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Kamal Fallahi

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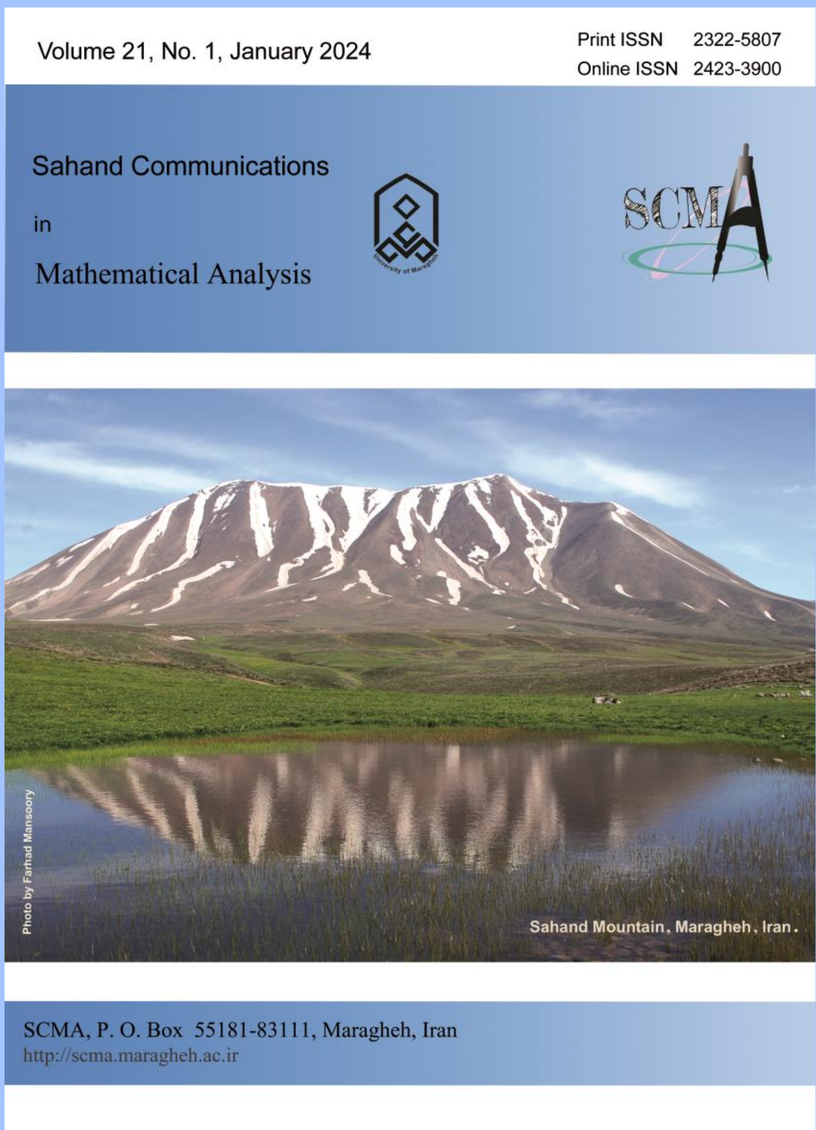

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Abstract. The underlying aim of this paper is first to state the Cyclic version of $\mathcal{K}$-quasi-contractive mappings introduced by Fallahi and Aghanians [On quasi-contractions in metric spaces with a graph, Hacet. J. Math. Stat. 45 (4) (2016), 1033-1047]. Secondly, it seeks to show to show the existence of fixed point and best proximity points for such contractive mappings in a metric space with a graph, which can entail a large number of former fixed point and best proximity point results. One fundamental issue that can be distinguished between this work and previous studies is that it can also involve all of results stated by taking comparable and η-close elements.

1. Introduction

Since 1922, metric fixed point (fp) theory and contractions have evolved into essential tools in nonlinear analysis. Numerous researchers have applied these concepts to address a wide range of problems in nonlinear functions and engineering, as evidenced by the works of [3, 4, 13, 16] and the references therein. For instance, in 2004, Ran and Reurings [15] considered a partial order set (POS) on a metric space (MS) discussing the existence and uniqueness of fps for contractive mappings, particularly for comparable elements.

Theorem 1.1 ([15]). *Consider a POS $(\mathcal{Y}, \lesssim)$, a complete MS $(\mathcal{Y}, \mathcal{D})$ and a nondecreasing mapping $\mathcal{G} : \mathcal{Y} \to \mathcal{Y}$ so that $\mathcal{D}(\mathcal{G}\mathfrak{a}, \mathcal{G}\mathfrak{b}) \leq \theta \mathcal{D}(\mathfrak{a}, \mathfrak{b})$ for any $\mathfrak{a}, \mathfrak{b} \in \mathcal{Y}$ with $\mathfrak{a} \lesssim \mathfrak{b}$, where $\theta \in [0, 1)$. Also, assume*

- *either $\mathcal{G}$ is continuous;*

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- *or if a nondecreasing sequence* $\mathfrak{a}_n$ *converges to a* $\mathfrak{a} \in \mathcal{Y}$*, then* $\mathfrak{a}_n \lesssim \mathfrak{a}$.

If there is $\mathfrak{a}_0 \in \mathcal{Y}$ *satisfying* $\mathfrak{a}_0 \lesssim \mathcal{G}\mathfrak{a}_0$*, then* $\mathcal{G}$ *has a fp. Furthermore, if each two fp(s) are comparable, then the fp is unique.*

Note that we say $\mathcal{G}$ in Theorem 1.1 is nondecreasing when $\mathfrak{a} \lesssim \mathfrak{b}$ implies $\mathcal{G}\mathfrak{a} \lesssim \mathcal{G}\mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{Y}$. Also, we say $\mathfrak{a}$ and $\mathfrak{b}$ are comparable whenever $\mathfrak{a} \lesssim \mathfrak{b}$ or $\mathfrak{b} \lesssim \mathfrak{a}$. In 2005, Nieto and Rodrıguez-López [12] used this definition and fp result to solve some differential equations. Moreover, in 2011, Abkar and Gabeleh [1] integrated Theorems 1.1 with the definition of cyclic mappings introduced by Kirk et al. [11] and established an fp result.

Theorem 1.2 ([1])**.** *Take a POS* $(\mathcal{Y}, \lesssim)$*, two closed subsets* $\mathcal{I}, \mathcal{J} \neq \emptyset$ *of a complete MS* $(\mathcal{Y}, \mathcal{D})$ *and a cyclic mapping* $\mathcal{G} : \mathcal{I} \cup \mathcal{J} \to \mathcal{I} \cup \mathcal{J}$ *so that* $\mathcal{D}(\mathcal{G}\mathfrak{a}, \mathcal{G}^2\mathfrak{b}) \leq \theta \mathcal{D}(\mathfrak{a}, \mathcal{G}\mathfrak{b})$ *for each* $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{I} \times \mathcal{I}$ *with* $\mathfrak{b} \lesssim \mathfrak{a}$*, where* $\theta \in (0, 1)$ *and* $\mathcal{G}^2$ *is nondecreasing on* $\mathcal{I}$*. Also, presume that*

- *either* $\mathcal{G}$ *is continuous;*
- *or if a nondecreasing sequence* $\mathfrak{a}_n$ *converges to a* $\mathfrak{a} \in \mathcal{Y}$*, then* $\mathfrak{a}_n \lesssim \mathfrak{a}$.

If there is $\mathfrak{a}_0 \in \mathcal{Y}$ *satisfying* $\mathfrak{a}_0 \lesssim \mathcal{G}^2\mathfrak{a}_0$*, then* $\mathcal{I} \cap \mathcal{J} \neq \emptyset$ *and* $\mathcal{G}$ *has a fp in* $\mathcal{I} \cap \mathcal{J}$*. Further, if* $\mathfrak{a}_{n+1} = \mathcal{G}(\mathfrak{a}_n)$*, then* $\mathfrak{a}_{2n} \to \mathfrak{p}$.

It should be noted that a mapping $\mathcal{G} : \mathcal{I} \cup \mathcal{J} \to \mathcal{I} \cup \mathcal{J}$ is named cyclic if $\mathcal{G}(\mathcal{I}) \subseteq \mathcal{J}$ and $\mathcal{G}(\mathcal{J}) \subseteq \mathcal{I}$.

Although the theory of fp is an important tool for obtaining such point for mapping $\mathcal{G}$ on $\mathcal{I} \subseteq \mathcal{Y}$, a non-self mapping $\mathcal{G} : \mathcal{I} \to \mathcal{J}$ does not essentially have an fp. Therefore, one may find a point $\mathfrak{a}$ that is closest to $\mathcal{G}\mathfrak{a}$. Hence, the best proximity point (bpp) results have became well-known in applied mathematics. Let $\mathcal{I}, \mathcal{J} \neq \emptyset$ be subsets of an MS, $\mathsf{dist}(\mathcal{I}, \mathcal{J}) = \inf\{\mathcal{D}(\mathfrak{a}, \mathfrak{b}) : \ \mathfrak{a} \in \mathcal{I}, \mathfrak{b} \in \mathcal{J}\}$ and $\mathcal{G} : \mathcal{I} \to \mathcal{J}$ is a non-self mapping. The bpp(s) of $\mathcal{G}$ is all $\mathfrak{a} \in \mathcal{I}$ with $\mathcal{D}(\mathfrak{a}, \mathcal{G}\mathfrak{a}) = \mathsf{dist}(\mathcal{I}, \mathcal{J})$. In the sequel, Eldred and Veeremani [6] and Suzuki et al. [19] presented the existence of bpp(s) of cyclic contractive mappings on various metric spaces regarding some properties such as the unconditionally Cauchy (UC) property.

