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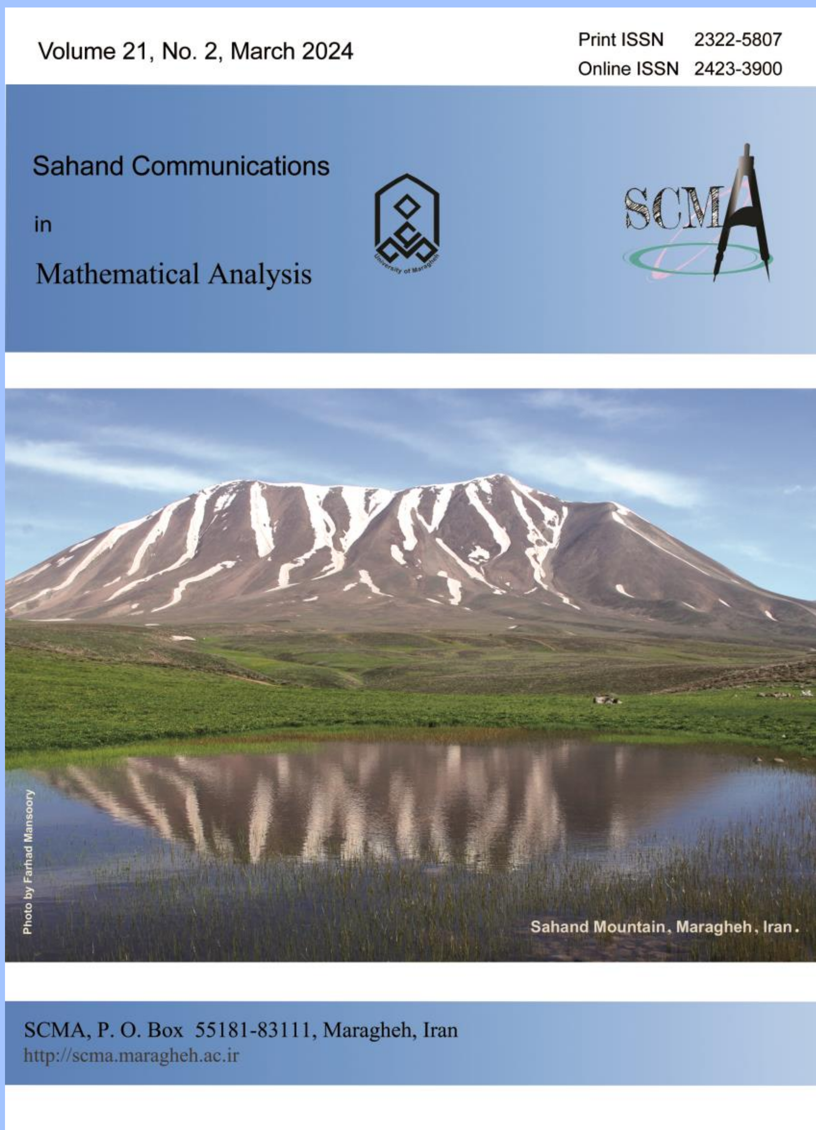

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Generalized Fractional Integral Inequalities for (h, m, s)-Convex Modified Functions of Second Type

Juan E. Nápoles[1] and Bahtiyar Bayraktar[2*]

ABSTRACT. New variants of the Hermite - Hadamard inequality within the framework of generalized fractional integrals for (h, m, s)-convex modified second type functions have been obtained in this article. To achieve these results, we used the Holder inequality and another form of it - power means. Some of the known results described in the literature can be considered as particular cases of the results obtained in our study.

1. Introduction

It is known that the theory of convexity occupies an essential place in optimization problems. In the last few decades, scientists in many countries have begun to pay more attention to evaluating and generalizing the results of estimating the mean value of the obtained function. Special classes of convex time functions are well known.

A function $\phi : [\varrho_1, \varrho_2] \to \mathbb{R}$ is said to be convex if $\phi\big(\sigma u + (1-\sigma)v\big) \leq \sigma\phi(u) + (1-\sigma)\phi(v)$ holds for all $u, v \in [\varrho_1, \varrho_2]$ and $\sigma \in [0, 1]$. A function ϕ is said to be concave if $-\phi$ is convex.

Convex functions have been generalized widely, highlighting the r–convex, m–convex, s–convex, (s, m)–convex, h–convex, (h, m)–convex functions and many others. Readers interested in exploring many of these generalizations and extensions of the basic concept of convexity may refer to [23].

