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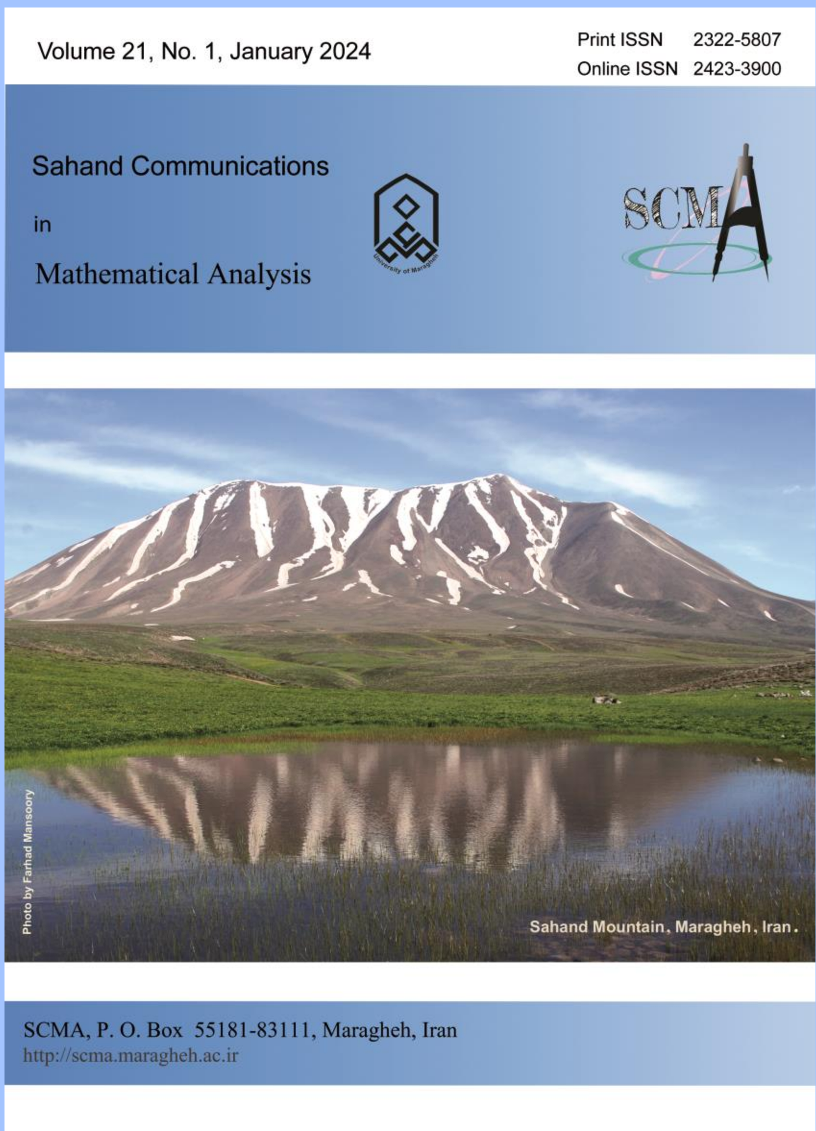

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Statistical Deferred Weighted Riemann Summability and Fuzzy Approximation Theorems

Priyadarsini Parida[1], Susanta Kumar Paikray[2*] and Bidu Bhusan Jena[3]

Abstract. The notion of statistical convergence has fascinated many researchers due mainly to the fact that it is more general than the well-established hypothesis of ordinary (classical) convergence. This work aims to investigate and present (presumably new) the statistical versions of deferred weighted Riemann integrability and deferred weighted Riemann summability for sequences of fuzzy functions. We first interrelate these two lovely theoretical notions by establishing an inclusion theorem. We then state and prove two fuzzy Korovkin-type theorems based on our proposed helpful and potential notions. We also demonstrate that our results are the nontrivial extensions of several known fuzzy Korovkin-type approximation theorems given in earlier works. Moreover, we estimate the statistically deferred weighted Riemann summability rate supported by another promising new result. Finally, we consider several interesting exceptional cases and illustrative examples supporting our definitions and the results presented in this paper.

1. Introduction and Motivation

Let $\mathcal{Z} = \{\vartheta : \mathbb{R} \to [0,1]\}$ satisfy the following assertions

(i) there exists $\mu_0 \in \mathbb{R}$ such that $\vartheta(\mu_0) = 1$, then ϑ is normal
(ii) ϑ is fuzzy convex
(iii) ϑ is upper semi-continuous
(iv) $[\vartheta]^0 = \{\overline{\mu \in \mathbb{R} \quad \text{and } \vartheta(\mu) > 0}\}$ is compact set.

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Then $\vartheta \in \mathcal{Z}$ is called a fuzzy number, and $\mathcal{Z}$ is called as a fuzzy number space.

If $\vartheta \in \mathcal{Z}$ and let $[\vartheta]^{\Lambda} = \{\mu \in \mathbb{R} : X(\mu) \geqq \Lambda\}$ be closed and bounded interval for $\Lambda \in [0,1]$.

Now, we recall some elementary properties of fuzzy numbers.

Let $\vartheta, \nu \in \mathcal{Z}$, $\Lambda \in [0,1]$ and $\lambda \in \mathbb{R}$. Then

(i) $(\vartheta + \nu)(\mu) = \sup_{\mu=t+s} \min\{\vartheta(t), \nu(s)\}$

(ii) $k\vartheta(\mu) = \vartheta(\mu/k) \quad (k \neq 0)$

(iii) $0\vartheta(x) = \bar{0}$, where

$$\bar{a}(x) = \begin{cases} 1 & (\mu = a) \\ 0 & (\text{otherwise}) \end{cases}$$

(iv) $[\vartheta + \nu]^{\Lambda} = [\vartheta]^{\Lambda} + [\nu]^{r} = \left[\vartheta_{\Lambda}^{-} + \nu_{\Lambda}^{-}, \vartheta_{\Lambda}^{+} + \nu_{\Lambda}^{+}\right]$

(v) $[k\vartheta]^{\Lambda} = k[u]^{\Lambda} = \left[k\vartheta_{\Lambda}^{-}, k\vartheta_{\Lambda}^{+}\right]$ for $(k \geqq 0)$

(vi) $\vartheta \leqq \nu \Leftrightarrow [\vartheta]^{\Lambda} \leqq [\nu]^{\Lambda}$.

Furthermore, the metric $\mathcal{D}$ is such that $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_{+}$ be defined as

$$\mathcal{D}(\vartheta, \nu) = \sup_{0 \leqq \Lambda \leqq 1} \max \left\{ \left|\vartheta_{-}^{\Lambda} - \nu_{-}^{\Lambda}\right|, \left|\vartheta_{+}^{\Lambda} - \nu_{+}^{\Lambda}\right| \right\},$$

where $d(\mathcal{Z}, \mathcal{D})$ is a metric space which is complete (see [17]).

Let $\mathcal{D}^{*}(\tilde{g}, \tilde{h})$ be the distance between two fuzzy number valued functions $\tilde{g}$ and $\tilde{h}$ such that

$$\mathcal{D}^{*}(\tilde{g}, \tilde{h}) = \sup_{\bar{x} \leqq \mu \leqq \bar{y}} \sup_{0 \leqq \Lambda \leqq 1} \max \left\{ \left|\tilde{g}_{-}^{\Lambda} - \tilde{h}_{-}^{\Lambda}\right|, \left|\tilde{g}_{+}^{\Lambda} - \tilde{h}_{+}^{\Lambda}\right| \right\}.$$

The *convergence analysis* on sequence space is one of the most important and exciting aspects of real and functional analysis. The gradual enrichment of this study leads to the blooming of *statistical convergence*, which is genuinely more general than the traditional convergence. The glory for independently defining this beautiful notion goes to both Fast [7] and Steinhaus [15]. Nowadays, this potential notion of statistical convergence has undoubtediy been a field of interest for many researchers and is becoming an active research area in various fields of pure and applied Mathematics. In particular, it is instrumental in the study of Machine Learning, Soft Computing, Number Theory, Measure theory, Probability theory, etc. For some current works in such direction, one may refer [2–4, 8–10, 13, 14, 19].

Suppose $\mathfrak{K}^{*} \subseteq \mathbb{N}$, and let $\mathfrak{K}_{k}^{*} = \{\xi : \xi \leqq k \quad \text{and} \quad \xi \in \mathfrak{K}^{*}\}$. Then the natural density $d(\mathfrak{K}^{*})$ of $\mathfrak{K}^{*}$ is defined by

$$d(\mathfrak{K}^{*}) = \lim_{k \to \infty} \frac{|\mathfrak{K}_{k}^{*}|}{k} = \ell,$$

where the number ℓ is real and finite, and $|\mathfrak{K}_k^*|$ is the cardinality of $\mathfrak{K}_k^*$.