Definition 1.3 ([19])**.** Taking $\mathcal{I}, \mathcal{J} \neq \emptyset$ two subsets of an MS $(\mathcal{Y}, \mathcal{D})$, we say the pair $(\mathcal{I}, \mathcal{J})$ has UC property whenever for two sequences $\{\mathfrak{a}_n\}$ and $\{\mathfrak{a}'_n\}$ in $\mathcal{I}$ and a sequence $\{\mathfrak{b}_n\}$ in $\mathcal{J}$, $\lim_{n\to\infty} \mathcal{D}(\mathfrak{a}_n, \mathfrak{b}_n) = \lim_{n\to\infty} \mathcal{D}(\mathfrak{a}'_n, \mathfrak{b}_n) = \mathsf{dist}(\mathcal{I}, \mathcal{J})$ implies $\lim_{n\to\infty} \mathcal{D}(\mathfrak{a}, \mathfrak{a}'_n) = 0$.

Lemma 1.4 ([19])**.** *Let* $\mathcal{I}, \mathcal{J} \neq \emptyset$ *be subsets of a MS* $(\mathcal{Y}, \mathcal{D})$ *and the pair* $(\mathcal{I}, \mathcal{J})$ *has the UC property. Additionally, assume that* $\{\mathfrak{a}_n\}$ *and*

$\{\mathfrak{b}_n\}$ are sequences in $\mathcal{I}$ and $\mathcal{J}$, respectively, provided that either

$$\lim_{m\to\infty}\sup_{n\geq m}\mathcal{D}(\mathfrak{a}_m,\mathfrak{b}_n)=\mathsf{dist}(\mathcal{I},\mathcal{J}),$$

or

$$\lim_{n\to\infty}\sup_{m\geq n}\mathcal{D}(\mathfrak{a}_m,\mathfrak{b}_n)=\mathsf{dist}(\mathcal{I},\mathcal{J}).$$

Then $\{\mathfrak{a}_n\}$ is a Cauchy sequence.

The theory of bpp for various mappings in different type of $MS(s)$ has been continued by many researchers (see also [8, 9, 14, 17, 18] and the references therein). On the other hand, if $\mathcal{I}\cap\mathcal{J}=\emptyset$ in Theorem 1.2, then $\mathcal{G}\mathfrak{a}=\mathfrak{a}$ has no solution. Hence, we may think about an approximate solution $\mathfrak{a}\in\mathcal{I}\cup\mathcal{J}$ so that the error $\mathsf{dist}(\mathfrak{a},\mathcal{G}\mathfrak{a})$ is minimum. As $\mathcal{G}$ is cyclic on $\mathcal{I}\cup\mathcal{J}$, we obtain $\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{a})\geq\mathsf{dist}(\mathcal{I},\mathcal{J})$. Hence, Abkar and Gabele introduced some useful tools for finding bpp of cyclic contractive and cyclic φ-contractive mapping, respectively.

Theorem 1.5 ([1]). *Let $(\mathcal{Y},\lesssim)$ be a PO set, $\mathcal{I},\mathcal{J}\neq\emptyset$ be two closed subsets of a complete MS $(\mathcal{Y},\mathcal{D})$ and $\mathcal{G}:\mathcal{I}\cup\mathcal{J}\to\mathcal{I}\cup\mathcal{J}$ be a cyclic mapping fulfilling*

$$\mathcal{D}(\mathcal{G}\mathfrak{a},\mathcal{G}^2\mathfrak{b})\leq\theta\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{b})-(1-\theta)\mathsf{dist}(\mathcal{I},\mathcal{J})$$

for each $(\mathfrak{a},\mathfrak{b})\in\mathcal{I}\times\mathcal{I}$ with $\mathfrak{b}\lesssim\mathfrak{a}$, where $\theta\in(0,1)$ and $\mathcal{G}^2$ is nondecreasing on $\mathcal{I}$. Also, presume that the following condition is held:

- *If a nondecreasing sequence $\mathfrak{a}_n$ converges to a $\mathfrak{a}$ in $\mathcal{Y}$, then $\mathfrak{a}_n\lesssim\mathfrak{a}$.*

If there is $\mathfrak{a}_0\in\mathcal{Y}$ satisfying $\mathfrak{a}_0\lesssim\mathcal{G}^2\mathfrak{a}_0$, $\mathfrak{a}_{n+1}=\mathcal{G}\mathfrak{a}_n$ for $n\geq 0$ and $\{\mathfrak{a}_{2n}\}$ possesses a convergent subsequence in $\mathcal{I}$, then $\mathcal{G}$ has a bpp in $\mathcal{I}$.

Theorem 1.6 ([2]). *Let $(\mathcal{Y},\lesssim)$ be a PO set, $\mathcal{I},\mathcal{J}\neq\emptyset$ be two closed subsets of a complete MS $(\mathcal{Y},\mathcal{D})$ and $\mathcal{G}:\mathcal{I}\cup\mathcal{J}\to\mathcal{I}\cup\mathcal{J}$ be a cyclic mapping fulfilling*

$$\mathcal{D}(\mathcal{G}\mathfrak{a},\mathcal{G}^2\mathfrak{b})\leq\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{b})-\varphi\left(\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{b})\right)+\varphi\left(\mathsf{dist}(\mathcal{I},\mathcal{J})\right)$$

for each $(\mathfrak{a},\mathfrak{b})\in\mathcal{I}\times\mathcal{I}$ with $\mathfrak{b}\lesssim\mathfrak{a}$, where $\varphi:\mathbb{R}^{\geq 0}\to\mathbb{R}^{\geq 0}$ is a strictly increasing function and $\mathcal{G}^2$ is nondecreasing on $\mathcal{I}$. Also, presume that the following condition is held:

- *If a nondecreasing sequence $\mathfrak{a}_n$ converges to a $\mathfrak{a}$ in $\mathcal{Y}$, then $\mathfrak{a}_n\lesssim\mathfrak{a}$.*

If there is $\mathfrak{a}_0\in\mathcal{Y}$ satisfying $\mathfrak{a}_0\lesssim\mathcal{G}^2\mathfrak{a}_0$, $\mathfrak{a}_{n+1}=\mathcal{G}\mathfrak{a}_n$ for $n\geq 0$ and $\{\mathfrak{a}_{2n}\}$ possesses a convergent subsequence in $\mathcal{I}$, then $\mathcal{G}$ has a bpp in $\mathcal{I}$.

To follow POS and fp subjects, in 2008, Jachymski [10] stated a graphical MS and introduced several concepts along with fp theorems. Afterward, many researchers working on both fP theory and bpp theorems extended Jachymski's idea in different directions regarding different spaces and various contraction (also, see [7]). Note that the results from these references can significantly expand the results regarding a PO relationship. Let $\mathcal{K}$ be a graph. A link is an edge of $\mathcal{K}$ in which its ends is different. Also, a loop is an edge of $\mathcal{K}$, where its ends is identical. Parallel edges of $\mathcal{K}$ are two or more links of $\mathcal{K}$ with same pairs of ends. Suppose $(\mathcal{Y}, \mathcal{D})$ is a MS and $\mathcal{K}$ is a directed graph, where $\mathsf{V}(\mathcal{K})$ is the vertex set coinciding with $\mathcal{Y}$ and $\mathcal{E}(\mathcal{K})$ is the edge set containing all loops and $\mathcal{K}$ has no parallel edges. Then, $(\mathcal{Y}, \mathcal{D})$ is named an MS with the graph $\mathcal{K}$ (or GMS). Additionally, suppose $\mathcal{K}^{-1}$ is a directed graph obtained from $\mathcal{K}$ by reversing the directions of its edges of $\mathcal{K}$ and $\tilde{\mathcal{K}}$ is the undirected graph gotten from $\mathcal{K}$ by removing the directions of its edges $\mathcal{K}$. It is clear that $\mathsf{V}(\mathcal{K}^{-1}) = \mathsf{V}(\tilde{\mathcal{K}}) = \mathsf{V}(\mathcal{K}) = \mathcal{Y}$, $\mathcal{E}(\mathcal{K}^{-1}) = \{(\mathfrak{a}, \mathfrak{b}) \in \mathcal{Y} \times \mathcal{Y} : (\mathfrak{b}, \mathfrak{a}) \in \mathcal{E}(\mathcal{K})\}$ and $\mathcal{E}(\tilde{\mathcal{K}}) = \mathcal{E}(\mathcal{K}) \cup \mathcal{E}(\mathcal{K}^{-1})$.

To show main results, some symbols and definitions, which is introduced in the following, are also required in next section.

