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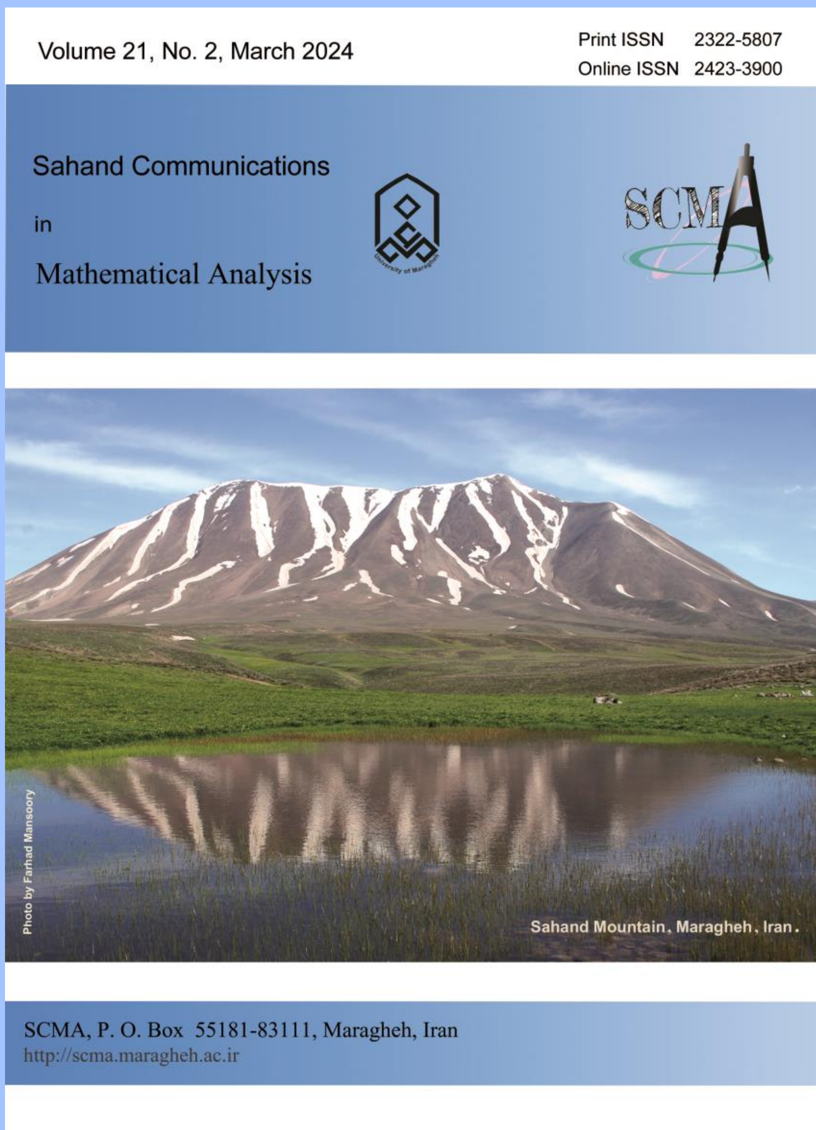

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$\mathcal{T}_M$-Amenability of Banach Algebras

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Abstract. We introduce the notions of $\mathcal{T}_M$-amenability and ϕ-$\mathcal{T}_M$-amenability. Then, we characterize ϕ-$\mathcal{T}_M$-amenability in terms of WAP-diagonals and ϕ-invariant means. Some concrete cases are also discussed.

1. Introduction

Amenable Banach algebras have since proved widely applicable in modern analysis. In many instances, the classical definition of *amenability* is, however, a too strong concept. For this reason, new concepts have been introduced by relaxing some of the constraints in the definition of amenability. In recent years, there has been considerable interest in harmonic analysis in the amenability of Banach algebras. In [16], Runde initiated and studied the notion of Connes amenability for dual Banach algebras. Let $\mathcal{A}$ be a Banach algebra, $\Delta(\mathcal{A})$ be the set of all homomorphisms from $\mathcal{A}$ onto $\mathbb{C}$ and $\phi \in \Delta(\mathcal{A})$. The concept of ϕ-amenability of Banach algebras was introduced recently by Kaniuth, Lau and Pym [8]. A linear functional $m \in \mathcal{A}^{**}$ a mean if $\langle m, \phi \rangle = 1$. A mean m is a ϕ-invariant mean if $\langle m, f.a \rangle = \phi(a)\langle m, f \rangle$ for all $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. A Banach algebra $\mathcal{A}$ is called ϕ-amenable if there exists a ϕ-invariant mean $m \in \mathcal{A}^{**}$ [8]. The Banach algebra $\mathcal{A}$ is called ϕ-contractible if an element $m \in \mathcal{A}$ satisfies $\phi(m) = 1$ and $am = \phi(a)m$ for all $a \in \mathcal{A}$. Hu, Monfared and Traynor introduced and studied these notions in [7]. Both of these concepts generalize the earlier concept of left amenability for F-algebras introduced by Lau. Recently, the authors in [4] have introduced the ϕ-version of Connes amenability of $\mathcal{A}$. In [5] we study Connes amenability

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of l^1-Munn algebras. We use the l^1-Munn algebras to investigate Connes amenability of semigroup algebras of weakly cancellative semigroups.