Definition 1.1. A given sequence (η_k) is statistically convergent to a fuzzy number ϑ if, for each $\varepsilon > 0$,

$$\mathfrak{K}_\varepsilon^* = \{\xi : \xi \in \mathbb{N} \text{ and } \mathcal{D}(\eta_\xi, \vartheta) \geqq \varepsilon\}$$

has zero natural density (see [12]). Thus, for each $\varepsilon > 0$, we have

$$d(\mathfrak{K}_\varepsilon^*) = \lim_{k\to\infty} \frac{|\mathfrak{K}_\varepsilon^*|}{k} = 0.$$

We write

$$\text{stat} \lim_{k\to\infty} \mathcal{D}(\eta_\xi, \vartheta) = 0.$$

Let $[\bar{x}, \bar{y}] \subset \mathcal{E}$, and for all $k \in \mathbb{N}$ there is a fuzzy number valued function $\tilde{h}_k : [\bar{x}, \bar{y}] \to \mathcal{Z}$ and it is called a sequence $(\tilde{h}_k)$ of fuzzy number valued functions on $[\bar{x}, \bar{y}]$.

We now describe the Riemann sum of a sequence $(\tilde{h}_k)$ of fuzzy number valued functions allied with a tagged partition $\dot{\mathcal{P}}$ which is of the following form

$$\delta\left(\tilde{h}_k; \dot{\mathcal{P}}\right) := \sum_{i=1}^{k} \tilde{h}(t_i)\mathcal{D}(z_i, z_{i-1}).$$

Next, we recall the definition for Riemann integrability of a sequence of fuzzy number valued functions over an interval $[\bar{x}, \bar{y}]$.

Definition 1.2. A sequence $(\tilde{h}_k)_{k\in\mathbb{N}}$ of fuzzy number valued functions is *Riemann integrable* to a fuzzy number valued function $\tilde{h}$ on $[\bar{x}, \bar{y}]$ if, for all $\varepsilon > 0$ there exists $\sigma_\varepsilon > 0$ and let $\dot{\mathcal{P}}$ be any tagged partition of $[\bar{x}, \bar{y}]$ with $\|\dot{\mathcal{P}}\| < \sigma_\varepsilon$ such that

$$\mathcal{D}\left(\delta\left(\tilde{h}_k; \dot{\mathcal{P}}\right), \tilde{h}\right) < \varepsilon.$$

We now outline the definition of statistical convergence of Riemann integrable fuzzy number valued functions.

Definition 1.3. A sequence $(\tilde{h}_k)_{k\in\mathbb{N}}$ of fuzzy number valued functions is *statistically Riemann integrable* to a fuzzy number valued function $\tilde{h}$ on $[\bar{x}, \bar{y}]$ if, for all $\varepsilon > 0$ and for each $\mu \in [a, b]$, there exists $\sigma_\varepsilon > 0$, and for $\dot{\mathcal{P}}$ be any tagged partition of $[\bar{x}, \bar{y}]$ with $\|\dot{\mathcal{P}}\| < \sigma_\varepsilon$, the set

$$\mathfrak{K}^*{}_\varepsilon = \left\{\xi : \xi \in \mathbb{N} \text{ and } \mathcal{D}\left(\delta\left(\tilde{h}_\xi; \dot{\mathcal{P}}\right), \tilde{h}\right) \geqq \varepsilon\right\}$$

has natural (asymptotic) density. That is, for every $\varepsilon > 0$,

$$d(\mathfrak{K}^*{}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathfrak{K}^*{}_\varepsilon|}{k} = 0.$$

We write

$$\mathrm{stat}_{\mathrm{Rie}} \lim_{k\to\infty} \mathcal{D}\left(\delta\left(\tilde{h}_k;\dot{\mathcal{P}}\right),\tilde{h}\right)=0.$$

The following example demonstrates that every Riemann integrable fuzzy number valued function is statistically Riemann integrable, while the converse is not usually trustworthy.

Example 1.4. Let $\tilde{h}_k : [0,1] \to \mathcal{Z}$ be a sequence of functions defined by

$$\tilde{h}_k(\mu)=\begin{cases}\sqrt{2} & (\mu\in\mathbb{Q}\cap[0,1];\ k=m^2,\ m\in\mathbb{N})\\ \\ \frac{1}{k+1} & (\text{otherwise}).\end{cases} \tag{1.1}$$

It is easy to see that the sequence $(\tilde{h}_k)$ of fuzzy number valued functions is statistically Riemann integrable to 0 over $[0,1]$, but not Riemann integrable over $[0,1]$.

The investigations mentioned above motivated us to explore the statistical versions of deferred weighted Riemann integrability and deferred weighted Riemann summability for sequences of fuzzy functions. We first connect these two lovely theoretical ideas by proving an inclusion theorem. We then demonstrate two fuzzy Korovkin-type theorems based on our proposed useful and potential conceptions. In addition, we give an example of sequences of fuzzy positive linear operators employing the Bernstein polynomials to demonstrate the usefulness of our findings. Finally, we estimate the statistical deferred weighted Riemann summability rate supported by another exciting result.

2. Deferred Weighted Statistical Riemann Integrability

Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$ be such that $\phi_k < \varphi_k$ and $\lim_{k\to\infty} \varphi_k = +\infty$, and let (p_k) be a sequence of non-negative real numbers with

$$P_k=\sum_{v=\phi_k+1}^{\varphi_k} p_v.$$

Then, we define the deferred weighted summability mean for the Riemann sum of a sequence of fuzzy number valued functions $\delta\left(\tilde{h}_k;\dot{\mathcal{P}}\right)$ allied with the tagged partition $\dot{\mathcal{P}}$ of the form

$$\mathcal{W}\left(\delta\left(\tilde{h}_k;\dot{\mathcal{P}}\right)\right)=\frac{1}{P_k}\sum_{v=\phi_k+1}^{\varphi_k} p_v\delta\left(\tilde{h}_v;\dot{\mathcal{P}}\right). \tag{2.1}$$

We now present the notions of statistical Riemann integrability and statistical Riemann summability of a sequence of fuzzy number valued functions via deferred weighted mean.

Definition 2.1. Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let (p_k) be a sequence of real numbers (non-negative). A sequence $(\tilde{h}_k)_{k\in\mathbb{N}}$ of fuzzy number valued functions is deferred weighted statistically Riemann ($\text{DWFR}_{\text{stat}}$) integrable to a fuzzy number valued function $\tilde{h}$ on $[\bar{x}, \bar{y}]$ if, for all $\varepsilon > 0$ there exists $\sigma_\varepsilon > 0$ allied with the tagged partition $\dot{\mathcal{P}}$ ($\|\dot{\mathcal{P}}\| < \sigma_\varepsilon$) of $[\bar{x}, \bar{y}]$, the set

$$\left\{\xi : \xi \leqq P_k \text{ and } p_\xi \mathcal{D}\left(\delta\left(\tilde{h}_\xi; \dot{\mathcal{P}}\right), \tilde{h}\right) \geqq \varepsilon\right\}$$

has zero natural (asymptotic) density. This implies that for each $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{\left|\left\{\xi : \xi \leqq P_k \text{ and } p_\xi \mathcal{D}\left(\delta\left(\tilde{h}_\xi; \dot{\mathcal{P}}\right), \tilde{h}\right) \geqq \varepsilon\right\}\right|}{P_k} = 0.$$

We write

$$\text{DWFR}_{\text{stat}} \lim_{k\to\infty} \mathcal{D}\left(\delta\left(\tilde{h}_k; \dot{\mathcal{P}}\right), \tilde{h}\right) = 0.$$

Definition 2.2. Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let (p_k) be a sequence of non-negative real numbers. A sequence $(\tilde{h}_k)_{k\in\mathbb{N}}$ of fuzzy number valued functions is statistically deferred weighted Riemann ($\text{stat}_{\text{DWFR}}$) summable to a fuzzy number valued function $\tilde{h}$ on $[\bar{x}, \bar{y}]$ if, for all $\varepsilon > 0$ there exists $\sigma_\varepsilon > 0$ allied with the tagged partition $\dot{\mathcal{P}}$ ($\|\dot{\mathcal{P}}\| < \sigma_\varepsilon$) of $[\bar{x}, \bar{y}]$, the set

$$\left\{\xi : \xi \leqq k \text{ and } |\mathcal{W}\left(\delta\left(\tilde{h}_\xi; \dot{\mathcal{P}}\right)\right) - \tilde{h}| \geqq \varepsilon\right\}$$

has zero natural (asymptotic) density. This implies that for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{\left|\left\{\xi : \xi \leqq k \text{ and } \mathcal{D}(\mathcal{W}\left(\delta\left(\tilde{h}_\xi; \dot{\mathcal{P}}\right)\right), \tilde{h}) \geqq \varepsilon\right\}\right|}{k} = 0.$$

We write

$$\text{stat}_{\text{DWFR}} \lim_{k\to\infty} \mathcal{D}\left(\delta\left(\tilde{h}_k; \dot{\mathcal{P}}\right), \tilde{h}\right) = 0.$$

We now develop an inclusion theorem between the above two potential and useful notions.