- Assume that $\mathcal{I}, \mathcal{J} \neq \emptyset$ are two subset of a GMS $(\mathcal{Y}, \mathcal{D})$, and define

$$\mathsf{dist}(\mathcal{I}, \mathcal{J}) = \inf \{\mathcal{D}(\mathfrak{a}, \mathfrak{b}) : \mathfrak{a} \in \mathcal{I}, \mathfrak{b} \in \mathcal{J}\}.$$

- Assume that $\mathcal{G} : \mathcal{Y} \to \mathcal{Y}$ is a mapping. We mean $C_{\mathcal{G}}$ by the set of all points $\mathfrak{a} \in \mathcal{Y}$ provided that $(\mathcal{G}^m\mathfrak{a}, \mathcal{G}^n\mathfrak{a})$ is an edge of $\tilde{\mathcal{K}}$ for each $m, n \in \mathbb{N} \cup \{0\}$; that is,

$$C_{\mathcal{G}} = \big\{\mathfrak{a} \in \mathcal{Y} : (\mathcal{G}^m\mathfrak{a}, \mathcal{G}^n\mathfrak{a}) \in \mathcal{E}(\tilde{\mathcal{K}}),\, m, n = 0, 1, \dots\big\}.$$

  Notice that $C_{\mathcal{G}}$ may become an empty set. For example, take $\mathbb{R}$ along with the usual Euclidean metric and a graph G given by $\mathsf{V}(\mathcal{K}) = \mathbb{R}$ and $\mathcal{E}(\mathcal{K}) = \{(\mathfrak{a}, \mathfrak{a}) : \mathfrak{a} \in \mathbb{R}\}$. If $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ is defined by $\mathcal{G}\mathfrak{a} = \mathfrak{a} + 1$ for any $\mathfrak{a} \in \mathbb{R}$, clearly $C_{\mathcal{G}} = \emptyset$.

Definition 1.7 ([10])**.** Presume that $(\mathcal{Y}, \mathcal{D})$ is a GMS. A mapping $\mathcal{G} : \mathcal{Y} \to \mathcal{Y}$ is known as an orbitally $\mathcal{K}$-continuous mapping on $\mathcal{Y}$ whenever $\mathcal{G}^{\mathfrak{b}_n}\mathfrak{a} \to \mathfrak{b}$ implies $\mathcal{G}(\mathcal{G}^{\mathfrak{b}_n}\mathfrak{a}) \to \mathcal{G}\mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{Y}$ and sequences $\{\mathfrak{b}_n\}$ of natural numbers so that $(\mathcal{G}^{\mathfrak{b}_n}\mathfrak{a}, \mathcal{G}^{\mathfrak{b}_n+1}\mathfrak{a}) \in \mathcal{E}(\mathcal{K})$ for every $n \in \mathbb{N}$.

Definition 1.8 ([10])**.** Taking $(\mathcal{Y}, \mathcal{D})$ is a GMS, we say $\mathcal{K}$ is a C-graph on $\mathcal{Y}$ if $\mathfrak{a} \in \mathcal{Y}$ and $\{\mathfrak{a}_n\}$ is a sequence in $\mathcal{Y}$ so that $\mathfrak{a}_n \to \mathfrak{a}$ and $(\mathfrak{a}_{n+1}, \mathfrak{a}_n) \in \mathcal{E}(\mathcal{K})$ for each $n \in \mathbb{N}$, then there is a subsequence $\{\mathfrak{a}_{2n_i}\}$ of $\{\mathfrak{a}_n\}$ such that $(\mathfrak{a}_{2n_i}, \mathfrak{a}) \in \mathcal{E}(\mathcal{K})$ for every $i \in \mathbb{N}$.

2. Best Proximity Points

In the sequel, note that $(\mathcal{I}, \mathcal{J})$ will be a pair of nonempty closed subsets of $\mathcal{Y}$. Now, we are ready to give the definition of $\mathcal{K}$-quasi-contractions in metric spaces with a graph which is motivated by [5, Definition 1] and [10, Definition 2.1].

Definition 2.1. Assume $(\mathcal{Y}, \mathcal{D})$ is a GMS. A mapping $\mathcal{G} : \mathcal{I}\cup\mathcal{J} \to \mathcal{I}\cup\mathcal{J}$ is known as cyclic $\mathcal{K}$-quasi-contractions on $\mathcal{I}$ if $\mathcal{G}$ is cyclic and

$$\mathcal{D}(\mathcal{G}\mathfrak{a}, \mathcal{G}^2\mathfrak{b}) \leq \xi\mathcal{N}^*_{\mathcal{G}}\mathcal{D}(\mathfrak{a}, \mathcal{G}\mathfrak{b}) \tag{2.1}$$

for all $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{I} \times \mathcal{I}$ with $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{E}(\mathcal{K})$ where

$$\begin{aligned}&\mathcal{N}^*_{\mathcal{G}}\mathcal{D}(\mathfrak{a}, \mathcal{G}\mathfrak{b})\\ &\quad= \max\left\{\mathcal{D}(\mathfrak{a}, \mathcal{G}\mathfrak{b}), \mathcal{D}(\mathfrak{a}, \mathcal{G}\mathfrak{a}) - \mathcal{D}(\mathcal{I}, \mathcal{J}), \mathcal{D}(\mathcal{G}\mathfrak{b}, \mathcal{G}^2\mathfrak{b}) - \mathcal{D}(\mathcal{I}, \mathcal{J})\right\}.\end{aligned}$$

Now, we are ready to state and prove the first fundamental theorem of this section.

Theorem 2.2. *Assume $(\mathcal{Y}, \mathcal{D})$ is a GMS, $\mathcal{I}$ is complete and $\mathcal{G} : \mathcal{I}\cup\mathcal{J} \to \mathcal{I}\cup\mathcal{J}$ is a cyclic $\mathcal{K}$-quasi-contractions, where $\mathcal{G}^2$ preserves the edges of $\mathcal{K}$ on $\mathcal{I}$, $C_{\mathcal{G}}|_{\mathcal{I}} \neq \emptyset$ and $\mathfrak{a}_{n+1} = \mathcal{G}\mathfrak{a}_n$. If $\mathcal{K}$ is C-graph on $\mathcal{I}$ and $\{\mathfrak{a}_{2n}\}$ has a convergent subsequence in $\mathcal{I}$, then $\mathcal{G}$ has a bpp $\mathfrak{a}^* \in \mathcal{I}$.*

Proof. As $C_{\mathcal{G}}|_{\mathcal{I}} \neq \emptyset$, assume $\mathfrak{a}_0 \in C_{\mathcal{G}}$ with $\mathfrak{a}_0 \in \mathcal{I}$. We have $(\mathfrak{a}_0, \mathcal{G}^2\mathfrak{a}_0) \in \mathcal{E}(\mathcal{K})$ and since $\mathcal{G}^2$ preserves the edges of $\mathcal{K}$ on $\mathcal{I}$, $(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+2}) \in \mathcal{E}(\mathcal{K})$ for $n = 0, 1, \ldots$. Since $(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+2}) \in \mathcal{E}(\mathcal{K})$ for every $n \in \mathbb{N} \cup \{0\}$ and by (2.1) on $\mathcal{I}$, we get

$$\begin{aligned}\mathcal{D}(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+1}) &= \mathcal{D}(\mathcal{G}\mathfrak{a}_{2n}, \mathcal{G}^2\mathfrak{a}_{2n-2})\\ &\leq \xi\mathcal{N}^*_{\mathcal{G}}\mathcal{D}(\mathfrak{a}_{2n}, \mathfrak{a}_{2n-1})\\ &= \xi\max\Big\{\mathcal{D}(\mathfrak{a}_{2n}, \mathfrak{a}_{2n-1}), \mathcal{D}(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+1}) - \mathcal{D}(\mathcal{I}, \mathcal{J}),\\ &\qquad \mathcal{D}(\mathfrak{a}_{2n-1}, \mathfrak{a}_{2n}) - \mathcal{D}(\mathcal{I}, \mathcal{J})\Big\}\\ &\leq \xi\mathcal{D}(\mathfrak{a}_{2n}, \mathfrak{a}_{2n-1}).\end{aligned} \tag{2.2}$$

So $\{\mathcal{D}(\mathfrak{a}_{2n-2}, \mathfrak{a}_{2n-1}\}$ is a decreasing sequence. Consider $\mathcal{D}(\mathfrak{a}_{2n-2}, \mathfrak{a}_{2n-1}) \to \mathfrak{u}$. Since for all $n = 1, 2, \ldots, \mathcal{D}(\mathcal{I}, \mathcal{J}) \leq \mathcal{D}(\mathfrak{a}_{2n-2}, \mathfrak{a}_{2n-1})$, we have $\mathcal{D}(\mathfrak{a}_{2n-2}, \mathfrak{a}_{2n-1}) \to \mathcal{D}(\mathcal{I}, \mathcal{J})$.