Let $\mathcal{A}$ be a Banach algebra. The Mackey topology for the dual pair $(\mathcal{A}^{**}, \mathcal{A}^*)$ is the topology of uniform convergence on sets $K \subseteq \mathcal{A}^*$ where K is absolutely convex and $\sigma(\mathcal{A}^*, \mathcal{A}^{**})$-compact [9]. We identify $\mathcal{A}$ with its canonical embedding in $\mathcal{A}^{**}$ and call the relative topology induced on $\mathcal{A}$ by $\mathcal{T}_M$. See [14], where this topology is called the quasi-Mackey topology and see [11], where this topology is called the right topology. It is known that the proper topology on a Banach space X coincides with the topology generated by the family of seminorms $\|x\|_T = \|T(x)\|$ that $x \in X$ and T is a bounded linear operator from X to a reflexive space. Also, in [12], the authors introduced the robust topology, that is, the topology generated by the family of seminorms $\|.\|_T$ where T is a bounded linear operator from X to a Hilbert space. In [1], it is shown that for Von Neumann algebras, the firm operator topology agrees with the Mackey topology on norm-bounded subsets. For more detailes, see [13], [3] and [12]. Recently, we and colleagues introduced and studied induced topology from $\mathcal{B}(L^1(G), L^\infty(G))$ on $L^\infty(H)$ for a hypergroup H [6].

In this paper, we define the ϕ-$\mathcal{T}_M$-amenability by vanishing of some cohomology groups. The Banach $\mathcal{A}$-bimodules E relevant to us are those where the module maps $a \to e.a$ and $a \to a.e$ are $\mathcal{T}_M$-norm-continuous. For each element $\phi \in \Delta(\mathcal{A})$, we introduce ϕ-$\mathcal{T}_M$-amenability of $\mathcal{A}$. We characterize ϕ-$\mathcal{T}_M$- amenability of certain Banach algebras in terms of $\mathcal{T}_M$-diagonals. We investigate the relation between the ϕ-$\mathcal{T}_M$-amenability and ϕ-amenability. For a certain Banach algebra $\mathcal{A}$, ϕ-$\mathcal{T}_M$-amenability is characterized by a ϕ-invariant mean $m \in \mathcal{A}$. We investigate the $\mathcal{T}_M$-amenability of Banach algebras, particulary those the algebras associated with a locally compact group G.

2. $\mathcal{T}_M$-Amenabinity of Banach Algebras

Definition 2.1. Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule E is called $\mathcal{T}_M$-bimodule if, for each $e \in E$, the maps $a \to e.a$ and $a \to a.e$ are $\mathcal{T}_M$-norm-continuous.

Definition 2.2. Let $\mathcal{A}$ be a Banach algebra. The algebra $\mathcal{A}$ is called $\mathcal{T}_M$-amenable if for each Banach $\mathcal{T}_M$-bimodule E, every $\mathcal{T}_M$-norm continuous derivation $D : \mathcal{A} \to E$ is inner.

It is known that a linear map T is $\mathcal{T}_M$-norm continuous if and only if T is weakly compact. Theorem 4 and Corollary 5 in [11] prove its proof. This allows us to introduce a quantified notion of definition 2.1:

Definition 2.3. Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule E is called $\mathcal{T}_M$-bimodule if, for each $e \in E$, the maps $a \to e.a$ and $a \to a.e$ are weakly compact.

Definition 2.4. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. An element $x \in E$ is called WAP if, for every $a \in \mathcal{A}$, $a \to a.x$ and $a \to x.a$ are weakly compact. The set of all WAP-elements of E is denoted by $WAP(E)$.

It is easy to see that a Banach algebra $\mathcal{A}$ is $\mathcal{T}_M$-amenable if, for evey Banach $\mathcal{A}$-bimodule E, every $\mathcal{T}_M$-norm continuous derivation $D : \mathcal{A} \to WAP(E)$ is inner. In the following, we study the ralation between $\mathcal{T}_M$-amenability and Johnson amenability. Recall that

$$WAP(\mathcal{A}^*) = \{f \in \mathcal{A}^*; a \to f.a \text{ is weakly compact}\}.$$

It is known that if $\mathcal{A}$ is Arens regular, then $WAP(\mathcal{A}^*) = \mathcal{A}^*$. If an Arens regular Banach algebra $\mathcal{A}$ is weakly amenable, then there is a Banach $\mathcal{A}$-bimodule E that each $\mathcal{T}_M$-norm continuous derivation $D : \mathcal{A} \to E$ is inner.