Theorem 2.3. *Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let (p_k) be a non-negative real numbers. If $(\tilde{h}_k)_{k\in\mathbb{N}}$ is deferred weighted statistically Riemann ($DWFR_{\text{stat}}$) integrable to $\tilde{h}$ on $[\bar{x}, \bar{y}]$, then it is statistically deferred weighted Riemann ($stat_{\text{DWFR}}$) summable to $\tilde{h}$ on $[\bar{x}, \bar{y}]$, but not conversely.*

Proof. Suppose $(\tilde{h}_k)_{k\in\mathbb{N}}$ is deferred weighted statistically Riemann ($\text{DWFR}_{\text{stat}}$) integrable to $\tilde{h}$ on $[\bar{x},\bar{y}]$, then by Definition 2.1, we have

$$\lim_{k\to\infty}\frac{\left|\left\{\xi:\xi\leqq P_k \text{ and } p_\xi\mathcal{D}\left(\delta\left(\tilde{h}_\xi;\dot{\mathcal{P}}\right),\tilde{h}\right)\geqq\varepsilon\right\}\right|}{P_k}=0.$$

Now assuming two sets as follows:

$$\mathcal{Y}_\epsilon=\left\{\xi:\xi\leqq P_k \text{ and } p_\xi\mathcal{D}\left(\delta\left(\tilde{h}_\xi;\dot{\mathcal{P}}\right),\tilde{h}\right)\geqq\varepsilon\right\}$$

and

$$\mathcal{Y}_\epsilon^c=\left\{\xi:\xi\leqq P_k \text{ and } p_\xi\mathcal{D}\left(\delta\left(\tilde{h}_\xi;\dot{\mathcal{P}}\right),\tilde{h}\right)<\varepsilon\right\},$$

we have

$$\begin{aligned}
\mathcal{D}\left(\mathcal{N}\left(\delta\left(\tilde{h}_k;\dot{\mathcal{P}}\right)\right),\tilde{h}\right)&=\frac{1}{P_k}\sum_{v=\phi_k+1}^{\varphi_k}p_v\mathcal{D}\left(\delta\left(\tilde{h}_v;\dot{\mathcal{P}}\right),\tilde{h}\right)\\
&\leqq\frac{1}{P_k}\sum_{v=\phi_k+1}^{\varphi_k}p_v\mathcal{D}\left(\delta\left(\tilde{h}_v;\dot{\mathcal{P}}\right),\tilde{h}\right)\\
&\quad+\frac{1}{P_k}\sum_{v=\phi_k+1}^{\varphi_k}\mathcal{D}\left(p_v\tilde{h},\tilde{h}\right)\\
&\leqq\frac{1}{P_k}\sum_{\substack{v=\phi_k+1\\(\xi\in\mathcal{Y}_\varepsilon)}}^{\varphi_k}p_v\mathcal{D}\left(\delta\left(\tilde{h}_v;\dot{\mathcal{P}}\right),\tilde{h}\right)\\
&\quad+\frac{1}{P_k}\sum_{\substack{v=\phi_k+1\\(\xi\in\mathcal{Y}_\varepsilon^c)}}^{\varphi_k}p_v\mathcal{D}\left(\delta\left(\tilde{h}_v;\dot{\mathcal{P}}\right),\tilde{h}\right)\\
&\quad+|\tilde{h}|\left(\frac{1}{P_k}\sum_{v=\phi_k+1}^{\varphi_k}p_v-1\right)\\
&\leqq\frac{1}{P_k}\left|\mathcal{Y}_\varepsilon\right|+\frac{1}{P_k}|\mathcal{Y}_\varepsilon^c|\\
&=0.
\end{aligned}$$

This implies that

$$\mathcal{D}(\mathcal{W}\left(\delta\left(\tilde{h}_k;\dot{\mathcal{P}}\right)\right),\tilde{h})<\varepsilon.$$

Thus, $(\tilde{h}_k)$ is statistically deferred weighted Riemann ($\text{stat}_{\text{DWFR}}$) summable to $\tilde{h}$ on $[\bar{x},\bar{y}]$. □

The next example shows that, a statistically deferred weighted Riemann ($\text{stat}_{\text{DWFR}}$) summable sequence of fuzzy number valued functions is not deferred weighted statistically Riemann ($DWFR_{stat}$) integrable in light of the invalidity of the converse statement.

Example 2.4. Let $\phi_k = 2k+1$, $\varphi_k = 4k+1$ and $p_k = 1$ and let $\tilde{h}_k : [0,1] \to \mathcal{Z}$ be a sequence of functions of the form given by

$$\tilde{h}_k(t) = \begin{cases} 0 & (t \in \mathbb{Q} \cap [0,1];\ k \text{ is even}) \\ \\ 1 & (t \in \mathbb{R} - \mathbb{Q} \cap [0,1];\ k \text{ is odd}). \end{cases} \tag{2.2}$$

It is obvious from the given sequence $(\tilde{h}_k)$ of functions that, it is neither Riemann integrable nor deferred weighted statistically Riemann ($DWFR_{stat}$) integrable. Nonetheless, it is clear from our suggested mean (2.1) that

$$\begin{aligned} \mathcal{W}\left(\delta\left(\tilde{h}_k; \dot{\mathcal{P}}\right)\right) &= \frac{1}{\varphi_k - \phi_k} \sum_{v=\phi_k+1}^{\varphi_k} \delta\left(\tilde{h}_v; \dot{\mathcal{P}}\right) \\ &= \frac{1}{2k} \sum_{v=2k+1}^{4k} \delta\left(\tilde{h}_v; \dot{\mathcal{P}}\right) \\ &= \frac{1}{2}. \end{aligned}$$

Thus, $(\tilde{h}_k)$ has deferred weighted Riemann sum $\frac{1}{2}$ allied with the tagged partition $\dot{\mathcal{P}}$. Hence, $(\tilde{h}_k)$ is statistically deferred weighted Riemann ($stat_{DWFR}$) summable to $\frac{1}{2}$ over the interval $[0,1]$. However, it is not deferred weighted statistically Riemann ($DWFR_{stat}$) integrable.

3. Korovkin-type Theorems via the $\mathcal{W}\left(\delta\left(h_k; \dot{\mathcal{P}}\right)\right)$-mean

Many researchers are working to extend (or generalise) the Korovkin-type approximation theorems in several other mathematical contexts, including sequence spaces, measurable spaces, Banach spaces, probability spaces, etc. These ideas are very useful in many related fields, including functional analysis, harmonic analysis, and real analysis, etc. For the attention of enthusiastic readers, we choose to [5, 6, 17].

Let $\tilde{h}$ be a fuzzy number valued function such that

$$\tilde{h} : [\bar{x}, \bar{y}] \to \mathcal{Z}.$$

$\tilde{h}$ is said to be a continuous fuzzy number valued function at a point $\mu_0 \in [\bar{x}, \bar{y}]$, if

$$\mathcal{D}(\mu_k, \mu_0) < \epsilon \quad (k \to \infty) \quad \text{whenever} \quad \mu_k \to \mu_0.$$

Moreover, if $\tilde{h}$ is fuzzy continuous at every point $\mu \in [\bar{x}, \bar{y}]$, then it is so also fuzzy continuous in the whole interval $[\bar{x}, \bar{y}]$.

Let $\mathcal{C}_{\mathcal{L}}[\bar{x}, \bar{y}]$ be the set of all fuzzy number valued continuous functions over $[\bar{x}, \bar{y}]$.