Now, suppose $\{\mathfrak{a}_{2n_j}\}$ is a subsequence of $\{\mathfrak{a}_{2n}\}$ converging to $\mathfrak{a}^* \in A$. Then

$$\mathcal{D}(\mathcal{I}, \mathcal{J}) \leq \mathcal{D}(\mathfrak{a}^*, \mathfrak{a}_{2n_j-1}) \leq \mathcal{D}(\mathfrak{a}^*, \mathfrak{a}_{2n_j}) + \mathcal{D}(\mathfrak{a}_{2n_j}, \mathfrak{a}_{2n_j-1}).$$

Now, taking limit, we get $\lim\limits_{j\to\infty} \mathcal{D}(\mathfrak{a}^*, \mathfrak{a}_{2n_j-1}) = \mathcal{D}(\mathcal{I}, \mathcal{J})$. As $\mathcal{G}^2$ preserves the edges of $\mathcal{K}$ and $\mathcal{K}$ is a C-graph, $(\mathfrak{a}_{2n_j}, \mathfrak{a}^*) \in \mathcal{E}(\mathcal{K})$ for all $j \in \mathbb{N}$. Using

(2.1), we obtain

$$\begin{aligned}\mathcal{D}(\mathfrak{a}_{2n_j+1},\mathcal{G}\mathfrak{a}^*) &= \mathcal{D}(\mathcal{G}\mathfrak{a}^*,\mathcal{G}^2\mathfrak{a}_{2n_j-1})\\ &\le \xi\mathcal{N}^*_{\mathcal{G}}(\mathfrak{a}^*,\mathfrak{a}_{2n_j})\\ &\le \xi\max\Big\{\mathcal{D}(\mathfrak{a}^*,\mathfrak{a}_{2n_j}),\mathcal{D}(\mathfrak{a}^*,\mathcal{G}\mathfrak{a}^*)-\mathcal{D}(\mathcal{I},\mathcal{J}),\\ &\quad \mathcal{D}(\mathfrak{a}_{2n_j},\mathfrak{a}_{2n_j+1})-\mathcal{D}(\mathcal{I},\mathcal{J})\Big\}.\end{aligned}$$

Now, we face three cases:

i) Getting $\max = \mathcal{D}(\mathfrak{a}^*,\mathfrak{a}_{2n_j})$, we have $\mathcal{D}(\mathfrak{a}_{2n_j+1},\mathcal{G}\mathfrak{a}^*)\le\xi\mathcal{D}(\mathfrak{a}^*,\mathfrak{a}_{2n_j})$.

ii) Getting $\max = \mathcal{D}(\mathfrak{a}^*,\mathcal{G}\mathfrak{a}^*)-\mathcal{D}(\mathcal{I},\mathcal{J})$, we have

$$\mathcal{D}(\mathfrak{a}_{2n_j+1},\mathcal{G}\mathfrak{a}^*)\le\xi\left(\mathcal{D}(\mathfrak{a}^*,\mathfrak{a}_{2n_k})+\mathcal{D}(\mathfrak{a}_{2n_j},\mathfrak{a}_{2n_j+1})-\mathcal{D}(\mathcal{I},\mathcal{J})+\mathcal{D}(\mathfrak{a}_{2n_j+1},\mathcal{G}\mathfrak{a}^*)\right).$$

Therefore,

$$\mathcal{D}(\mathfrak{a}_{2n_j+1},\mathcal{G}\mathfrak{a}^*)\le\frac{\xi}{1-\xi}\left(\mathcal{D}(\mathfrak{a}^*,\mathfrak{a}_{2n_j})+\mathcal{D}(\mathfrak{a}_{2n_j},\mathfrak{a}_{2n_j+1})-\mathcal{D}(\mathcal{I},\mathcal{J})\right).$$

iii) Getting $\max = \mathcal{D}(\mathfrak{a}_{2n_j},\mathfrak{a}_{2n_j+1})-\mathcal{D}(\mathcal{I},\mathcal{J})$, we have

$$\mathcal{D}(\mathfrak{a}_{2n_j+1},\mathcal{G}\mathfrak{a}^*)\le\xi\left(\mathcal{D}(\mathfrak{a}_{2n_j},\mathfrak{a}_{2n_j+1})-\mathcal{D}(\mathcal{I},\mathcal{J})\right).$$

As one case happens, $\{\mathfrak{a}_{2n_j+1}\}$ possesses a subsequence converging to $\mathcal{G}\mathfrak{a}^*$, which concludes

$$\begin{aligned}\mathcal{D}(\mathfrak{a}^*,\mathcal{G}\mathfrak{a}^*) &= \lim_{j\to\infty}\mathcal{D}(\mathfrak{a}_{2n_j},\mathfrak{a}_{2n_j+1})\\ &= \mathcal{D}(\mathcal{I},\mathcal{J}).\end{aligned}$$ □

Theorem 2.3. *Assume $(\mathcal{Y},\mathcal{D})$ is a GMS, $\mathcal{I}$ is complete and $(\mathcal{J},\mathcal{I})$ satisfies the property UC. In addition, assume $\mathcal{G}:\mathcal{I}\cup\mathcal{J}\to\mathcal{I}\cup\mathcal{J}$ is a cyclic $\mathcal{K}$-quasi-contractions on $\mathcal{I}$ (and $\mathcal{J}$) in which $\mathcal{G}$ and $\mathcal{G}^2$ preserve the edges of $\mathcal{K}$ on $\mathcal{I}$. If $\mathcal{G}$ is orbitally $\mathcal{K}$-continuous on $\mathcal{I}$ or $\mathcal{K}$ is a* C*-graph on $\mathcal{I}$, $\mathcal{G}$ has a bpp $\mathfrak{a}^*\in\mathcal{I}$ whenever there is $\mathfrak{a}_0\in\mathcal{I}$ with $\mathfrak{a}_0\in C_{\mathcal{G}}$.*

Proof. Assume $\mathfrak{a}_0\in C_{\mathcal{G}}$ with $\mathfrak{a}_0\in A$. Since $\mathcal{G}$ and $\mathcal{G}^2$ preserve the edges of $\mathcal{K}$ on $\mathcal{I}$ and $(\mathfrak{a}_0,\mathcal{G}^2\mathfrak{a}_0)\in\mathcal{E}(\mathcal{K})$ on $\mathcal{A}$, we have $(\mathfrak{a}_{2n},\mathfrak{a}_{2n+2})\in\mathcal{E}(\mathcal{K})$ and $(\mathfrak{a}_{2n+1},\mathfrak{a}_{2n+3})\in\mathcal{E}(\mathcal{K})$ for $n=0,1,\dots$.

As similar proof is done in Theorem 2.2, we have $\mathcal{D}(\mathfrak{a}_{2n},\mathfrak{a}_{2n+1})\to\mathcal{D}(\mathcal{I},\mathcal{J})$ and $\mathcal{D}(\mathfrak{a}_{2n+2},\mathfrak{a}_{2n+1})\to\mathcal{D}(\mathcal{I},\mathcal{J})$. From the property UC for $(\mathcal{I},\mathcal{J})$, we obtain $\mathcal{D}(\mathfrak{a}_{2n},\mathfrak{a}_{2n+2})\to 0$. Also, as $(\mathcal{J},\mathcal{I})$ has the property UC we conclude that $\mathcal{D}(\mathfrak{a}_{2n+1},\mathfrak{a}_{2n+3})\to 0$. We show that for all $\eta>0$, there is a $n\in\mathbb{N}$ so that

$$\mathcal{D}^*(\mathfrak{a}_{2m},\mathfrak{a}_{2n+1})<\eta, \tag{2.3}$$

for every $m>n\ge N$, where $\mathcal{D}^*(\mathfrak{a},\mathfrak{b})=\mathcal{D}(\mathfrak{a},\mathfrak{b})-\mathcal{D}(\mathcal{I},\mathcal{J})$ for all $(\mathfrak{a},\mathfrak{b})\in\mathcal{I}\times\mathcal{J}$. To contrary, assume there is $\eta_0>0$ such that for each $j \geq 1$, there is $m_j > n_j \geq j$ satisfying $\mathcal{D}^*(\mathfrak{a}_{2m_j}, \mathfrak{a}_{2n_j+1}) \geq \eta_0$ and $\mathcal{D}^*(\mathfrak{a}_{2m_j-2}, \mathfrak{a}_{2n_j+1}) < \eta_0$. Then

$$\begin{aligned}\eta_0 \leq \mathcal{D}^*(\mathfrak{a}_{2m_j}, \mathfrak{a}_{2n_j+1}) &\leq \mathcal{D}(\mathfrak{a}_{2m_j-2}, \mathfrak{a}_{2m_j}) + \mathcal{D}^*(\mathfrak{a}_{2m_j-2}, \mathfrak{a}_{2n_j+1})\\ &\leq \mathcal{D}(\mathfrak{a}_{2m_j-2}, \mathfrak{a}_{2m_j}) + \eta_0,\end{aligned}$$

so $\lim_{k\to\infty} \mathcal{D}^*(\mathfrak{a}_{2m_j}, \mathfrak{a}_{2n_j+1}) = \eta_0$. Since $\mathcal{G}$ and $\mathcal{G}^2$ preserve the edges of $\mathcal{K}$ on $\mathcal{I}$,