For convex functions, the Hermite–Hadamard inequality is known, undoubtedly one of the most famous in mathematics, for its multiple

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connections and applications:

$$\phi\left(\frac{\varrho_1+\varrho_2}{2}\right) \leq \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\phi(x)\,dx \leq \frac{\phi(\varrho_1)+\phi(\varrho_2)}{2}. \tag{1.1}$$

In this work, we will use the following notion of convexity:

Definition 1.1 ([3, 4])**.** Let $h:[0,1]\to(0,1]$ and $\phi: I=[0,+\infty)\to[0,+\infty)$. If inequality

$$\phi\left(\sigma\xi+(1-\sigma)m\varsigma\right) \leq h^s(\sigma)\phi(\xi)+(1-h(\sigma))^s m\phi(\varsigma) \tag{1.2}$$

is fulfilled for all $\xi,\varsigma\in I$ and $\sigma\in[0,1]$, where $m\in[0,1]$, $s\in(0,1]$. Then is said function ϕ is a (m,h,s)-convex modified of the second type on I.

Remark 1.2. From Definition 1.1, we can define $N_{h,m}^{s,2}[\varrho_1,\varrho_2]$, where $\varrho_1,\varrho_2\in[0,+\infty)$, as the set of (h,m,s)-convex modified functions of the second type, for $\phi(\varrho_1)\geq 0$. In [3, 4] you can see the convex classes obtained from the special cases of this triple.

We use the functions Γ and Γ_k ([11, 27]) in study:

$$\Gamma(\zeta)=\int_0^\infty \sigma^{\zeta-1}e^{-\sigma}\,d\sigma, \quad \Re(\zeta)>0,$$
$$\Gamma_k(\zeta)=\int_0^\infty \sigma^{\zeta-1}e^{\frac{-\sigma^k}{k}}\,d\sigma, \quad k>0.$$

Note that $\Gamma_k(\zeta)= (k)^{\frac{\zeta}{k}-1}\Gamma\left(\frac{\zeta}{k}\right)$, $\Gamma_k(\zeta+k)=z\Gamma_k(\zeta)$ and $\lim\limits_{k\to 1}\Gamma_k(\zeta)=\Gamma(\zeta)$.

As an extension of classical analysis, Fractional Calculus has become the focus of attention of many researchers both with its theory and applications in applied sciences. For example, in mathematical biology, physics, and mathematical modelling of various processes and phenomena ([19, 29]).

To make the subject easy to understand, we provide several definitions of fractional integrals (with $0\leq\varrho_1<\sigma<\varrho_2\leq\infty$), some of which are new.

The first of these are the classical Riemann-Liouville(**RL**) fractional integrals.

Definition 1.3. Let $\phi\in L_1[\varrho_1,\varrho_2]$ and let $\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$. The **RL** fractional integrals of order α are defined by (right and left respectively):

$$^{\alpha}I_{\varrho_1+}\phi(u)=\frac{1}{\mathit{\Gamma}(\alpha)}\int_{\varrho_1}^{u}(u-\sigma)^{\alpha-1}\phi(\sigma)\,d\sigma, \quad u>\varrho_1$$

$$^{\alpha}I_{\varrho_2-}\phi(u)=\frac{1}{\Gamma(\alpha)}\int_u^{\varrho_2}(\sigma-u)^{\alpha-1}\phi(\sigma)\,d\sigma,\quad u<\varrho_2.$$

Definition 1.4 ([18])**.** Let $\phi\in L_1[\varrho_1,\varrho_2]$ and let $\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$. The **RL** $k-$fractional integrals of order α for $k>0$ are given by the expressions (right and left respectively):

$$^{\alpha}I^k_{\varrho_1+}\phi(u)=\frac{1}{k\Gamma_k(\alpha)}\int_{\varrho_1}^{u}(u-\sigma)^{\frac{\alpha}{k}-1}\phi(\sigma)\,d\sigma,\quad u>\varrho_1,$$
$$^{\alpha}I^k_{\varrho_2-}\phi(u)=\frac{1}{k\Gamma_k(\alpha)}\int_u^{\varrho_2}(\sigma-u)^{\frac{\alpha}{k}-1}\phi(\sigma)\,d\sigma,\quad u<\varrho_2.$$

In 2011 Katugampola defined a new integral operator, as a generalization of the n-integral, as follows.