Suppose that, $\mathfrak{L} : \mathcal{C}_{\mathcal{L}}[\bar{x}, \bar{y}] \to \mathcal{C}_{\mathcal{L}}[\bar{x}, \bar{y}]$ be a fuzzy linear operator, if for each $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\tilde{h}_1, \tilde{h}_2 \in \mathcal{C}_{\mathcal{L}}[\bar{x}, \bar{y}]$,

$$\mathfrak{L}\left(\lambda_1 \odot \tilde{h}_1 \oplus \lambda_2 \odot \tilde{h}_2; t\right) = \lambda_1 \odot \mathfrak{L}(\tilde{h}_1) \oplus \lambda_2 \odot \mathfrak{L}(\tilde{h}_2).$$

Next, a fuzzy linear operator $\mathfrak{L}$ is said to be a positive fuzzy linear operator, if

$$\tilde{h}_1(\mu) \preceq \tilde{h}_2(t)) \Rightarrow \mathfrak{L}(\tilde{h}_1; \mu) \preceq \mathfrak{L}(\tilde{h}_2; \mu) \quad (\because \tilde{h}_1, \tilde{h}_2 \in \mathcal{C}_{\mathcal{L}}[\bar{x}, \bar{y}]; \mu \in [\bar{x}, \bar{y}]).$$

Theorem 3.1. *Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let $\mathfrak{L}_k : \mathcal{C}_{\mathcal{L}}[\bar{x}, \bar{y}] \to \mathcal{C}_{\mathcal{L}}[\bar{x}, \bar{y}] \quad (k \in \mathbb{N})$ be the fuzzy number valued sequence of positive linear operators. Also, let $\{\mathfrak{L}_k^*\}_{k \in \mathbb{N}}$ be the analogous sequence of positive linear operators from $\mathcal{C}[\bar{x}, \bar{y}]$ into $\mathcal{C}[\bar{x}, \bar{y}]$ such that*

$$\left\{\mathfrak{L}_k\left(\tilde{h}; \mu\right)\right\}_{\pm}^{\Lambda} = \mathfrak{L}_k^*\left(\tilde{h}_{\pm}^{\Lambda}; \mu\right) \tag{3.1}$$

for all $\mu \in [\bar{x}, \bar{y}]$, $\Lambda \in [0, 1]$, $k \in \mathbb{N}$. Then, for $\tilde{h} \in \mathcal{C}_{\mathcal{L}}[\bar{x}, \bar{y}]$

$$\mathrm{DWFR}_{\mathrm{stat}} \lim_{k \to \infty} \mathcal{D}^*\left(\mathfrak{L}_k\left(\tilde{h}; \mu\right), \tilde{h}(\mu)\right) = 0 \tag{3.2}$$

if and only if

$$\mathrm{DWFR}_{\mathrm{stat}} \lim_{k \to \infty} \mathcal{D}\left(\mathfrak{L}_k^*(1; \mu), 1\right) = 0, \tag{3.3}$$

$$\mathrm{DWFR}_{\mathrm{stat}} \lim_{k \to \infty} \mathcal{D}\left(\mathfrak{L}_k^*(\mu; \mu), \mu\right) = 0 \tag{3.4}$$

and

$$\mathrm{DWFR}_{\mathrm{stat}} \lim_{k \to \infty} \mathcal{D}\left(\mathfrak{L}_k^*(\mu^2; \mu), \mu^2\right) = 0. \tag{3.5}$$

Proof. Let $\tilde{h} \in \mathcal{C}_{\mathcal{L}}[\bar{x}, \bar{y}]$, $\mu \in [\bar{x}, \bar{y}]$ and $\Lambda \in [0, 1]$. Since $h_{\pm}^{\Lambda}(\mu) \in \mathcal{C}[\bar{x}, \bar{y}]$, so for each $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\left|\tilde{h}_{\pm}^{\Lambda}(\kappa) - \tilde{h}_{\pm}^{\Lambda}(\mu)\right| < \varepsilon \tag{3.6}$$

$$\text{whenever} \quad |\kappa - \mu| < \delta, \quad (\forall\ \mu, \kappa \in [\bar{x}, \bar{y}]).$$

Next, for $\tilde{h}$ is fuzzy bounded, $\left|\tilde{h}_{\pm}^{\Lambda}(\mu)\right| \leq \mathcal{M}_{\pm}^{\Lambda}$ $(\bar{x} < \mu < \bar{y})$. Clearly, we have

$$\left|\tilde{h}_{\pm}^{\Lambda}(\kappa) - \tilde{\tilde{h}}_{\pm}^{\Lambda}(\mu)\right| \leqq 2\mathcal{M}_{\pm}^{\Lambda}, \quad (\bar{x} < \mu, \kappa < \bar{y}).$$

Let us choose $\theta(\kappa, \mu) = (\kappa - \mu)^2$. Then,

$$\left|\tilde{h}_{\pm}^{\Lambda}(\kappa) - \tilde{h}_{\pm}^{\Lambda}(\mu)\right| < \varepsilon + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\theta(\kappa, \mu)$$

which yields

$$(3.7)\qquad -\varepsilon - \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\theta(\kappa,\mu) < \left(\tilde{h}_{\pm}^{\Lambda}(\kappa) - \tilde{h}_{\pm}^{\Lambda}(\mu)\right) < \varepsilon + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\theta(\kappa,\mu).$$

Now the operator $\mathfrak{L}_k^*$ is linear and monotone, by applying the operator $\mathfrak{L}_k^*(1,\mu)$ in (3.7), we get

$$\begin{aligned}\mathfrak{L}_k^*(1,\mu)\left(-\varepsilon - \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\theta(\kappa,\mu)\right) &< \mathfrak{L}_k^*(1,\mu)\left(\tilde{h}_{\pm}^{\Lambda}(\kappa) - \tilde{h}_{\pm}^{\Lambda}(\mu)\right)\\ &< \mathfrak{L}_k^*(1,\mu)\left(\varepsilon + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\theta(\kappa,\mu)\right).\end{aligned}$$

We note that μ is fixed and $\tilde{h}_{\pm}^{\Lambda}(\mu)$ is a constant number, we get

$$(3.8)\qquad \begin{aligned}-\varepsilon\mathfrak{L}_k^*(1,\mu) - \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\mathfrak{L}_k^*(\theta,\mu) &< \varepsilon\mathfrak{L}_k^*\left(\tilde{h}_{\pm}^{\Lambda},\mu\right) - \tilde{h}_{\pm}^{\Lambda}(\mu)\mathfrak{L}_k^*(1,\mu)\\ &< \varepsilon\mathfrak{L}_k^*(1,\mu) + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\mathfrak{L}_k^*(\theta,\mu).\end{aligned}$$

Also, we know that

(3.9)

$$\mathfrak{L}_k^*\left(\tilde{h}_{\pm}^{\Lambda},\mu\right) - \tilde{h}_{\pm}^{\Lambda}(\mu) = \left[\mathfrak{L}_k^*\left(\tilde{h}_{\pm}^{\Lambda},\mu\right) - \tilde{h}_{\pm}^{\Lambda}(\mu)\mathfrak{L}_k^*(1,\mu)\right] + \tilde{h}_{\pm}^{\Lambda}(\mu)[\mathfrak{L}_k^*(1,\mu) - 1].$$

Using (3.8) and (3.9), we get

(3.10)

$$\mathfrak{L}_k^*\left(\tilde{h}_{\pm}^{\Lambda},\mu\right) - \tilde{h}_{\pm}^{\Lambda}(\mu) < \varepsilon\mathfrak{L}_k^*(1,\mu) + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\mathfrak{L}_k^*(\theta,\mu) + \tilde{h}_{\pm}^{\Lambda}(\mu)\left[\mathfrak{L}_k^{\Lambda}(1,\mu) - 1\right].$$

We now, compute $\mathfrak{L}_k^*(\theta,\mu)$ as follows:

$$\begin{aligned}\mathfrak{L}_k^*(\theta,\mu) &= \mathfrak{L}_k^*\left(\kappa^2 - 2\mu\kappa + \mu^2,\mu\right)\\ &= \mathfrak{L}_k^*\left(\kappa^2,\mu\right) - 2\mu\mathfrak{L}_k^*(\kappa,\mu) + \mu^2\mathfrak{L}_k^*(1,\mu)\\ &= \left[\mathfrak{L}_k^*\left(\kappa^2,\mu\right) - \mu^2\right] - 2\mu[\mathfrak{L}_k^*(\kappa,\mu) - \mu] + \mu^2[\mathfrak{L}_k^*(1,\mu) - 1].\end{aligned}$$