(2.4)

$$\begin{aligned}
&\lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2m_j+2}, \mathfrak{a}_{2n_j+3})\\
&\quad = \lim_{j\to\infty} \mathcal{D}\left(\mathcal{G}(\mathfrak{a}_{2m_j+1}), \mathcal{G}^2(\mathfrak{a}_{2n_j+1})\right)\\
&\quad \leq \xi \lim_{j\to\infty} \mathcal{N}_{\mathcal{G}}^*\mathcal{D}(\mathfrak{a}_{2m_j+1}, \mathfrak{a}_{2n_j+2})\\
&\quad \leq \xi \lim_{j\to\infty} \max\Bigg\{\mathcal{D}(\mathfrak{a}_{2m_j+1}, \mathfrak{a}_{2n_j+2}), \underbrace{\mathcal{D}(\mathfrak{a}_{2m_j+1}, \mathfrak{a}_{2m_j+2}) - \mathcal{D}(\mathcal{I}, \mathcal{J})}_{=0},\\
&\qquad \underbrace{\mathcal{D}(\mathfrak{a}_{2n_j+2}, \mathfrak{a}_{2n_j+3}) - \mathcal{D}(\mathcal{I}, \mathcal{J})}_{=0}\Bigg\}\\
&\quad \leq \xi \lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2m_j+1}, \mathfrak{a}_{2n_j+2})\\
&\quad \leq \lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2m_j+1}, \mathfrak{a}_{2n_j+2})\\
&\quad = \lim_{j\to\infty} \mathcal{D}\left(\mathcal{G}(\mathfrak{a}_{2m_j}), \mathcal{G}^2(\mathfrak{a}_{2n_j})\right)\\
&\quad \leq \xi \lim_{j\to\infty} \cdot\mathcal{N}_{\mathcal{G}}^*\mathcal{D}(\mathfrak{a}_{2m_j}, \mathfrak{a}_{2n_j+1})\\
&\quad = \xi \lim_{j\to\infty} \max\Bigg\{\mathcal{D}(\mathfrak{a}_{2m_j}, \mathfrak{a}_{2n_j+1}), \underbrace{\mathcal{D}(\mathfrak{a}_{2m_j}, \mathfrak{a}_{2m_j+1}) - \mathcal{D}(\mathcal{I}, \mathcal{J})}_{=0},\\
&\qquad \underbrace{\mathcal{D}(\mathfrak{a}_{2n_j+1}, \mathfrak{a}_{2n_j+2}) - \mathcal{D}(\mathcal{I}, \mathcal{J})}_{=0}\Bigg\}\\
&\quad = \xi \lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2m_j}, \mathfrak{a}_{2n_j+1})\\
&\quad \leq \lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2m_j}, \mathfrak{a}_{2n_j+1}).
\end{aligned}$$

We get $\lim_{j\to\infty}\mathcal{D}(\mathfrak{a}_{2m_j+2},\mathfrak{a}_{2n_j+3}) \le \lim_{j\to\infty}\mathcal{D}(\mathfrak{a}_{2m_j},\mathfrak{a}_{2n_j+1})$, and so by (2.4), we obtain

$$\begin{aligned}
\underbrace{\lim_{j\to\infty}\mathcal{D}^*(\mathfrak{a}_{2m_j},\mathfrak{a}_{2n_j+1})}_{=\eta_0} &\le \underbrace{\lim_{j\to\infty}\mathcal{D}(\mathfrak{a}_{2m_j},\mathfrak{a}_{2m_j+2})}_{=0} + \lim_{j\to\infty}\mathcal{D}^*(\mathfrak{a}_{2m_j+2},\mathfrak{a}_{2n_j+3})\\
&\quad + \underbrace{\lim_{j\to\infty}\mathcal{D}(\mathfrak{a}_{2n_j+1},\mathfrak{a}_{2n_j+3})}_{=0}\\
&\le \underbrace{\lim_{j\to\infty}\mathcal{D}(\mathfrak{a}_{2m_j},\mathfrak{a}_{2m_j+2})}_{=0} + \underbrace{\lim_{j\to\infty}\mathcal{D}^*(\mathfrak{a}_{2m_j},\mathfrak{a}_{2n_j+1})}_{\eta_0}\\
&\quad + \underbrace{\lim_{j\to\infty}\mathcal{D}(\mathfrak{a}_{2n_j+1},\mathfrak{a}_{2n_j+3})}_{=0}.
\end{aligned}$$

This implies $\lim_{j\to\infty}\mathcal{D}^*(\mathfrak{a}_{2m_j+2},\mathfrak{a}_{2n_j+3}) = \eta_0$ and $\lim_{j\to\infty}\mathcal{D}(\mathfrak{a}_{2m_j+2},\mathfrak{a}_{2n_j+3}) = \eta_0 + \mathcal{D}(\mathcal{I},\mathcal{J})$. Now,

$$\begin{aligned}
\eta_0 + \mathcal{D}(\mathcal{I},\mathcal{J}) &= \lim_{j\to\infty}\mathcal{D}(\mathfrak{a}_{2m_j+2},\mathfrak{a}_{2n_j+3})\\
&= \lim_{j\to\infty}\mathcal{D}\left(\mathcal{G}(\mathfrak{a}_{2m_j+1}),\mathcal{G}^2(\mathfrak{a}_{2n_j+1})\right)\\
&\le \xi\lim_{j\to\infty}\mathcal{N}^*_{\mathcal{G}}\mathcal{D}(\mathfrak{a}_{2m_j+1},\mathfrak{a}_{2n_j+2})\\
&\le \xi\lim_{j\to\infty}\max\Bigg\{\mathcal{D}(\mathfrak{a}_{2m_j+1},\mathfrak{a}_{2n_j+2}),\underbrace{\mathcal{D}(\mathfrak{a}_{2m_j+1},\mathfrak{a}_{2m_j+2})-\mathcal{D}(\mathcal{I},\mathcal{J})}_{=0},\\
&\qquad \underbrace{\mathcal{D}(\mathfrak{a}_{2n_j+2},\mathfrak{a}_{2n_j+3})-\mathcal{D}(\mathcal{I},\mathcal{J})}_{=0}\Bigg\}\\
\xi &\le \xi\lim_{j\to\infty}\mathcal{D}(\mathfrak{a}_{2m_j+1},\mathfrak{a}_{2n_j+2})\\
\xi &\le \lim_{j\to\infty}\mathcal{D}(\mathfrak{a}_{2m_j+1},\mathfrak{a}_{2n_j+2})\\
&= \xi\lim_{j\to\infty}\mathcal{D}\left(\mathcal{G}(\mathfrak{a}_{2m_j}),\mathcal{G}^2(\mathfrak{a}_{2n_j})\right)\\
&\le \xi^2\lim_{j\to\infty}\mathcal{N}^*_{\mathcal{G}}\mathcal{D}(\mathfrak{a}_{2m_j},\mathfrak{a}_{2n_j+1})\\
&= \xi^2\Bigg(\lim_{j\to\infty}\max\Bigg\{\mathcal{D}(\mathfrak{a}_{2m_j},\mathfrak{a}_{2n_j+1}),\underbrace{\mathcal{D}(\mathfrak{a}_{2m_j},\mathfrak{a}_{2m_j+1})-\mathcal{D}(\mathcal{I},\mathcal{J})}_{=0},\\
&\qquad \underbrace{\mathcal{D}(\mathfrak{a}_{2n_j+1},\mathfrak{a}_{2n_j+2})-\mathcal{D}(\mathcal{I},\mathcal{J})}_{=0}\Bigg\}\Bigg)
\end{aligned}$$

$$= \xi^2 \lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2m_j}, \mathfrak{a}_{2n_j+1})$$
$$\leq \xi^2 \lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2m_j}, \mathfrak{a}_{2n_j+1})$$
$$= \xi^2 \left(\eta_0 + \mathcal{D}(\mathcal{I}, \mathcal{J})\right),$$

concluding $\eta_0 + \mathcal{D}(\mathcal{I}, \mathcal{J}) \leq \xi^2 (\eta_0 + \mathcal{D}(\mathcal{I}, \mathcal{J}))$, which is impossible as $\xi \in [0, 1)$, so (2.3) holds and

$$\lim_{m\to\infty} \sup_{n\geq m} \mathcal{D}^*(\mathfrak{a}_{2m}, \mathfrak{a}_{2n+1}) = 0.$$

Since $(\mathcal{I}, \mathcal{J})$ has the property UC and by Lemma 1.4, $\{\mathfrak{a}_{2n}\}$ is a Cauchy sequence in $\mathcal{I}$. Because $\mathcal{I}$ is complete, $\{\mathfrak{a}_{2n}\}$ converges to some point $\mathfrak{a}^* \in \mathcal{I}$.

