from Theorem 4.13.

Theorem 4.16. *Let* $\mathrm{r}, \mathrm{t} \in \mathbb{R}; \mathrm{r} \leq \mathrm{t}$.

(i) *If* $\mathrm{t} \geq 0$, *then*

$$\left(\hat{\mathrm{M}}_n^{[\mathrm{r}]}\right)^{\mathrm{t}} \leq W_{\mathrm{I}} \left(\hat{\mathrm{M}}_{\mathrm{I}}^{[\mathrm{r}]}\right)^{\mathrm{t}} + W_{\bar{\mathrm{I}}} \left(\hat{\mathrm{M}}_{\bar{\mathrm{I}}}^{[\mathrm{r}]}\right)^{\mathrm{t}} \leq \left(\hat{\mathrm{M}}_n^{[\mathrm{t}]}\right)^{s} \tag{4.10}$$

(ii) *For* $\mathrm{t} < 0$, (4.10) *reversed.*

Proof. Let $\mathrm{t} \geq 0$. Applying Theorem 3.5 to the convex function $\Upsilon(\varsigma) = \varsigma^{\frac{\mathrm{t}}{\mathrm{r}}}$ and replace σ_j with σ_j^{r} and ς_{ij} with $(\varsigma_{ij})^{\mathrm{r}}$ for all $i \in \mathrm{I}$ and $j \in \{1, \ldots, m\}$, we obtain (4.10).

If $\mathrm{t} < 0$, then (4.10) reversed since $\Upsilon(\varsigma) = \varsigma^{\frac{\mathrm{t}}{\mathrm{r}}}$ is concave. □

5. Conclusion

In this article, we have generalized the result of Niezgoda [21], which gives the extension of Jensen–Mercer inequality. We have obtained a generalised Niezgoda inequality by using the Jensen–Steffensen inequality and its generalization as defined in [18]. At last, we have presented refinements and applications of our main results.

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