Definition 1.5. Let $\phi:[\varrho_1,\varrho_2]\to\mathbb{R}$ be an integrable function. The general Katugampola fractional integrals of a function ϕ of order $\alpha\in\mathbb{R}$, and $s\neq-1$ is expressed by:

$$^{\beta}_{\varrho}I^{\alpha}_{u}\phi(u)=\frac{(\beta+1)^{1-\alpha}}{\Gamma(\alpha)}\int_{\varrho}^{u}(u^{\beta+1}-\sigma^{\beta+1})^{\alpha-1}\sigma^{\beta}\phi(\sigma)\,d\sigma.$$

Definition 1.6 ([16])**.** Let $\phi:[\varrho_1,\varrho_2]\to\mathbb{R}$ and $\phi\in L_1[\varrho_1,\varrho_2]$, fractional integrals of a function ϕ with respect to function ψ on $[\varrho_1,\varrho_2]$ of order $\alpha\in\mathbb{C}$, $\Re(\alpha)>0$ are expressed by (right and left sided respectively):

$$^{\alpha}_{g}I_{\varrho_1+}\phi(u)=\frac{1}{\Gamma(\alpha)}\int_{\varrho_1}^{u}\big(\psi(u)-\psi(\sigma)\big)^{\alpha-1}\psi'(\sigma)\phi(\sigma)\,d\sigma,\quad u>\varrho_1,$$
$$^{\alpha}_{g}I_{\varrho_2-}\phi(u)=\frac{1}{\Gamma(\alpha)}\int_u^{\varrho_2}\big(\psi(\sigma)-\psi(u)\big)^{\alpha-1}\psi'(\sigma)\phi(\sigma)\,d\sigma,\quad u<\varrho_2.$$

Here $g(\sigma)$ positive and be an increasing function on $[\varrho_1,\varrho_2]$, and $\psi'\in C^1(\varrho_1,\varrho_2)$.

Below is a $k-$fractional analog of Definition 1.6:

Definition 1.7 ([2, 28])**.** Let $\phi:[\varrho_1,\varrho_2]\to\mathbb{R}$ and $\phi\in L_1[\varrho_1,\varrho_2]$. k-fractional integrals of a function ϕ with respect to function ψ on $[\varrho_1,\varrho_2]$ of order $\alpha\in\mathbb{C}$, $\Re(\alpha)>0$ and $k>0$ are expressed by(the right and left sided respectively):

$$^{\alpha}I^k_{\varrho_1+}\phi(u)=\frac{1}{k\Gamma_k(\alpha)}\int_{\varrho_1}^{u}\big(\psi(u)-\psi(\sigma)\big)^{\frac{\alpha}{k}-1}\psi'(\sigma)\phi(\sigma)\,d\sigma,\quad u>\varrho_1,$$
$$^{\alpha}I^k_{\varrho_2-}\phi(u)=\frac{1}{k\Gamma_k(\alpha)}\int_u^{\varrho_2}\big(\psi(\sigma)-\psi(u)\big)^{\frac{\alpha}{k}-1}\psi'(\sigma)\phi(\sigma)\,d\sigma,\quad u<\varrho_2.$$

Here $\psi(\sigma)$ positive and be an increasing function on $[\varrho_1,\varrho_2]$, and $g'\in C^1(\varrho_1,\varrho_2)$.

Next, we will define the operators that we will use in our work.

Definition 1.8. Let $\phi : [0, +\infty) \to [0, +\infty)$ and $\phi \in L_1[0, +\infty)$. Generalized fractional **RL** integral of order α with $\alpha \in \mathbb{R}$, and $\beta \neq -1$ are given as follows:

$$ {}^{\beta}J^{\frac{\alpha}{k}}_{F,\varrho}\phi(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{\varrho}^{u} \frac{\phi(\sigma)d\sigma}{[\mathbb{F}(u,\sigma)]^{1-\frac{\alpha}{k}} F(\sigma,\beta)}, \tag{1.3}$$

with $F(\sigma, \beta) > 0$, $F(\sigma, 0) = 1$ and $\mathbb{F}(u, \sigma) = \int_{\sigma}^{u} \frac{d\theta}{F(\theta,\beta)}$. Obviously $\mathbb{F}(u, \sigma) = -\mathbb{F}(\sigma, u)$

In the following remark, we will establish some relationships between our generalized operator and some of the operators presented in the previous definitions.

Remark 1.9. Let us consider the kernel $F(\sigma, \beta) = \sigma^{-\beta}$, then we will have successively:

$$\begin{aligned} \mathbb{F}(u,\sigma) &= \frac{u^{\beta+1} - \sigma^{\beta+1}}{\beta+1}, \\ [\mathbb{F}(u,\sigma)]^{1-\frac{\alpha}{k}} &= \left[\frac{u^{\beta+1} - \sigma^{\beta+1}}{\beta+1}\right]^{1-\frac{\alpha}{k}}, \end{aligned}$$

what is the (k, β)-**RL** fractional integral in Definition 2.1 of [31], and from here we have the integral of the Definition 1.5 with $k \equiv 1$.

Analogously, if $\beta \equiv 0$ and $k \equiv 1$, we obtain the classic **RL** operator.