Using (3.10), we get

$$\begin{aligned}\mathfrak{L}_k^*\left(\tilde{h}_{\pm}^{\Lambda},\mu\right) - \tilde{h}_{\pm}^{\Lambda}(\mu) &< \varepsilon\mathfrak{L}_k^*(1,\mu) + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\Big\{\left[\mathfrak{L}_k^*\left(\kappa^2,\mu\right) - \mu^2\right]\\ &\quad - 2\mu[\mathfrak{L}_k^*(\kappa,\mu) - \mu] + \mu^2[\mathfrak{L}_k^*(1,\mu) - 1]\Big\}\\ &\quad + \tilde{h}_{\pm}^{\Lambda}(\mu)[\mathfrak{L}_k^*(1,\mu) - 1]\\ &= \varepsilon[\mathfrak{L}_k^*(1,\mu) - 1] + \varepsilon + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\Big\{\left[\mathfrak{L}_k^*\left(\kappa^2,\mu\right) - \mu^2\right]\\ &\quad - 2\mu\left[\mathfrak{L}_k^{\Lambda}(\kappa,\mu) - \mu\right] + \mu^2[\mathfrak{L}_k^*(1,\mu) - 1]\Big\}\\ &\quad + \tilde{h}_{\pm}^{\Lambda}(\mu)[\mathfrak{L}_k^*(1,\mu) - 1].\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can write

$$\left|\mathfrak{L}_k^*\left(\tilde{h}_\pm^\Lambda,\mu\right)-\tilde{h}_\pm^\Lambda(\mu)\right| \leqq \varepsilon+\left(\varepsilon+\frac{2\mathcal{M}_\pm^\Lambda c^2}{\delta^2}+\mathcal{M}_\pm^\Lambda\right) \times |\mathfrak{L}_k^*(1,\mu)-1|+\frac{4\mathcal{M}_\pm^\Lambda c}{\delta^2}|\mathfrak{L}_k^\Lambda(\kappa,\mu)-\mu| + \frac{2\mathcal{M}_\pm^\Lambda}{\delta^2}\left|\mathfrak{L}_k^\Lambda\left(\kappa^2,\mu\right)-\mu^2\right|,$$

where $c=\max\{|\bar{x}|,|\bar{y}|\}$.

Consequently, we get

$$|\mathfrak{L}_k^*\left(\tilde{h}_\pm^\Lambda,\mu\right)-\tilde{h}_\pm^\Lambda(\mu)| \leqq \epsilon+\mathcal{H}_\pm^\Lambda(\epsilon)\Big(|\mathfrak{L}_k^*(1,\mu)-1| + |\mathfrak{L}_k^*(\kappa,\mu)-\mu|+|\mathfrak{L}_k^*\left(\kappa^2,\mu\right)-\mu^2|\Big), \tag{3.11}$$

where

$$\mathcal{H}_\pm^\kappa(\epsilon)=\max\left(\epsilon+\frac{2\mathcal{M}_\pm^\Lambda c^2}{\delta^2}+\mathcal{M}_\pm^\Lambda,\frac{4\mathcal{M}_\pm^\Lambda c}{\delta^2},\ \frac{2\mathcal{M}_\pm^\Lambda}{\delta^2}\right).$$

Now it clearly follows from (3.1) that,

$$\begin{aligned}\mathcal{D}^*&\left(\mathfrak{L}_k(\tilde{h}),\tilde{h}\right)\\ &=\sup_{\mu\in[\bar{x},\bar{y}]}\mathcal{D}\left(\mathfrak{L}_k\left(\tilde{h};\mu\right),\tilde{h}\right)\\ &=\sup_{\mu\in[\bar{x},\bar{y}]}\sup_{\Lambda\in[0,1]}\max\left\{\left|\mathfrak{L}_k^*(\tilde{h}_-^\Lambda;\mu)-\tilde{h}_-^\Lambda\right|,\left|\mathfrak{L}_k^*(\tilde{h}_+^\Lambda;\mu)-\tilde{h}_+^\Lambda(\mu)\right|\right\}.\end{aligned}$$

Considering (3.11) with the last equality, one can easily write

$$\mathcal{D}^*\left(\mathfrak{L}_k(\tilde{h}),\tilde{h}\right)\leqq\sup_{\mu\in[\bar{x},\bar{y}]}\varepsilon+\mathcal{M}(\varepsilon)\left(\sup_{\mu\in[\bar{x},\bar{y}]}|\mathfrak{L}_k^*(1,\mu)-1| + \sup_{\mu\in[\bar{x},\bar{y}]}|\mathfrak{L}_k^*(\kappa,\mu)-\mu|+\sup_{\mu\in[\bar{x},\bar{y}]}\left|\mathfrak{L}_k^*\left(\kappa^2,\mu\right)-\mu^2\right|\right),$$

where

$$\mathcal{M}(\epsilon)=\sup_{\Lambda\in[0,1]}\max\left\{\mathcal{M}_-^\Lambda(\epsilon),\mathcal{M}_+^\Lambda(\epsilon)\right\}.$$

Therefore,

(3.12)

$$p_v\mathcal{D}^*\left(\mathfrak{L}_k(\tilde{h}),\tilde{h}\right)\leqq p_v\sup_{\mu\in[\bar{x},\bar{y}]}\varepsilon+\mathcal{H}(\varepsilon)\Bigg(p_v\sup_{\mu\in[\bar{x},\bar{y}]}|\mathfrak{L}_k^*(1,\mu)-1|$$

$$+ p_v \sup_{\mu\in[\bar{x},\bar{y}]} \left|\mathfrak{L}_k^*(\kappa,\mu)-\mu\right| + p_v \sup_{\mu\in[\bar{x},\bar{y}]} \left|\mathfrak{L}_k^*\left(\kappa^2,\mu\right)-\mu^2\right| \Bigg).$$

Next, for given $\kappa > 0$, choose $\varepsilon > 0$ such that $p_v \sup_{\mu\in[\bar{x},\bar{y}]} \varepsilon < \omega$.

Then, we can write

$$\Theta_k(\mu;\varepsilon) = \left|\left\{k : k \leqq P_k \quad \text{and} \quad p_v \mathcal{D}^*\left(\mathfrak{L}_k(\tilde{h}), \tilde{h}\right) \geqq \varepsilon'\right\}\right|$$

and

$$\Theta_{j,k}(\mu,\varepsilon) = \left|\left\{k : k \leqq P_k \quad \text{and} \quad p_v \mathcal{D}\left(\mathfrak{L}_k^* \tilde{h}_j(\mu), \tilde{h}_j(\mu)\right) \geqq \frac{\varepsilon'-\varepsilon}{3\mathcal{H}_{\pm}^{\Lambda}}\right\}\right|,$$

we easily obtain from (3.12) that

$$\Theta_k(\mu,\varepsilon) \leqq \sum_{j=0}^{2} \Theta_{j,k}(\mu,\varepsilon).$$

Thus, we fairly have

$$\frac{\|\Theta_k(\mu,\varepsilon)\|}{P_k} \leqq \sum_{j=0}^{2} \frac{\|\Theta_{j,k}(\mu,\varepsilon)\|}{P_k}. \tag{3.13}$$

Consequently, by Definition 2.1 and under the above assumption for the implications in (3.3) to (3.5), the right-hand side of (3.13) are zero as $k \to \infty$. We, thus get

$$\lim_{k\to\infty} \frac{\|\Theta_k(\mu,\varepsilon)\|}{P_k} = 0, \quad (\varepsilon > 0).$$

Hence, the implication in (3.2) is fairly true. □

Theorem 3.2. *Let* (ϕ_k) *and* $(\varphi_k) \in \mathbb{Z}^{0+}$*, and let* $\mathfrak{L}_k : \mathcal{C}_{\mathcal{L}}[\bar{x},\bar{y}] \to \mathcal{C}_{\mathcal{L}}[\bar{x},\bar{y}] \quad (k \in \mathbb{N})$ *be the fuzzy number valued sequence of positive linear operators. Also, let* $\{\mathfrak{L}_k^*\}_{k\in\mathbb{N}}$ *be the analogous sequence of positive linear operators from* $\mathcal{C}[\bar{x},\bar{y}]$ *into* $\mathcal{C}[\bar{x},\bar{y}]$ *such that*

$$\left\{\mathfrak{L}_k\left(\tilde{h};\mu\right)\right\}_{\pm}^{\Lambda} = \mathfrak{L}_k^*\left(\tilde{h}_{\pm}^{\Lambda};\mu\right) \tag{3.14}$$

for all $\mu \in [\bar{x},\bar{y}]$*,* $\Lambda \in [0,1]$*,* $k \in \mathbb{N}$*. Then, for* $\tilde{h} \in \mathcal{C}_{\mathcal{L}}[\bar{x},\bar{y}]$

$$\text{stat}_{\text{DWFR}} \lim_{k\to\infty} \mathcal{D}^*\left(\mathfrak{L}_k\left(\tilde{h};\mu\right), \tilde{h}(\mu)\right) = 0 \tag{3.15}$$

if and only if

$$\text{stat}_{\text{DWFR}} \lim_{k\to\infty} \mathcal{D}\left(\mathfrak{L}_k^*(1;\mu), 1\right) = 0, \tag{3.16}$$

$$\text{stat}_{\text{DWFR}} \lim_{k\to\infty} \mathcal{D}\left(\mathfrak{L}_k^*(\mu;\mu), \mu\right) = 0 \tag{3.17}$$

and

$$\mathrm{stat}_{\mathrm{DWFR}} \lim_{k\to\infty} \mathcal{D}\left(\mathfrak{L}_k^*(\mu^2;\mu),\mu^2\right)=0. \tag{3.18}$$

Proof. Here we skip the details proof of Theorem 3.2 as it can be proved in the similar lines of the proof of Theorem 3.1. □

Given the applicability of Theorem 3.2 over Theorem 3.1, here we consider a numerical example and analyze a sequence of positive linear operators that does not escalate the functioning of Theorem 3.1. However, it does well work on Theorem 3.2. In this sense, we call that Theorem 3.2 is a non-trivial generalization of Theorem 3.1.