Theorem 2.5. *Let $\mathcal{A}$ be a Banach algebra with $WAP(\mathcal{A}) = \mathcal{A}$. If $\mathcal{A}$ is $\mathcal{T}_M$-amenable, $\mathcal{A}$ has an identity.*

Proof. Let $\mathcal{A}$ be the Banach $\mathcal{A}$-bimodule with the following module actions:

$$a.'x := a.x, \qquad x.a = 0.$$

Let $D : \mathcal{A} \to \mathcal{A}$ be the identity map. Since $WAP(\mathcal{A}) = \mathcal{A}$, $\mathcal{A}$ is a $\mathcal{T}_M$-bimodule and D is a $\mathcal{T}_M$-norm-continuous derivation. Now since D is inner, this follows that $\mathcal{A}$ is unital. □

Let G be a compact group. Then, $L^1(G)$ is amenable. It is known that if G is a compact group, then each element of the group algebra $L^1(G)$ is compact [2]. This means that $WAP(L^1(G)) = L^1(G)$. By Theorem 2.5, if $L^1(G)$ is $\mathcal{T}_M$-amenable, then $L^1(G)$ has an identity. Therefore, G is discrete and so it is finite. It means that there exist many amenable Banach algebras that are not $\mathcal{T}_M$-amenable. It is easy to see that if $\mathcal{A}$ is reflexive, then $\mathcal{T}_M$-topology is compatible with norm topology. Then, each reflexive Banach algebra $\mathcal{A}$ is contractible if and only if $\mathcal{A}$ is $\mathcal{T}_M$-amenable. However, Runde's theorem about Connes amenable Banach algebras does not hold for $\mathcal{T}_M$-amenable Banah algebras. It sounds like the concept of $\mathcal{T}_M$- amenability is too strong. In the next section, we consider the left module action on a $\mathcal{T}_M$-bimodule E with $a.e = \phi(a)e$ that ϕ is a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$ and we characterize ϕ-$\mathcal{T}_M$-amenability of certain Banach algebras through the existence of a ϕ-invariant mean.

3. ϕ-$\mathcal{T}_M$-Amenability of Banach Algebras

Let $\mathcal{A}$ be a Banach algebra. Banach $\mathcal{A}$-bimodule E is ϕ-bimodule if

$$a.e = \phi(a)e, \quad e.a = e.a \quad (a \in \mathcal{A}, e \in E).$$

If E is a Banach ϕ-bimodule, then the dual E^* is a Banach $\mathcal{A}$-bimodule with the module actions given by

$$\begin{aligned}\langle a.e^*, e\rangle &= \langle e^*, e.a\rangle \\ \langle e^*.a, e\rangle &= \langle e^*, a.e\rangle \\ &= \phi(a)\langle e^*, e\rangle \quad (a \in \mathcal{A}, e^* \in E^*, e \in E).\end{aligned}$$

Moreover, the seconde dual E^{**} is a ϕ-bimodule with the module actions given by

$$\begin{aligned}\langle a.e^{**}, e^*\rangle &= \langle e^{**}, e^*.a\rangle \\ &= \phi(a)\langle e^{**}, e^*\rangle \\ \langle e^{**}.a, e^*\rangle &= \langle e^{**}, a.e^*\rangle \quad (a \in \mathcal{A}, e^{**} \in E^{**}, e^* \in E^*).\end{aligned}$$

Lau and colleagues characterized ϕ-amenability through vanishing of the cohomology groups $\mathcal{H}(\mathcal{A}, E^*)$ for each Banach ϕ-bimodule E [8].

Definition 3.1. Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Banach $\mathcal{A}$-bimodule E is called ϕ-$\mathcal{T}_M$-bimodule if E is ϕ-bimodule and for each $e \in E$, the module map $a \to e.a$ is $\mathcal{T}_M$-norm-continuous.

Definition 3.2. Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. $\mathcal{A}$ is called ϕ-$\mathcal{T}_M$-amenable if, for each Banach ϕ-$\mathcal{T}_M$-bimodule E, every $\mathcal{T}_M$-norm continuous derivation $D : \mathcal{A} \to E$ is inner.