In many studies (for example, see [1, 3–7, 7, 10, 21, 22, 30–32] and references therein), the upper bound estimate of the Hermite-Hadamard type inequality and other integral inequalities has been obtained using the fractional integration operators tool. For example, In [1], Abdeljawad proposes and discusses some rules (integration by parts, Taylor power series expansions) of classical analysis in the version of conformal fractional calculus. In [2], Akkurt et al. obtained Hadamard-type inequalities for fractional integrals using synchronous and monotone functions. Bayraktar and Özdemir [5] showed that the upper limit of the absolute error of the Hadamard-type inequality decreases by about n^2 times, where n is the number of intermediate points of the integration interval at which the convex second derivative of the function takes value. In [6] Budak et al. established some trapezoidal and midpoint-type inequalities for generalized fractional integrals using functions whose second derivatives are bounded. Butt et al., in [7]presented new and general integral inequalities for the convex functions using Atangana-Baleanu integral operators. In [8] Butt et al. presented an article in which they obtained new Hermite-Hadamard inequalities of the Jensen-Mercer type for a harmonically convex function through fractional integrals. In [10], Chen and Katugampola obtained the Hadamard–Fejér type and Hadamard type inequalities for fractional integrals, which generalize the Hadamard and the Riemann–Liouville fractional integrals into a single form. In [22], Nápoles et al. obtained variants of Hadamard-type inequalities for convex and quasi-convex functions using weighted integral operators. Du et al., in [13] use generalized fractional integrals; some Bullen-type inequalities are obtained where the first derivative of functions is Lipschitzian, bounded or generalized (s,m)-preinvex and also give the applications. Du et al., in [14] are defined and developed the conceptions of the interval-valued fractional double integrals having exponential kernels. Zhou et al, in [34] established specific fractional integral inclusions having exponential kernels, which are related to the Hermite–Hadamard, Hermite–Hadamard–Fejér, and Pachpatte type inequalities and give the graphical representations for the results.

In this work, we obtain new variants of the classical Hermite-Hadamard Inequality for functions (h, m, s)-convex modified of the second type via the generalized fractional integral operators of Definition 1.8.

2. Main Results

As a first result, we get the generalized fractional integral inequality of Hermite-Hadamard type for functions (h, m, s)-convex of the second type.

Theorem 2.1. *Let $\phi : [0, \infty) \to [0, +\infty)$ and $\phi \in N_{h,m}^{s,2}[\varrho_1, \varrho_2]$ with $0 \le \varrho_1 < \varrho_2$, $m \in (0, 1]$. If $\phi' \in L_1[\varrho_1, \varrho_2]$ and $\frac{\varrho_1}{m} \in [\varrho_1, \varrho_2]$, then we will have*

(2.1)
$$\begin{aligned}
&\phi\left(\frac{\varrho_1+\varrho_2}{2}\right)\int_0^1 \frac{d\sigma}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}\,F(\sigma,\beta)} \\
&\quad\le \frac{1}{\varrho_2-\varrho_1}\left[h^s\left(\frac{1}{2}\right)J^{\frac{\alpha}{k}}_{F,\varrho_1+}\phi(\varrho_2) + \left(1-h\left(\frac{1}{2}\right)\right)^s J^{\frac{\alpha}{k}}_{F,\varrho_2-}\phi(\varrho_1)\right] \\
&\quad\le \left[h^s\left(\frac{1}{2}\right)\phi(\varrho_2) + \left(1-h\left(\frac{1}{2}\right)\right)^s\phi(\varrho_1)\right]\int_0^1 \frac{h^s(\sigma)d\sigma}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}\,F(\sigma,\beta)} \\
&\qquad + m\left[h^s\left(\frac{1}{2}\right)\phi\left(\frac{\varrho_1}{m}\right) + \left(1-h\left(\frac{1}{2}\right)\right)^s\phi\left(\frac{\varrho_2}{m}\right)\right] \\
&\qquad \times \int_0^1 \frac{(1-h(\sigma))^s d\sigma}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}\,F(\sigma,\beta)}.
\end{aligned}$$

Proof. As $\phi \in N^{s,2}_{h,m}[\varrho_1,\varrho_2]$ for $u,v \in [\varrho_1,\varrho_2]$ with $\sigma = \frac{1}{2}$ and $m = 1$ in (1.2), we get

$$\phi\left(\frac{u+v}{2}\right) \le h^s\left(\frac{1}{2}\right)\phi(u) + \left(1 - h\left(\frac{1}{2}\right)\right)^s \phi(v)$$

By setting $u = (1-\sigma)\varrho_1 + \sigma\varrho_2$ and $v = \sigma\varrho_1 + (1-\sigma)\varrho_2$ we obtain from the above equation

$$\phi\left(\frac{\varrho_1+\varrho_2}{2}\right) \le h^s\left(\frac{1}{2}\right)\phi((1-\sigma)\varrho_1 + \sigma\varrho_2) + \left(1 - h\left(\frac{1}{2}\right)\right)^s \phi(\sigma\varrho_1 + (1-\sigma)\varrho_2).$$