To continue, note first that from $\mathfrak{a} \in C_{\mathcal{G}}$, we get $(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+1}) \in \mathcal{E}(\mathcal{K})$ for every $n \in \mathbb{N}$. When $\mathcal{G}$ is orbitally $\mathcal{K}$-continuous on $\mathcal{I}$, $\mathfrak{a}_{2n} \to \mathfrak{a}^*$ implies $\mathcal{G}(\mathfrak{a}_{2n}) \to \mathcal{G}\mathfrak{a}^*$. Thus,

$$\begin{aligned}\mathcal{D}(\mathfrak{a}^*, \mathcal{G}\mathfrak{a}^*) &= \lim_{n\to\infty} \mathcal{D}(\mathfrak{a}_{2n}, \mathfrak{a}_{2n+1}) \\ &= \mathcal{D}(\mathcal{I}, \mathcal{J}),\end{aligned}$$

i.e. $\mathfrak{a}^*$ is a bpp. Second, let $\mathcal{K}$ be a C-graph. Since $\mathfrak{a}_{2n} \to \mathfrak{a}^*$, there is a strictly increasing sequence $\{n_j\}$ of positive integers such that $(\mathfrak{a}_{2n_j}, \mathfrak{a}^*) \in \mathcal{E}(\mathcal{K})$ for all $k \in \mathbb{N}$. As $\mathcal{G}$ satisfies (2.1) for the graph $\mathcal{K}$, we get

$$\begin{aligned}\lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2n_j+1}, \mathcal{G}\mathfrak{a}^*) &\leq \lim_{j\to\infty} \mathcal{D}(\mathcal{G}\mathfrak{a}^*, \mathcal{G}^2\mathfrak{a}_{2n_j-1}) \\ &\leq \xi \lim_{j\to\infty} \mathcal{N}^*_{\mathcal{G}}(\mathfrak{a}^*, \mathfrak{a}_{2n_j}) \\ &= \xi \lim_{j\to\infty} \max \Big\{ \mathcal{D}(\mathfrak{a}^*, \mathfrak{a}_{2n_j}), \mathcal{D}(\mathfrak{a}^*, \mathcal{G}\mathfrak{a}^*) - \mathcal{D}(\mathcal{I}, \mathcal{J}), \\ &\qquad \mathcal{D}(\mathfrak{a}_{2n_j}, \mathfrak{a}_{2n_j+1}) - \mathcal{D}(\mathcal{I}, \mathcal{J}) \Big\}.\end{aligned}$$

Again, we face three cases.

i) When maximum equals to $\mathcal{D}(\mathfrak{a}^*, \mathfrak{a}_{2n_j})$, we have

$$\lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2n_j+1}, \mathcal{G}\mathfrak{a}^*) \leq \xi \lim_{j\to\infty} \mathcal{D}(\mathfrak{a}^*, \mathfrak{a}_{2n_j}).$$

ii) When maximum equals to $\mathcal{D}(\mathfrak{a}^*, \mathcal{G}\mathfrak{a}^*) - \mathcal{D}(\mathcal{I}, \mathcal{J})$, have

$$\begin{aligned}\lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2n_j+1}, \mathcal{G}\mathfrak{a}^*) \leq \xi \lim_{j\to\infty} \mathcal{D}(\mathfrak{a}^*, \mathfrak{a}_{2n_j}) &+ \mathcal{D}(\mathfrak{a}_{2n_j}, \mathfrak{a}_{2n_j+1}) \\ &- \mathcal{D}(\mathcal{I}, \mathcal{J}) + \mathcal{D}\left(\mathfrak{a}_{2n_j+1}, \mathcal{G}\mathfrak{a}^*\right).\end{aligned}$$

Therefore,

$$\lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2n_j+1}, \mathcal{G}\mathfrak{a}^*) \leq \frac{\xi}{1-\xi} \left(\lim_{j\to\infty} \mathcal{D}(\mathfrak{a}^*, \mathfrak{a}_{2n_j}) + \mathcal{D}(\mathfrak{a}_{2n_j}, \mathfrak{a}_{2n_j+1}) - \mathcal{D}(\mathcal{I}, \mathcal{J}) \right).$$

iii) When maximum equals to the third term, we have

$$\lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2n_j+1}, \mathcal{G}\mathfrak{a}^*) \le \xi \left(\lim_{j\to\infty} \mathcal{D}(\mathfrak{a}_{2n_j}, \mathfrak{a}_{2n_j+1}) - \mathcal{D}(\mathcal{I}, \mathcal{J}) \right).$$

Evidently, at least one of three cases mentioned above will happen for infinitely many indices j. Thus, the sequence $\{\mathfrak{a}_{2n_j+1}\}$ has a subsequence converging to $\mathcal{G}\mathfrak{a}^*$. This implies that

$$\begin{aligned} \mathcal{D}(\mathfrak{a}^*, \mathcal{G}\mathfrak{a}^*) &= \lim_{n\to\infty} \mathcal{D}(\mathfrak{a}_{2n_j}, \mathfrak{a}_{2n_j+1}) \\ &= \mathcal{D}(\mathcal{I}, \mathcal{J}). \end{aligned}$$ □

Example 2.4. Take $\mathcal{Y} = \mathbb{R}^2$ and usual metric

$$\mathcal{D}\left((\mathfrak{a}_1, \mathfrak{b}_1), (\mathfrak{a}_2, \mathfrak{b}_2)\right) = \sqrt{(\mathfrak{a}_1 - \mathfrak{a}_2)^2 + (\mathfrak{b}_1 - \mathfrak{b}_2)^2}$$

for $(\mathfrak{a}_1, \mathfrak{b}_1), (\mathfrak{a}_2, \mathfrak{b}_2) \in \mathbb{R}^2$ and set

$$\mathcal{I} = \{(\mathfrak{a}, 1) : \mathfrak{a} \in [0,1]\}, \qquad \mathcal{J} = \{(\mathfrak{b}, 0) : \mathfrak{b} \in [0,1]\}.$$

Also, define $\mathcal{G} : \mathcal{I} \cup \mathcal{J} \to \mathcal{I} \cup \mathcal{J}$ by

$$\mathcal{G}(\mathfrak{a}, 1) = \begin{cases} (0,0), & 0 \le \mathfrak{a} < 1 \\ \left(\dfrac{2}{3}, 0\right), & \mathfrak{a} = 1 \end{cases}$$

for $(\mathfrak{a}, 1) \in \mathcal{I}$ and

$$\mathcal{G}(\mathfrak{b}, 0) = \begin{cases} (0,1), & 0 \le \mathfrak{b} < 1 \\ \left(\dfrac{2}{3}, 1\right), & \mathfrak{b} = 1 \end{cases}$$

for $(\mathfrak{b}, 0) \in \mathcal{J}$. Note that for $(1,1), (\frac{1}{2}, 1) \in \mathbb{R}^2$, we have

$$\mathcal{N}^*_{\mathcal{G}}\left((1,1), \left(\frac{1}{2}, 1\right)\right) = \frac{1}{2}$$

and again, by (2.1), we have

$$\mathcal{D}\left(\mathcal{G}(1,1), \mathcal{G}\left(\frac{1}{2}, 1\right)\right) > \xi \cdot \mathcal{N}^*_{\mathcal{G}}\left((1,1), \left(\frac{1}{2}, 1\right)\right).$$

Consequently, (2.1) is not true for the mapping $\mathcal{G}$ when we take a usual metric (non a GMS) on $\mathcal{I}$.

Now, take a graph $\mathcal{K}$ by $\mathsf{V}(\mathcal{K}) = \mathbb{R}^2$ and

$$\begin{aligned} \mathcal{E}(\mathcal{K}) = \big\{ ((\mathfrak{a}_1, \mathfrak{a}_2), (\mathfrak{a}_1, \mathfrak{a}_2)) : (\mathfrak{a}_1, \mathfrak{a}_2) \in \mathbb{R}^2 \big\} \cup \big\{ ((0,1), (1,1)), \\ ((1,1), (0,1)), ((0,0), (1,0)), ((1,0), (0,0)) \big\}. \end{aligned}$$

Then $(\mathbb{R}^2, \mathcal{D})$ is a complete GMS endowed by $\mathcal{K}$. Evidently, $\mathcal{G}$ is orbitally $\mathcal{K}$-continuous. Also, it is clear for $\mathfrak{a}, \mathfrak{b} \in [0,1]$ that

$$\mathcal{D}\left(\mathcal{G}(\mathfrak{a},1), \mathcal{G}(\mathfrak{a},1)\right) = 0 \leq \xi \cdot \mathcal{N}^*_{\mathcal{G}}\left((\mathfrak{b},0),(\mathfrak{b},0)\right)$$

and

$$\mathcal{D}\left(\mathcal{G}(\mathfrak{b},0), \mathcal{G}(\mathfrak{b},0)\right) = 0 \leq \xi \cdot \mathcal{N}^*_{\mathcal{G}}\left((\mathfrak{b},0),(\mathfrak{b},0)\right).$$

Moreover, $\mathcal{N}^*_{\mathcal{G}}\left((0,1),(1,1)\right) = 1$, $\mathcal{N}^*_{\mathcal{G}}\left((0,0),(1,0)\right) = 1$ and clearly,

$$\mathcal{D}\left(\mathcal{G}(0,1), \mathcal{G}(1,1)\right) \leq \xi \cdot \mathcal{N}^*_{\mathcal{G}}\left((0,1),(1,1)\right)$$

and

$$\mathcal{D}\left(\mathcal{G}(0,0), \mathcal{G}(1,0)\right) \leq \xi \cdot \mathcal{N}^*_{\mathcal{G}}\left((0,0),(1,0)\right)$$

where $\xi = \frac{3}{4}$. Thus, (2.1) is valid for the mapping $\mathcal{G}$ on $\mathcal{I}($ and $\mathcal{J})$. Therefore, all hypotheses of Theorem 2.3 fulfill and $\mathcal{G}$ has a bpp, being $\vartheta = (0,1)$ and $\gamma = (0,0)$.