We now consider the operator that was used by Al-Salam [1] and, more recently by Viskov and Srivastava [16] as follows:

$$\chi(1+\chi D), \qquad \left(D=\frac{d}{d\chi}\right). \tag{3.19}$$

Example 3.3. Consider the *Bernstein polynomial* $\mathfrak{B}_k\left(\tilde{h};\alpha\right)$ on $\mathcal{C}[0,1]$ given by

$$\mathfrak{B}_k\left(\tilde{h};\alpha\right)=\sum_{\rho=0}^{k}\tilde{h}\left(\frac{\rho}{k}\right)\binom{k}{\rho}\alpha^{\rho}(1-b)^{k-\rho}, \quad (\alpha\in[0,1];k=0,1,\ldots). \tag{3.20}$$

We now propose the following positive linear operators on $\mathcal{C}[0,1]$ under the composition of Bernstein polynomial and the operators given by (3.19)

$$\mathfrak{L}_\rho\left(\tilde{h};\alpha\right)=\left[1+\tilde{h}_\rho\right]\alpha(1+\alpha D)\mathfrak{B}_\rho\left(\tilde{h};\alpha\right), \quad \left(\forall\ \tilde{h}\in\mathcal{C}[0,1]\right), \tag{3.21}$$

where $(\tilde{h}_\rho)$ is the same as intimated in Example 2.4.

We now apprise the values of the individual testing functions 1, α and α^2 by using our designated operators (3.21) as follows:

$$\mathfrak{L}_\rho(1;\alpha)=\left[1+\tilde{h}_\rho\right]\alpha(1+\alpha D)1=\left[1+\tilde{h}_\rho\right]\alpha,$$
$$\mathfrak{L}_\rho(\mu;\alpha)=\left[1+\tilde{h}_\rho\right]\alpha(1+\alpha D)\alpha=\left[1+\tilde{h}_\rho\right]\alpha(1+\alpha)$$

and

$$\begin{aligned}\mathfrak{L}_\rho\left(\mu^2;\alpha\right)&=\left[1+\tilde{h}_\rho\right]\alpha(1+\alpha D)\left\{\alpha^2+\frac{\alpha(1-\alpha)}{\rho}\right\}\\&=\left[1+\tilde{h}_\rho\right]\left\{\alpha^2\left(2-\frac{3\alpha}{\rho}\right)\right\}.\end{aligned}$$

Consequently, we have

$$\text{stat}_{\text{DWFR}} \lim_{\rho\to\infty} \mathcal{D}(\mathfrak{L}_\rho^*(1;\alpha),1) = 0, \tag{3.22}$$

$$\text{stat}_{\text{DWFR}} \lim_{\rho\to\infty} \mathcal{D}(\mathfrak{L}_\rho^*(\alpha;\alpha),\alpha) = 0 \tag{3.23}$$

and

$$\text{stat}_{\text{DWFR}} \lim_{\rho\to\infty} \mathcal{D}\left(\mathfrak{L}_\rho^*\left(\alpha^2;\alpha\right),\alpha^2\right) = 0, \tag{3.24}$$

that is, the sequence $\mathfrak{L}_\rho^*\left(\tilde{h};\alpha\right)$ satisfies the conditions (3.16) to (3.18). Hence, by Theorem 3.2, we fairly have

$$\text{stat}_{\text{DWFR}} \lim_{\rho\to\infty} \mathcal{D}^*\left(\mathfrak{L}_\rho\left(\tilde{h};\alpha\right),\tilde{h}\right) = 0.$$

Clearly, $(\tilde{h}_k)$, as specified in Example 2.4 is statistically deferred weighted Riemann ($\text{stat}_{\text{DWFR}}$) summable. However, it does not fairly deferred weighted statistically Riemann ($\text{DWFR}_{\text{stat}}$) integrable. Thus, our recommended operators in (3.21) satisfy Theorem 3.2. But the same is not true for the Theorem 3.1. Thus, unquestionably, we can say that the statistical versions of deferred weighted Riemann summability is well behave over the deferred weighted Riemann integrability for sequences of fuzzy number valued functions.

4. Fuzzy Rate of Deferred Weighted Riemann Summable

In this section, we wish to study the fuzzy rate of statistically deferred weighted Riemann summability of sequences of fuzzy number valued positive linear operators $\mathcal{C}_{\mathcal{L}}(\mathcal{Z})$ into itself.

Definition 4.1. Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let (ζ_n) be a non-increasing positive sequence. A fuzzy sequence of functions $(\tilde{h}_k)$ is statistically deferred weighted Riemann ($\text{stat}_{\text{DWFR}}$) summable to a fuzzy number $\tilde{h}$ on $\mathcal{Z}$ with rate $o(\zeta_k)$, if for each $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{\Omega_k(\mu;\varepsilon)}{\zeta_k P_k} = 0$$

uniformly with regards to $\mu \in \mathcal{Z}$ or, otherwise if

$$\lim_{k\to\infty} \frac{\|\Omega_k(\mu;\varepsilon)\|_{\mathcal{C}_{\mathcal{L}}[0,1]}}{\zeta_k P_k} = 0,$$

where

$$\Omega_k(\mu,\varepsilon) = \left|\left\{\xi : \xi \leqq k \text{ and } \mathcal{D}(\mathcal{W}\left(\delta\left(\tilde{h}_\xi;\dot{\mathcal{P}}\right)\right),\tilde{h}) \geqq \varepsilon\right\}\right| = 0.$$

We write

$$\text{stat}_{\text{RDWFR}} \mathcal{D}^*\left(\tilde{h}_k(\mu),\tilde{h}(\mu)\right) = o(\zeta_k) \quad \text{on } \mathcal{Z}.$$

We next wish to prove the following Lemma.

Lemma 4.2. *Let* (a'_k) *and* (b'_k) *be two non-increasing positive sequences, and let* $(\tilde{h}_k)$ *and* $(\tilde{g}_k) \in \mathcal{C}_{\mathcal{F}}(\mathcal{Z})$ *satisfy the conditions:*

$$\text{stat}_{\text{RDWFR}}\mathcal{D}^*\left(\tilde{h}_k(\mu), h(\mu)\right) = o(a'_k) \quad \text{on } \mathcal{Z}.$$

and

$$\text{stat}_{\text{RDWFR}}\mathcal{D}^*\left(\tilde{g}_k(\mu), \tilde{g}(\mu)\right) = o(b'_k) \quad \text{on } \mathcal{Z},$$

then all the following assertions are true:

(i) $\text{stat}_{\text{RDWFR}}\mathcal{D}^*\left(\tilde{h}_k(\mu) + \tilde{g}_k(\mu), \tilde{h}(\mu) + \tilde{g}(\mu)\right) = o(c'_k)$ *on* $\mathcal{Z}$;

(ii) $\text{stat}_{\text{RDWFR}}\mathcal{D}^*\left(\tilde{h}_k(\mu), \tilde{h}(\mu)\right)\mathcal{D}^*\left(\tilde{g}_k(\mu), \tilde{g}(\mu)\right) = o(a'_k b'_k)$ *on* $\mathcal{Z}$;

(iii) $\text{stat}_{\text{RDWFR}}K\mathcal{D}^*\left(\tilde{h}_k(\mu), \tilde{h}(\mu)\right) = o(a'_k)$ *on* $\mathcal{Z}$, *for any scalar* K;

(iv) $\text{stat}_{\text{RDWFR}}\left\{\mathcal{D}^*\left(\tilde{h}_k(\mu), \tilde{h}(\mu)\right)\right\}^{\frac{1}{2}} = o(a'_k)$ *on* $\mathcal{Z}$,

where $c'_k = \max\{a'_k, b'_k\}$.