By integrating this inequality, over $[0,1]$ after multiplying by $\frac{1}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}F(\sigma,\beta)}$, we have

$$\begin{aligned}\phi\left(\frac{\varrho_1+\varrho_2}{2}\right)\int_0^1 \frac{d\sigma}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}F(\sigma,\beta)} &\le h^s\left(\frac{1}{2}\right)\int_0^1 \frac{\phi((1-\sigma)\varrho_1+\sigma\varrho_2)}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}F(\sigma,\beta)}d\sigma \\ &+ \left(1-h\left(\frac{1}{2}\right)\right)^s\int_0^1 \frac{\phi(\sigma\varrho_1+(1-\sigma)\varrho_2)}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}F(\sigma,\beta)}d\sigma.\end{aligned} \tag{2.2}$$

Denoting $I_1 = \int_0^1 \frac{\phi((1-\sigma)\varrho_1+\sigma\varrho_2)}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}F(\sigma,\beta)}d\sigma$ and $I_2 = \int_0^1 \frac{\phi(\sigma\varrho_1+(1-\sigma)\varrho_2)}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}F(\sigma,\beta)}d\sigma$, we obtain

$$\begin{aligned} I_1 &= \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} \frac{\phi(z)dz}{\left[\mathbb{F}\left(\frac{z-\varrho_1}{\varrho_2-\varrho_1},0\right)\right]^{1-\frac{\alpha}{k}}F\left(\frac{z-\varrho_1}{\varrho_2-\varrho_1},\beta\right)} \\ &= \frac{1}{\varrho_2-\varrho_1}J^{\frac{\alpha}{k}}_{F,\varrho_1+}\phi(\varrho_2), \\ I_2 &= \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} \frac{\phi(z)dz}{\left[\mathbb{F}\left(\frac{\varrho_2-z}{\varrho_2-\varrho_1},0\right)\right]^{1-\frac{\alpha}{k}}F\left(\frac{\varrho_2-z}{\varrho_2-\varrho_1},\beta\right)} \\ &= \frac{1}{\varrho_2-\varrho_1}J^{\frac{\alpha}{k}}_{F,\varrho_2-}\phi(\varrho_1). \end{aligned}$$

So, we have

$$\begin{aligned}\phi\left(\frac{\varrho_1+\varrho_2}{2}\right)\int_0^1 \frac{d\sigma}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}F(\sigma,\beta)} \le \frac{1}{\varrho_2-\varrho_1}\left[h^s\left(\frac{1}{2}\right)J^{\frac{\alpha}{k}}_{F,\varrho_1+}\phi(\varrho_2) + \left(1-h\left(\frac{1}{2}\right)\right)^s J^{\frac{\alpha}{k}}_{F,\varrho_2-}\phi(\varrho_1)\right].\end{aligned} \tag{2.3}$$

In other words, the first part of the inequality sought.

To obtain the right hand side of (2.1) we use the (h, m, s)-convexity of ϕ:

$$\begin{aligned}\phi((1-\sigma)\varrho_1+\sigma\varrho_2) &= \phi(\sigma\varrho_2+(1-\sigma)\varrho_1)\\ &\leq h^s(\sigma)\phi(\varrho_2)+m(1-h(\sigma))^s\phi\left(\frac{\varrho_1}{m}\right),\end{aligned}$$

and

$$\phi(\sigma\varrho_1+(1-\sigma)\varrho_2) \leq h^s(\sigma)\phi(\varrho_1)+m(1-h(\sigma))^s\phi\left(\frac{\varrho_2}{m}\right).$$

Multiplying the first of the above inequalities by $h^s\left(\frac{1}{2}\right)$ and the second by $\left(1-h\left(\frac{1}{2}\right)\right)^s$ we obtain, after multiplying by $\frac{1}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}F(\sigma,\beta)}$ and integrating between 0 and 1:

$$\begin{aligned}&h^s\left(\frac{1}{2}\right)\int_0^1 \phi(\sigma\varrho_2+(1-\sigma)\varrho_1)d\sigma\\ &\leq h^s\left(\frac{1}{2}\right)\left[\phi(\varrho_2)\int_0^1 \frac{h^s(\sigma)}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}F(\sigma,\beta)}d\sigma\right.\\ &\left.+m\phi\left(\frac{\varrho_1}{m}\right)\int_0^1 \frac{(1-h(\sigma))^s}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}F(\sigma,\beta)}d\sigma\right]\end{aligned}$$

and

$$\begin{aligned}&\left(1-h\left(\frac{1}{2}\right)\right)^s\int_0^1 \phi(\sigma\varrho_1+(1-\sigma)\varrho_2)d\sigma\\ &\leq \left(1-h\left(\frac{1}{2}\right)\right)^s\left[\phi(\varrho_1)\int_0^1 \frac{h^s(\sigma)}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}F(\sigma,\beta)}d\sigma\right.\\ &\left.+m\phi\left(\frac{\varrho_2}{m}\right)\int_0^1 \frac{(1-h(\sigma))^s}{[\mathbb{F}(\sigma,0)]^{1-\frac{\alpha}{k}}F(\sigma,\beta)}d\sigma\right].\end{aligned}$$