Taking only the condition orbitally $\mathcal{K}$-continuity version of the mapping $\mathcal{G}$ from Theorem2.3, we can deduce some attractive corollaries. First, First, take $\mathcal{K} = \mathcal{K}_0$ in which $\mathcal{K}_0$ is a complete graph, i.e. $\mathcal{K}_0$ is a graph with $\mathsf{V}(\mathcal{K}_0) = \mathcal{Y}$ and $\mathcal{E}(\mathcal{K}_0) = \mathcal{Y} \times \mathcal{Y}$.

Corollary 2.5. *Let $(\mathcal{Y}, \mathcal{D})$ be a GMS, $\mathcal{I}$ be complete and $(\mathcal{I}, \mathcal{J})$ and $(\mathcal{J}, \mathcal{I})$ satisfy the property UC. Assume $\mathcal{G} : \mathcal{I} \cup \mathcal{J} \to \mathcal{I} \cup \mathcal{J}$ is a cyclic quasi-contractions on $\mathcal{I}$ (and $\mathcal{J}$). Then whenever $\mathcal{G}$ is continuous on $\mathcal{I}$, $\mathcal{G}$ has a bpp $\mathfrak{a}^* \in \mathcal{I}$.*

Second, presume $(\mathcal{Y}, \preceq)$ is a POS and $\mathcal{K}_1$ is a graph on $\mathcal{Y}$ in which $\mathsf{V}(\mathcal{K}_1) = \mathcal{Y}$ and $\mathcal{E}(\mathcal{K}_1) = \{(\mathfrak{a}, \mathfrak{b}) \in \mathcal{Y} \times \mathcal{Y} : \mathfrak{a} \preceq \mathfrak{b}\}$. If $\mathcal{K} = \mathcal{K}_1$ in Theorem 2.3, then we gain the second corollary.

Corollary 2.6. *Let $(\mathcal{Y}, \mathcal{D})$ be a Poset MS, $\mathcal{I}$ be complete and $(\mathcal{I}, \mathcal{J})$ and $(\mathcal{J}, \mathcal{I})$ satisfy the property UC. Assume that $\mathcal{G} : \mathcal{I} \cup \mathcal{J} \to \mathcal{I} \cup \mathcal{J}$ is a cyclic $\mathcal{K}_1$-quasi-contractions on $\mathcal{I}$ (and $\mathcal{J}$) such that $\mathcal{G}$ and $\mathcal{G}^2$ are nondecersing on $\mathcal{I}$.*

Then whenever $\mathcal{G}$ is orbitally $\mathcal{K}_1$-continuous on $\mathcal{I}$ or $\mathcal{K}_1$ is a C-graph on $\mathcal{I}$, $\mathcal{G}$ has a bpp $\mathfrak{a}^* \in \mathcal{I}$ if there exists $\mathfrak{a}_0 \in \mathcal{I}$ with $\mathfrak{a}_0 \in C_{\mathcal{G}}$.

For our next consequence, presume $(\mathcal{Y}, \preceq)$ is a POS and $\mathcal{K}_2$ is a graph on $\mathcal{Y}$ in which $\mathsf{V}(\mathcal{K}_2) = \mathcal{Y}$ and $\mathcal{E}(\mathcal{K}_2) = \{(\mathfrak{a}, \mathfrak{b}) \in \mathcal{Y} \times \mathcal{Y} : \mathfrak{a} \preceq \mathfrak{b} \vee \mathfrak{b} \preceq \mathfrak{a}\}$. If we set $\mathcal{K} = \mathcal{K}_2$ in Theorem 2.3, then the following version of our bpp theorem in metric spaces endowed with endowed with graph $\mathcal{K}_2$.

Corollary 2.7. *Presume $(\mathcal{Y}, \preceq)$ is POS, $(\mathcal{Y}, \mathcal{D})$ is an MS such that $\mathcal{I}$ is complete and let $(\mathcal{I}, \mathcal{J})$ and $(\mathcal{J}, \mathcal{I})$ satisfy the property UC. Let $\mathcal{G} : \mathcal{I} \cup \mathcal{J} \to \mathcal{I} \cup \mathcal{J}$ be a cyclic G_2-$(\varphi - \psi)$-contractions and for $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$, if $\mathfrak{a}$ and $\mathfrak{b}$ are comparable we have $\mathcal{G}^2\mathfrak{a}$ and $\mathcal{G}^2\mathfrak{b}$ are comparable. Then whenever $\mathcal{G}$ is orbitally $\mathcal{K}_2$-continuous on $\mathcal{I}$ or $\mathcal{K}_2$ is a C-graph on $\mathcal{I}$, $\mathcal{G}$ has a bpp $\mathfrak{a}^* \in \mathcal{I}$ if there exists $\mathfrak{a}_0 \in \mathcal{I}$ with $\mathfrak{a}_0 \in C_{\mathcal{G}}$.*

Finally, consider a fixed value $\eta > 0$. Recall that $\mathfrak{a}, \mathfrak{b} \in \mathcal{Y}$ are said to be η-close if $\mathcal{D}(\mathfrak{a}, \mathfrak{b}) \preceq \eta$. Taking $\mathcal{K}_\eta$ by $\mathsf{V}(\mathcal{K}_\eta) = \mathcal{Y}$ and $\mathcal{E}(\mathcal{K}_\eta) = \{(\mathfrak{a}, \mathfrak{b}) \in \mathcal{Y} \times \mathcal{Y} : \mathcal{D}(\mathfrak{a}, \mathfrak{b}) \preceq \eta\}$, we get the latest corollary of this section regarding $\mathcal{K} = \mathcal{K}_\eta$ in Theorem 2.2.

Corollary 2.8. *Let $(\mathcal{Y}, \mathcal{D})$ be a GMS endowed with graph $\mathcal{K}_\eta$ and $\mathcal{I}$ be complete. Assume $\mathcal{G} : \mathcal{I} \cup \mathcal{J} \to \mathcal{I} \cup \mathcal{J}$ is a cyclic $\mathcal{K}_\eta$-$(\varphi - \psi)$-contractions and if $\mathfrak{a}$ and $\mathfrak{b}$ are η-close for $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$, we have $\mathcal{G}^2\mathfrak{a}$ and $\mathcal{G}^2\mathfrak{b}$ are η-close. Then whenever $\mathcal{G}$ is orbitally $\mathcal{K}_\eta$-continuous on $\mathcal{I}$ or $\mathcal{K}_\eta$ is a C-graph on $\mathcal{I}$, $\mathcal{G}$ has a bpp $\mathfrak{a}^* \in \mathcal{I}$ if there exists $\mathfrak{a}_0 \in \mathcal{I}$ with $\mathfrak{a}_0 \in C_{\mathcal{G}}$.*

Using Theorem 2.3, we obtain the bpp result for former type of $\mathcal{K}$-contractions in a GMS.

Corollary 2.9. *Let $(\mathcal{Y}, \mathcal{D})$ be a GMS, $\mathcal{I}$ be complete and $(\mathcal{J}, \mathcal{I})$ satisfy the property UC. Assume $\mathcal{G} : \mathcal{I} \cup \mathcal{J} \to \mathcal{I} \cup \mathcal{J}$ is a cyclic mapping provided that*

(i) $\mathcal{G}$ is orbitally $\mathcal{K}$-continuous on $\mathcal{I}$ or $\mathcal{K}$ is a C-graph on $\mathcal{I}$;
(ii) $\mathcal{G}$ and $\mathcal{G}^2$ preserve the edges of $\mathcal{K}$ on $\mathcal{I}$ and there is $\mathfrak{a}_0 \in \mathcal{I}$ with $\mathfrak{a}_0 \in C_{\mathcal{G}}$.

If any of the following contractions holds for $\mathcal{G}$, then $\mathcal{G}$ has a bpp in $\mathcal{I}$.

(C_1) ***(cyclic Banach-type $\mathcal{K}$-contraction):*** *there exists an $\alpha \in (0, 1)$ so that*

$$\mathcal{D}(\mathcal{G}\mathfrak{a}, \mathcal{G}^2\mathfrak{b}) \leq \alpha \mathcal{D}(\mathfrak{a}, \mathcal{G}\mathfrak{b})$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ (and $\mathcal{J}$) with $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{E}(\mathcal{K})$.

(C_2) ***(cyclic Kannan-type $\mathcal{K}$-contraction):*** *there is an $\alpha \in \left(0, \frac{1}{2}\right)$ provided that*

$$\mathcal{D}(\mathcal{G}\mathfrak{a}, \mathcal{G}^2\mathfrak{b}) \leq \alpha \left(\mathcal{D}(\mathfrak{a}, \mathcal{G}\mathfrak{a}) + \mathcal{D}(\mathcal{G}\mathfrak{b}, \mathcal{G}^2\mathfrak{b})\right) - 2\alpha \mathcal{D}(\mathcal{I}, \mathcal{J})$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ (and $\mathcal{J}$) with $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{E}(\mathcal{K})$.