Proof. For the assertion (i) of Lemma 4.2, we consider the following sets for which $\epsilon > 0$ and $\mu \in \mathcal{Z}$:

$$\mathfrak{G}_k(\mu,\varepsilon) = \left|\left\{\xi : \xi \leqq k \text{ and } \mathcal{D}\left[\mathcal{W}\left(\delta\left(\tilde{h}_\xi + \tilde{g}_\xi; \dot{\mathcal{P}}\right)\right), \left(\tilde{h} + \tilde{g}\right)\right] \geqq \varepsilon\right\}\right|,$$

$$\mathfrak{G}_{0,k}(\mu,\varepsilon) = \left|\left\{\xi : \xi \leqq k \text{ and } \mathcal{D}(\mathcal{W}\left(\delta\left(\tilde{h}_\xi; \dot{\mathcal{P}}\right)\right), \tilde{h}) \geqq \frac{\varepsilon}{2}\right\}\right|$$

and

$$\mathfrak{G}_{1,k}(\mu,\varepsilon) = \left|\left\{\xi : \xi \leqq k \text{ and } \mathcal{D}(\mathcal{W}\left(\delta\left(\tilde{h}_\xi; \dot{\mathcal{P}}\right)\right), \tilde{h}) \geqq \frac{\varepsilon}{2}\right\}\right|.$$

Clearly, we have

$$\mathfrak{G}_k(\mu,\varepsilon) \subseteq \mathfrak{G}_{0,k}(\mu,\epsilon) \cup \mathfrak{G}_{1,k}(\mu,\varepsilon).$$

Moreover, since

$$c'_k = \max\{a'_k, b'_k\}, \tag{4.1}$$

by using the assertion (i) of Theorem 3.2, we obtain

$$\frac{\|\mathfrak{G}_k(\mu,\varepsilon)\|_{\mathcal{C}_{\mathcal{L}}(\mathcal{Z})}}{c'_k P_v} \leqq \frac{\|\mathfrak{G}_{0,k}(\mu,\varepsilon)\|_{\mathcal{C}_{\mathcal{L}}(\mathcal{Z})}}{a'_k P_v} + \frac{\|\mathfrak{G}_{1,k}(\mu,\varepsilon)\|_{\mathcal{C}_{\mathcal{L}}(\mathcal{Z})}}{b'_k P_v}. \tag{4.2}$$

Also, by using the assertion (i) of Theorem 3.2, we obtain

$$\frac{\|\mathfrak{G}_k(\mu,\varepsilon)\|_{\mathcal{C}_{\mathcal{L}}(\mathcal{Z})}}{c'_k P_v} = 0. \tag{4.3}$$

Thus, assertion (i) of this Lemma is proved.

Furthermore, the remaining assertions (ii) to (iv) of Lemma 4.2 are resembling the assertion (i), so these can be proved in a similar manner to establish the proof of the Lemma 4.2. □

Next, for $\tilde{h} : [\bar{x}, \bar{y}] \to \mathcal{Z}$, the *fuzzy modulus of continuity* is defined by

(4.4)
$$\omega\left(\tilde{h}, \tilde{\delta}\right) = \sup_{x,y \in [\bar{x}, \bar{y}]} \left\{ \mathcal{D}^* \left(\tilde{h}(y), \tilde{h}(x) \right) : |y - x| \leqq \tilde{\delta} \quad (0 < \tilde{\delta} \leq \bar{x} - \bar{y}) \right\}.$$

We now establish a theorem on fuzzy rates of statistically deferred weighted Riemann ($\mathrm{stat}_{\mathrm{DWFR}}$) summable sequences of fuzzy number-valued positive linear operators via the fuzzy modulus of continuity.

Theorem 4.3. *Let* (ϕ_k) *and* $(\varphi_k) \in \mathbb{Z}^{0+}$*, and let* $\mathfrak{L}_k : \mathcal{C}_{\mathcal{L}}[\bar{x}, \bar{y}] \to \mathcal{C}_{\mathcal{F}}[\bar{x}, \bar{y}] \quad (k \in \mathbb{N})$ *be a sequence of fuzzy positive linear operators. Also, suppose that* $\{\mathfrak{L}_k^*\}_{k \in \mathbb{N}}$ *be the analogous sequence of positive linear operators from* $\mathcal{C}[\bar{x}, \bar{y}]$ *into* $\mathcal{C}[\bar{x}, \bar{y}]$ *such that (3.1) holds. Further assume that* (a_k') *and* (b_k') *be two non-increasing positive sequences and the operators* $\{\mathfrak{L}_k^*\}_{k \in \mathbb{N}}$ *such that*

(i) $\mathrm{stat}_{\mathrm{RDWFR}} \mathfrak{L}_k^*(1, \mu) - 1 = o(a_n')$ *on* $\mathcal{Z}$*,*

(ii) $\mathrm{stat}_{\mathrm{RDWFR}} \omega\left(\tilde{h}, \tilde{\delta}_k\right) = o(b_k')$ *on* $\mathcal{Z}$*,*

where
$$\tilde{\delta}_k(\mu) = \left\{ \mathfrak{L}_k^* \left(\theta^2; \mu \right) \right\}^{\frac{1}{2}} \text{ and } \theta(\kappa) = (\kappa - \mu),$$
then for each $\tilde{h} \in \mathcal{C}_{\mathcal{L}}(\mathcal{Z})$*, the assertion as below holds true:*

$$\mathrm{stat}_{\mathrm{RDWFR}} \mathcal{D}^* \left(\mathfrak{L}_k(\tilde{h}), \tilde{h} \right) = o(c_k') \text{ on } \mathcal{Z}, \tag{4.5}$$

where (c_k') *defined by (4.1).*

Proof. Suppose $\mathcal{Z} \subset \mathbb{R}$ be compact, and let $\tilde{h} \in \mathcal{C}_{\mathcal{L}}(\mathcal{Z})$, $\mu \in \mathcal{Z}$. Then,

$$\mathcal{D}^* \left(\mathfrak{L}_k(\tilde{h}, \mu); \tilde{h} \right) \leqq \mathcal{Q} |\mathfrak{L}_k^*(1; \mu) - 1| + \left(\mathfrak{L}_k^*(1; \mu) + \sqrt{\mathfrak{L}_k^*(1; \mu)} \omega\left(\tilde{h}, \tilde{\delta}_k\right) \right),$$

where
$$\mathcal{Q} = \|\tilde{h}\|_{\mathcal{C}_{\mathcal{L}}(\mathcal{Z})}.$$

Which yields

(4.6)
$$\begin{aligned} \mathcal{D}^* \left(\mathfrak{L}_k(\tilde{h}), \tilde{h} \right) \leqq \mathcal{Q} \mathfrak{L}_k^*(1; \mu) - 1 + 2\omega\left(\tilde{h}, \tilde{\delta}_k\right) + \omega\left(\tilde{h}, \tilde{\delta}_k\right) \left(\mathfrak{L}_k^*(1; \mu) - 1 \right) \\ + \omega\left(\tilde{h}, \tilde{\delta}_k\right) \sqrt{\left(\mathfrak{L}_k^*(1; \mu) - 1 \right)}. \end{aligned}$$

Thus, for the conditions (i) and (ii) (of Theorem 4.3) inclusive of Lemma 4.2, the last inequality (4.6) assists us to settle the assertion (4.5). Hence, the proof of Theorem 4.3 is completed. □

5. Remarkable Conclusion

In this conclusive section of our investigation, we further recognize some special remarks on Theorem 3.2 and Theorem 3.1 and of specific earlier published classical versions of the Korovkin-type theorems.