Changing variables in the integrals on the left-hand side of the above inequalities leads us easily to the second part of (2.1), which is the required inequality. This completes the proof. □

Remark 2.2. Next we will present several results reported in the literature, which can be obtained from the Theorem previously proven.

1) With $F \equiv 1$, $\alpha = k = 1$, $m = s = 1$ and $h(z) = z$, that is, working with the Riemann Integral and with convex functions, we obtain the Classical Hermite-Hadamard Inequality (1.1).
2) For the case of **RL** Integrals, $F \equiv 1$, $k = 1$ and in the framework of convex functions, Theorem 2 of [32] is derived from the

previous result. For the case of s-convex functions, with $F \equiv 1$, Theorem 2.1 of [33] is easily obtained.

3) For k-fractional integrals, $F \equiv 1$, and convex functions, Theorem 2.1 of [15] follows easily from the above.
4) For the case of Katugampola fractional integrals, $k = 1$, and convex functions, we have the Theorem 2.1 of [10]. With the Katugampola Fractional Integral, and working with s-convex functions, that is, $m = 1$ and $h(z) = z$, it is not difficult to derive Theorem 2.1 of [17]. Also see Theorem 2.1 of [4].
5) With $\alpha = k = 1$, the above result generalizes Theorem 2.1 of [20].
6) Under the condition $\alpha = k = 1$ and $F(t, \beta) = t^\beta$, the operator (1.3) becomes the non-conformable integral operator used in [24]. In this way, our result completes the results obtained in said paper, so we have:

Corollary 2.3. *Let $\phi : [0, \infty) \to \mathbb{R}$ and $\phi \in L_1[\varrho_1, \varrho_2]$ with $0 \le \varrho_1 < \varrho_2$, If ϕ be a convex function, then we have*

$$\begin{aligned}\phi\left(\frac{\varrho_1+\varrho_2}{2}\right) &\le \frac{1-\beta}{(\varrho_2-\varrho_1)^{1-\beta}}\left[\int_{\varrho_1}^{\varrho_2}\frac{\phi(z)dz}{(z-\varrho_1)^\beta}+\int_{\varrho_1}^{\varrho_2}\frac{\phi(z)dz}{(\varrho_2-z)^\beta}\right]\\ &\le \frac{\phi(\varrho_1)+\phi(\varrho_2)}{2}.\end{aligned}$$

This inequality was obtained by Özdemir et al. in [26] (see Theorem 2.6).

The following result is basic in the subsequence.

Lemma 2.4. *Let $\phi : [0, \infty) \to [0, +\infty)$ be a differentiable and $\phi \in N^{s,2}_{h,m}[\varrho_1, \varrho_2]$ with $0 \le \varrho_1 < \varrho_2$ and $m \in (0, 1]$. If $\phi' \in L_1[\varrho_1, \varrho_2]$ and $\frac{\varrho_1}{m} \in [\varrho_1, \varrho_2]$, then we will have*

$$\begin{aligned}&\frac{1}{\varrho_2-\varrho_1}\left[\mathbb{F}(1,0)\right]^{\frac{\alpha}{k}}\left(\phi(\varrho_1)+\phi(\varrho_2)\right)\\ &\qquad -\frac{\Gamma_k(1+\alpha)}{(\varrho_2-\varrho_1)^2}\left[J^{\frac{\alpha}{k}}_{F,\varrho_1+}\phi(\varrho_2)+J^{\frac{\alpha}{k}}_{F,\varrho_2-}\phi(\varrho_1)\right]\\ &\qquad =\int_0^1\left[\mathbb{F}(r,0)\right]^{\frac{\alpha}{k}}\left[\phi'(r\varrho_1+(1-r)\varrho_2)-\phi'((1-r)\varrho_1+r\varrho_2)\right]dr.\end{aligned} \tag{2.4}$$