(C_3) ***(cyclic Ćirić-Reich-Rus-type $\mathcal{K}$-contraction)*** *there are $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$ provided that*

$$\mathcal{D}(\mathcal{G}\mathfrak{a}, \mathcal{G}^2\mathfrak{b}) \leq \alpha \mathcal{D}(\mathfrak{a}, \mathcal{G}\mathfrak{b}) + \beta \mathcal{D}(\mathfrak{a}, \mathcal{G}\mathfrak{a}) + \gamma \mathcal{D}(\mathcal{G}\mathfrak{b}, \mathcal{G}^2\mathfrak{b}) - (\beta + \gamma) \mathcal{D}(\mathcal{I}, \mathcal{J})$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ (and $\mathcal{J}$) with $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{E}(\mathcal{K})$.

(C_4) ***(cyclic λ-generalized $\mathcal{K}$-contraction in sense of Ćirić):*** *there are functions $\mathcal{Q}, \mathcal{R}, \mathcal{G} : \mathcal{Y} \times \mathcal{Y} \to [0, +\infty)$ with*

$$\sup\{\mathcal{Q}(\mathfrak{a}, \mathfrak{b}) + \mathcal{R}(\mathfrak{a}, \mathfrak{b}) + \mathcal{U}(\mathfrak{a}, \mathfrak{b}) : \mathfrak{a}, \mathfrak{b} \in \mathcal{Y} \times \mathcal{Y}\} = \lambda < 1$$

such that

$$\begin{aligned}\mathcal{D}(\mathcal{G}\mathfrak{a}, \mathcal{G}^b\mathfrak{b}) \leq \mathcal{Q}(\mathfrak{a}, \mathfrak{b}) \mathcal{D}(\mathfrak{a}, \mathcal{G}\mathfrak{b}) + \mathcal{R}(\mathfrak{a}, \mathfrak{b}) \mathcal{D}(\mathfrak{a}, \mathcal{G}\mathfrak{a}) + \mathcal{U}(\mathfrak{a}, \mathfrak{b}) \mathcal{D}(\mathcal{G}\mathfrak{b}, \mathcal{G}^2\mathfrak{b}) \\ - \left(\mathcal{Q}(\mathfrak{a}, \mathfrak{b}) + \mathcal{R}(\mathfrak{a}, \mathfrak{b}) + \mathcal{U}(\mathfrak{a}, \mathfrak{b})\right) \mathcal{D}(\mathcal{I}, \mathcal{J}).\end{aligned}$$

In addition, if $(\mathfrak{u},\mathfrak{v}) \in \mathcal{E}(\mathcal{K})$ *for any two bpp(s)* $\mathfrak{u},\mathfrak{v} \in \mathcal{I}$, *then* $\mathcal{G}$ *has a unique bpp in* $\mathcal{I}$.

Proof. If $\mathcal{G}$ satisfies (C_1), then there is an $\alpha \in (0,1)$ provided that

$$\mathcal{D}(\mathcal{G}\mathfrak{a},\mathcal{G}^2\mathfrak{b}) \leq \alpha\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{b})$$

for all $\mathfrak{a},\mathfrak{b} \in \mathcal{I}$ (and $\mathcal{J}$) with $(\mathfrak{a},\mathfrak{b}) \in \mathcal{E}(\mathcal{K})$. Consequently,

$$\begin{aligned}\mathcal{D}(\mathcal{G}\mathfrak{a},\mathcal{G}^2\mathfrak{b}) &\leq \alpha\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{b})\\ &\leq \alpha\max\Big\{\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{b}),\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{a})-\mathcal{D}(\mathcal{I},\mathcal{J}),\mathcal{D}(\mathcal{G}\mathfrak{b},\mathcal{G}^2\mathfrak{b})-\mathcal{D}(\mathcal{I},\mathcal{J})\Big\},\end{aligned}$$

which induces that $\mathcal{G}$ satisfies (2.1) and so $\mathcal{G}$ is a cyclic $\mathcal{K}$-quasi-contractions. In conclusion, every cyclic Banach-type $\mathcal{K}$-contraction is a $\mathcal{K}$-quasi-contractions.

If $\mathcal{G}$ satisfies (C_2), then there is an $\alpha \in \left(0,\frac{1}{2}\right)$ provided that

$$\mathcal{D}(\mathcal{G}\mathfrak{a},\mathcal{G}^2\mathfrak{b}) \leq \alpha\left(\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{a})+\mathcal{D}(\mathcal{G}\mathfrak{b},\mathcal{G}^2\mathfrak{b})\right)-2\alpha\mathcal{D}(\mathcal{I},\mathcal{J})$$

for all $\mathfrak{a},\mathfrak{b} \in \mathcal{I}$ (and $\mathcal{J}$) with $(\mathfrak{a},\mathfrak{b}) \in \mathcal{E}(\mathcal{K})$. Consequently,

$$\begin{aligned}\mathcal{D}(\mathcal{G}\mathfrak{a},\mathcal{G}^2\mathfrak{b}) &\leq \alpha(\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{a})+\mathcal{D}\left(\mathcal{G}\mathfrak{b},\mathcal{G}^2\mathfrak{b}\right))-2\alpha\mathcal{D}(\mathcal{I},\mathcal{J})\\ &\leq 2\alpha\max\left\{\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{a})-\mathcal{D}(\mathcal{I},\mathcal{J}),\mathcal{D}(\mathcal{G}\mathfrak{b},\mathcal{G}^2\mathfrak{b})-\mathcal{D}(\mathcal{I},\mathcal{J})\right\}\\ &\leq 2\alpha\max\left\{\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{b}),\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{a})-\mathcal{D}(\mathcal{I},\mathcal{J}),\mathcal{D}(\mathcal{G}\mathfrak{b},\mathcal{G}^2\mathfrak{b})-\mathcal{D}(\mathcal{I},\mathcal{J})\right\},\end{aligned}$$

which induces that $\mathcal{G}$ satisfies (2.1) and so $\mathcal{G}$ is a cyclic $\mathcal{K}$-quasi-contractions. In conclusion, every cyclic Kannan-type $\mathcal{K}$-contraction is a cyclic $\mathcal{K}$-quasi-contractions.

Assume (C_3) holds for $\mathcal{G}$ and $(\mathfrak{a},\mathfrak{b}) \in \mathcal{E}(\mathcal{K})$, similar to what appeared in condition (C_2), one can establish that

$$\begin{aligned}\mathcal{D}(\mathcal{G}\mathfrak{a},\mathcal{G}^2\mathfrak{b}) \leq (\alpha+\beta+\gamma)\max\Big\{&\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{b}),\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{a})-\mathcal{D}(\mathcal{I},\mathcal{J}),\\ &\mathcal{D}(\mathcal{G}\mathfrak{b},\mathcal{G}^2\mathfrak{b})-\mathcal{D}(\mathcal{I},\mathcal{J})\Big\},\end{aligned}$$

which implies that $\mathcal{G}$ satisfies (2.1) and thus, $\mathcal{G}$ is a $\mathcal{K}$-quasi-contractions. In conclusion, every Cirić-Reich-Rus-type $\mathcal{K}$-contraction is a $\mathcal{K}$-quasi-contractions.

Assume (C_4) holds for $\mathcal{G}$ and $(\mathfrak{a},\mathfrak{b}) \in \mathcal{E}(\mathcal{K})$, similar to what appeared in condition (C_2), one can establish that

$$\begin{aligned}\mathcal{D}(\mathcal{G}a,\mathcal{G}^2\mathfrak{b}) \leq (\mathcal{Q}(\mathfrak{a},\mathfrak{b})+\mathcal{R}(\mathfrak{a},\mathfrak{b})+\mathcal{U}(\mathfrak{a},\mathfrak{b}))\max\Big\{&\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{b}),\\ &\mathcal{D}(\mathfrak{a},\mathcal{G}\mathfrak{a})-\mathcal{D}(\mathcal{I},\mathcal{J}),\mathcal{D}(\mathcal{G}\mathfrak{b},\mathcal{G}^2\mathfrak{b})-\mathcal{D}(\mathcal{I},\mathcal{J})\Big\},\end{aligned}$$

which implies that $\mathcal{G}$ satisfies (2.1) and $\mathcal{G}$ is a cyclic $\mathcal{K}$-quasi-contractions. In conclusion, any cyclic λ-generalized $\mathcal{K}$-contraction is a cyclic $\mathcal{K}$-quasi-contractions. Thus, $\mathcal{G}$ is a cyclic $\mathcal{K}$-quasi-contractions in case $C_1,\ldots,C_4$ and it has a bpp in $\mathcal{I}$. □

Similarly, all the corollaries stated above hold for Theorem 2.2.

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[1] Department of Mathematics, Payame Noor University, P.O. Box 19395-4697, Tehran, Iran.

Email address: k.fallahi@pnu.ac.ir, fallahi1361@gmail.com