Remark 5.1. Considering the sequence $(\tilde{h}_\rho)_{\rho\in\mathbb{N}}$ of functions in our Example 2.4, it is statistically deferred weighted Riemann ($\text{stat}_{\text{DWFR}}$) summable, and that

$$\text{stat}_{\text{DWFR}} \lim_{\rho\to\infty} \mathcal{D}\left(\delta(\tilde{h}_\rho; \dot{\mathcal{P}}), \frac{1}{2}\right) \text{ on } [0,1].$$

Then, we have

$$\text{stat}_{\text{DWFR}} \lim_{\rho\to\infty} \mathcal{D}(\mathfrak{L}^*_\rho(\tilde{h}_\nu; \chi), \tilde{h}_\nu(\chi)) = 0, \quad (\nu = 0,1,2). \tag{5.1}$$

Thus, by virtue of Theorem 3.2, we immediately obtain

$$\text{stat}_{\text{DWFR}} \lim_{\rho\to\infty} \mathcal{D}^*(\mathfrak{L}_\rho(\tilde{h}; \chi), \tilde{h}(\chi)) = 0, \tag{5.2}$$

where

$$\tilde{h}_0(\chi) = 1, \qquad \tilde{h}_1(\chi) = \chi \quad \text{and} \quad \tilde{h}_2(\chi) = \chi^2.$$

Clearly, $(\tilde{h}_k)$ is statistically deferred weighted Riemann ($\text{stat}_{\text{DWFR}}$) summable, but neither deferred weighted statistically Riemann ($\text{DWFR}_{\text{stat}}$) integrable nor classically Riemann integrable. Thus, our proposed Theorem 3.2 appropritaly works over the operators specified in equation (3.21). But, neither the traditional nor the statistical versions of deferred weighted Riemann ($\text{DWFR}_{\text{stat}}$) integrable sequence of fuzzy number valued functions work on (3.21). In this context, we claim that our Theorem 3.2 is a non-trivial generalization of Theorem 3.1 as well as the previously published classical Korovkin-type theorem [11].

Remark 5.2. We suppose replace the conditions (i) and (ii) of our Theorem 4.3 by the following condition:

$$\mathcal{D}^*\left(\mathfrak{L}_k(\tilde{h}_\nu; \mu), \tilde{h}_\nu(\mu)\right)_{\mathcal{C}_{\mathcal{L}}(\mathcal{Z})} = \text{stat}_{\text{RDWFR}} o(\zeta_{k_\nu}), \quad (\nu = 0,1,2). \tag{5.3}$$

Then, since

$$\mathfrak{L}^*_k(\theta,\mu) = \left[\mathfrak{L}^*_k\left(\kappa^2,\mu\right) - \mu^2\right] - 2\mu[\mathfrak{L}^*_k(\kappa,\mu) - \mu] + \mu^2[\mathfrak{L}^*_k(1,\mu) - 1].$$

We can write

$$\mathfrak{L}^*_k(\theta,\mu) \leqq \mathcal{G} \sum_{i=0}^{2} \left|\mathfrak{L}_m(\tilde{h}_i; x) - \tilde{h}_i(x)\right|_{2\pi}, \tag{5.4}$$

where

$$\mathcal{G} = 1 + \|\tilde{h}_1\|_{\mathcal{C}_{\mathcal{L}}(\mathcal{Z})} + \|\tilde{h}_2\|_{\mathcal{C}_{\mathcal{L}}(\mathcal{Z})}.$$

It clearly follows from (5.3), (5.4) and Lemma 4.2 that

$$\tilde{\delta}_k(\mu) = \sqrt{\mathfrak{L}_k^*(\theta^2)} = \text{stat}_{\text{RDWFR}}\, o(d'_k), \tag{5.5}$$

where

$$o(d'_k) = \max\{\zeta_{k_0}, \zeta_{k_1}, \zeta_{k_2}\}.$$

This implies that

$$\omega\left(\tilde{h}, \tilde{\delta}\right) = \text{stat}_{\text{RDWFR}}\, o(d'_k).$$

Now, by using (5.3) in Theorem 4.3, we immediately see for $\tilde{h} \in \mathcal{C}_{\mathcal{L}}(\mathcal{Z})$ that

$$\mathfrak{L}_k^*\left(\tilde{h}; \mu\right), \tilde{h}(\mu) = \text{stat}_{\text{RDWFR}}\, o(d'_k).$$

Hence, if we use condition (5.3) in Theorem 4.3 instead of the conditions (i) and (ii), then we easily obtain the rates of the statistically deferred weighted Riemann ($\text{stat}_{\text{DWFR}}$) summablity of the sequence of fuzzy valued positive linear operators in Theorem 3.2.

Remark 5.3. If we substitute $(\phi_k) = 0$ and $(\varphi_k) = k$ into our main Theorem 3.2, then the earlier-published results by Yavuz [18] and by Das *et al.* [5] are deduced. In this sense, we say that Theorem 3.2 is a non-trivial unification and generalization of the earlier-published results (see [5, 18]).

Remark 5.4. Through this study, we have precluded the notion of statistical convergence in the sense of the deferred weighted Riemann summability technique and presented some new definitions. We have after that established some new theorems. Furthermore, by considering the modulus of continuity, we have estimated the fuzzy rate of statistically deferred weighted Riemann summability of sequences of fuzzy number valued positive linear operators.

Many researchers have considered various summability means on a various fuzzy sequence spaces to prove several fuzzy approximation results. A list of some articles has been mentioned in the references. Thus, by combining the existing concepts and directions of the fuzzy sequence spaces associated with our proposed mean, many new fuzzy Korvokin-type approximation theorems can be proved under different settings of algebraic and trigonometric functions.

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References

1. W.A. Al-Salam, *Operational representations for the Laguerre and other polynomials*, Duke Math. J., 31 (1964), pp. 127–142.
2. P. Baliarsingh, *On certain dynamic properties of difference sequences and the fractional derivatives*, Math. Methods Appl. Sci., 44 (2021), pp. 3023–3035.
3. N.L. Braha, V. Loku and H.M. Srivastava, Λ^2-*Weighted statistical convergence and Korovkin and Voronovskaya type theorems*, Appl. Math. Comput., 266 (2015), pp. 675–686.
4. N.L. Braha, H.M. Srivastava and S.A. Mohiuddine, *A Korovkin-type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean*, Appl. Math. Comput., 228 (2014), pp. 162–169.
5. A.A. Das, S.K. Paikray, T. Pradhan and H. Dutta, *Statistical $(C,1)(E,\mu)$-summablity and associated fuzzy approximation theorems with statistical fuzzy rates*, Soft Comput., 24 (2020), pp. 10883–10892.
6. H. Dutta, S.K. Paikray and B.B. Jena, *On statistical deferred Cesàro summability*, in "Current Trends in Mathematical Analysis and Its Interdisciplinary Applications" (H. Dutta, D. R. Ljubiša Kočinac and H. M. Srivastava, Editors), pp. 885–909, Springer Nature Switzerland AG, Cham, Switzerland, 2019.
7. H. Fast, *Sur la convergence statistique*, Colloq. Math., 2 (1951), pp. 241–244.
8. B.B. Jena and S.K. Paikray, *Product of deferred Cesàro and deferred weighted statistical probability convergence and its applications to Korovkin-type theorems*, Univ. Sci., 25 (2020), pp. 409–433.
9. B.B. Jena, S.K. Paikray and H. Dutta, *On various new concepts of statistical convergence for sequences of random variables via deferred Cesàro mean*, J. Math. Anal. Appl., 487 (2020), Article ID 123950, pp. 1–18.
10. B.B. Jena, S.K. Paikray, P. Parida and H. Dutta, *Results on Tauberian theorem for Cesàro summable double sequences of fuzzy numbers*, Kragujevac J. Math., 44 (2020), pp. 495–508.

11. P.P. Korovkin, *Convergence of linear positive operators in the spaces of continuous functions* (in Russian), Doklady Akad. Nauk. SSSR (New Ser.), 90 (1953), pp. 961–964.
12. F. Nuray and E. Savaş, *Statistical convergence of sequences of fuzzy numbers*, Math. Slovaca, 45 (1995), pp. 269–273.
13. P. Parida, S.K. Paikray and B.B. Jena, *Generalized deferred Cesàro equi-statistical convergence and analogous approximation theorems*, Proyecciones J. Math., 39 (2020), pp. 307–331.
14. S. Saha, B.C. Tripathy and S. Roy, *On Riesz mean of complex uncertain sequences*, J. Math. Anal. Appl., 499 (2020), Article ID: 125017.
15. H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., 2 (1951), pp. 73–74.
16. O.V. Viskov and H.M. Srivastava, *New approaches to certain identities involving differential operators*, J. Math. Anal. Appl., 186 (1994), pp. 1–10.
17. C.X. Wu and M. Ma, *Embedding problem of fuzzy number space I*, Fuzzy Sets Syst., 44 (1991), pp. 33–38.
18. E. Yavuz, *Basic trigonometric Korovkin approximation for fuzzy valued functions of two variables*, Theory of fuzzy sets, 74 (2021), pp. 1276–1288.
19. A. Zraiqat, S.K. Paikray and H. Dutta, *A certain class of deferred weighted statistical B-summability involving (p, q)-integers and analogous approximation theorems*, Filomat, 33 (2019), pp. 1425–1444.

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