Proof. Denoting

$$I_1 = \int_0^1 \left[\mathbb{F}(r,0)\right]^{\frac{\alpha}{k}} \phi'((1-r)\varrho_1 + r\varrho_2)]dr$$

and

$$I_2 = \int_0^1 \left[\mathbb{F}(r,0)\right]^{\frac{\alpha}{k}} \phi'(r\varrho_1 + (1-r)\varrho_2)dr,$$

we have

$$I_1 = -\frac{1}{\varrho_2 - \varrho_1}\left[\mathbb{F}(1,0)\right]^{\frac{\alpha}{k}} \phi(\varrho_2) + \frac{\Gamma_k(1+\alpha)}{(\varrho_2-\varrho_1)^2} J^{\frac{\alpha}{k}}_{F,\varrho+}\phi(\varrho_2)$$

$$I_2 = \frac{1}{\varrho_2 - \varrho_1}\left[\mathbb{F}(1,0)\right]^{\frac{\alpha}{k}} \phi(\varrho_2) - \frac{\Gamma_k(1+\alpha)}{(\varrho_2-\varrho_1)^2} J^{\frac{\alpha}{k}}_{F,b-}\phi(\varrho_1).$$

The previous results were obtained after integrating parts and making a simple change of variables.

It only remains to add both equalities and reorder. This completes the proof of the Lemma. □

Remark 2.5. Consider $F \equiv 1$, then the right hand side of (2.4) becomes

$$\int_0^1 t^{\frac{\alpha}{k}} \left[\phi'(r\varrho_1 + (1-r)\varrho_2) - \phi'((1-r)\varrho_1 + r\varrho_2)\right] dr$$

and, by a simple change of variables, we obtain

$$\int_0^1 r^{\frac{\alpha}{k}} \left[\phi'(r\varrho_1 + (1-r)\varrho_2) - \phi'((1-r)\varrho_1 + r\varrho_2)\right] dr$$
$$= \int_0^1 \left[(1-r)^{\frac{\alpha}{k}} - r^{\frac{\alpha}{k}}\right] \phi'(r\varrho_1 + (1-r)\varrho_2) dr.$$

Taking into account the above, if $\alpha = k = 1$, the Lemma 2.1 of [12] is derived. In the same way, with $k = 1$ of the previous result, the Lemma 2 of [32] is obtained (see also Lemma 2.1 of [30]).

Remark 2.6. Putting $F \equiv 1$, we obtain the following result for $k-$**RL** Integrals.

Let $\phi : [0,\infty) \to \mathbb{R}$ be a differentiable and $\phi \in N^{s,2}_{h,m}[\varrho_1,\varrho_2]$ with $0 \le \varrho_1 < \varrho_2$, $m \in (0,1]$. If $\phi' \in L_1[\varrho_1,\varrho_2]$ and $\frac{\varrho_1}{m} \in [\varrho_1,\varrho_2]$, then we will have

$$\frac{\phi(\varrho_1)+\phi(\varrho_2)}{2} - \frac{\Gamma_k(1+\alpha)}{2(\varrho_2-\varrho_1)}\left[J^{\frac{\alpha}{k}}_{\varrho_1+}\phi(\varrho_2) + J^{\frac{\alpha}{k}}_{\varrho_2-}\phi(\varrho_1)\right]$$
$$= \frac{\varrho_2-\varrho_1}{2}\int_0^1 r^{\frac{\alpha}{k}} \left[\phi'(r\varrho_1 + (1-r)\varrho_2) - \phi'((1-r)\varrho_1 + r\varrho_2)\right] dr.$$

Theorem 2.7. *Let $\phi : [0,\infty) \to [0,+\infty)$ be a differentiable and $\phi \in N^{s,2}_{h,m}[\varrho_1,\varrho_2]$ with $0 \le \varrho_1 < \varrho_2$, $m \in (0,1]$. If $\phi' \in L_1[\varrho_1,\varrho_2]$ and $\frac{\varrho_1}{m} \in [\varrho_1,\varrho_2]$, then for $q > 1$ we have the following inequality*

(2.5)

$$\left|\left[\mathbb{F}(1,0)\right]^{\frac{\alpha}{k}} \frac{\phi(\varrho_1)+\phi(\varrho_2)}{2} - \frac{\Gamma_k(1+\alpha)}{2(\varrho_2-\varrho_1)}\left[J^{\frac{\alpha}{k}}_{F,\varrho_1+}\phi(\varrho_2) + J^{\frac{\alpha}{k}}_{F,\varrho_2+}\phi(\varrho_1)\right]\right|$$

$$
\begin{aligned}
&\leq \frac{\varrho_2-\varrho_1}{2}\left(\int_0^1 [\mathbb{F}(r,0)]^{\frac{\alpha p}{k}}\,dr\right)^{\frac{1}{p}} \\
&\quad\times\left(\left[\left|\phi'(\varrho_2)\right|^q\int_0^1 h^s(r)dr+m\left|\phi'\left(\frac{\varrho_1}{m}\right)\right|^q\int_0^1(1-h(r))^s\,dr\right]\right. \\
&\quad\left.+\left[\left|\phi'(\varrho_1)\right|^q\int_0^1 h^s(r)dr+m\left|\phi'\left(\frac{\varrho_2}{m}\right)\right|^q\int_0^1(1-h(r))^s\,dr\right]\right)^{\frac{1}{q}}.
\end{aligned}
$$

Proof. After multiplying (2.4) by $\frac{\varrho_2-\varrho_1}{2}$, use Hölder's inequality and consider the (h,m,s)-convexity of ϕ' on the member right, the desired result is obtained. □

Remark 2.8. Readers will have no difficulty in verifying that the previous result follows, under different notions of convexity and appropriate choice of F, α and k: Theorem 2.3 of [12]; Theorems 2.6 and 2.7, Corollaries 2.7 and 2.8 of [30]; Theorems 3.2, 3.6 and Corollary 3.3 of [33].

Theorem 2.9. *Let $\phi:[0,\infty)\to[0,+\infty)$ be a differentiable and $\phi\in N^{s,2}_{h,m}[\varrho_1,\varrho_2]$ with $0\leq\varrho_1<\varrho_2$, $m\in(0,1]$. If $\phi'\in L_1[\varrho_1,\varrho_2]$ and $\frac{\varrho_1}{m}\in[\varrho_1,\varrho_2]$, then for $q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$ we have the following inequality*

(2.6)

$$
\begin{aligned}
&\left|[\mathbb{F}(1,0)]^{\frac{\alpha}{k}}\frac{\phi(\varrho_1)+\phi(\varrho_2)}{2}-\frac{\Gamma_k(1+\alpha)}{2(\varrho_2-\varrho_1)}\left[J^{\frac{\alpha}{k}}_{F,\varrho_1+}\phi(\varrho_2)+J^{\frac{\alpha}{k}}_{F,\varrho_2-}\phi(\varrho_1)\right]\right| \\
&\quad\leq\frac{\varrho_2-\varrho_1}{2}\left(\int_0^1[\mathbb{F}(r,0)]^{\frac{\alpha}{k}}\,dr\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\left[\int_0^1[\mathbb{F}(r,0)]^{\frac{\alpha}{k}}\left(\left|\phi'(\varrho_2)\right|^q h^s(r)+m\left|\phi'\left(\frac{\varrho_1}{m}\right)\right|^q(1-h(r))^s\right)dr\right]^{\frac{1}{q}}\right. \\
&\quad\left.+\left[\int_0^1[\mathbb{F}(r,0)]^{\frac{\alpha}{k}}\left(\left|\phi'(\varrho_1)\right|^q h^s(r)+m\left|\phi'\left(\frac{\varrho_2}{m}\right)\right|^q(1-h(r))^s\right)dr\right]^{\frac{1}{q}}\right).
\end{aligned}
$$

Proof. The test follows a similar path to the previous one, although power mean inequality is used instead of Hölder's inequality. □

Remark 2.10. Using different notions of convexity and with a suitable choice of F, α and k, they are derived from the previous result: Theorem 2.10 and Corollaries 2.11, 2.12 of [30]; Theorem 3.4 and Corollary 3.5 of [33].

3. Conclusions

In this study, we gave a formulation of the generalized fractional integral operator. Several operators described in the literature are a particular case of this definition.

The strength of Definition 1.8 lies in the fact that if we represent the kernel in the form $F(\sigma, s) = \sigma^{1-s}$, then we obtain a variant of the fractional integral $(k, s)-\mathbf{RL}$ from [31]

$$ {}_{\xi}^{\beta}I_u^{\alpha}\phi(u) = \frac{(2-\beta)^{1-\frac{\alpha}{k}}}{k\ \Gamma_k(\alpha)} \int_{\xi}^{u} \left(u^{2-\beta} - \sigma^{2-\beta} \right)^{\frac{\alpha}{k}-1} \sigma^{1-\beta}\phi(\sigma)\, d\sigma. $$

This opens up vast possibilities for obtaining new integral inequalities. For example, in [9], the following function was considered:

$$ \begin{aligned} T(f,g) &= \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} f(x)g(x)dx \\ &\quad - \left(\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} f(x)dx\right)\left(\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} g(x)dx\right) \end{aligned} $$

The study of this function can be generalized using our integral operator.

The generality of our results can also be checked if we apply our integral operator to the results of [25], which can be easily generalized, as readers can check if we consider $F(\sigma, \beta) = (\sigma - \rho_1)^{\beta-1}$ and $F(\sigma, \beta) = (\rho_2 - \sigma)^{\beta-1}$, to left and right-sided integral.

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