Theorem 3.3. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Let $\mathcal{A}$ be ϕ-amenable, then $\mathcal{A}$ is ϕ-$\mathcal{T}_M$-amenable*

Proof. Suppose that $\mathcal{A}$ is ϕ-amenable. Then there exists $m \in \mathcal{A}^{**}$ such that $\langle m, \phi\rangle = 1$ and $\langle m, f.a\rangle = \phi(a)\langle m, f\rangle$ for all $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Let E be a ϕ-$\mathcal{T}_M$-bimodule and $D : \mathcal{A} \to E$ be a $\mathcal{T}_M$-norm continuous derivation. By [11, Theorem 1] D^{**} is a $\mathcal{T}_M$-norm continuous map from $\mathcal{A}^{**}$ to E. Put $g = D^{**}(m)$. Then for all $a \in \mathcal{A}$ and $x \in E^*$

$$\begin{aligned}\langle x, a.g\rangle &= \langle x, a.D^{**}(m)\rangle \\ &= \langle D^*(x.a), m\rangle \\ &= \phi(a)\langle D^*(x), m\rangle \\ &= \phi(a)\langle x, g\rangle\end{aligned}$$

and hence $a.g = \phi(a)g$. Since D is a derivation, we get for the left action of $\mathcal{A}$ on E

$$\langle D^*(a.x), b\rangle = \langle a.x, D(b)\rangle$$

$$= \langle x, D(b).a\rangle$$
$$= \langle x, D(ba)\rangle - \langle x, b.D(a)\rangle$$
$$= \langle ba, D^*(x)\rangle - \phi(b)\langle x, D(a)\rangle$$

for all $a, b \in \mathcal{A}$ and $x \in E^*$. This implies that

$$D^*(a.x) = D^*(x).a - \langle x, D(a)\rangle\phi.$$

It followes that

$$\begin{aligned}\langle x, g.a\rangle &= \langle a.x, g\rangle \\ &= \langle a.x, D^{**}(m)\rangle \\ &= \langle m, D^*(a.x)\rangle \\ &= \langle m, D^*(x).a\rangle - \phi(m)\langle x, D(a)\rangle \\ &= \phi(a)\langle m, D^*(x)\rangle - \langle x, D(a)\rangle \\ &= \phi(a)\langle x, D^{**}(m)\rangle - \langle x, D(a)\rangle.\end{aligned}$$

This shows that $D(a) = \phi(a)g - g.a = a.g - g.a$. □

In general, not every norm-norm continuous operator is $\mathcal{T}_M$-norm continuous. Then, the converse of Theorem 3.3 is not valid. We show below that for introverted subalgebras, ϕ-$\mathcal{T}_M$-amenability and ϕ-amenability are equivalent. Let $\mathcal{A}$ be a Banach algebra and X be a closed $\mathcal{A}$-submodule of $\mathcal{A}^*$. Then X is left introverted (respectively, right introverted) if $f.x \in X$ (respectively, $x.f \in X$) for each $f \in X^*, x \in X$. Further, the submodule X is introverted if it is both left introverted and right introverted in $\mathcal{A}^*$.

Theorem 3.4. *Let $\mathcal{A}$ be a Banach algebra, $\mathcal{X}$ be an introverted subalgebra of $\mathcal{A}^*$ and $\phi \in \Delta(\mathcal{A})$. If $\mathcal{X}$ is ϕ-$\mathcal{T}_M$-amenable, then there exists an invariant ϕ-mean $m \in \mathcal{X}$.*

Proof. Let $\mathcal{X}$ be ϕ-$\mathcal{T}_M$-amenable. Consider $\mathcal{X}$ as an $\mathcal{A}$-bimodule whose underling space is $\mathcal{X}$, but on which $\mathcal{X}$ acts via

$$a.u = \phi(a)u, \quad u.a = ua \quad (a, u \in \mathcal{X}).$$

We show that $\mathcal{X}$ is $\mathcal{T}_M$-bimodule. In fact, let $a_\alpha \to a$ with $\mathcal{T}_M$-topology and fix $u \in \mathcal{X}$. We show that $u.a_\alpha \to u.a$ in the norm-topology. Since $a_\alpha \to a$ in the $\mathcal{T}_M$-topology, $a_\alpha \to a$ in the weak*-topology and so for each $f \in \mathcal{X}^*$, $\langle f.u, a_\alpha\rangle \to \langle f.u, a\rangle$. Let $K = \{f \in \mathcal{X}^* : \|f\| \leq 1\}$. By Banach-Alaoghlu theorem, K is a weak*- compact set of $\mathcal{X}^*$. Since $\mathcal{X}$ is introverted subalgebra, the module map $f \to f.u$ from $\mathcal{X}^*$ into $\mathcal{X}$ is weak*-weak-continuous. This implies that $K_u = \{f.u : f \in K\}$ is weakly compact. Then $a_\alpha \to a$ uniformly on weakly compact set K_u and so

$$\|u.a_\alpha - u.a\| = \sup\{\langle u.a_\alpha - u.a, f\rangle : f \in K\}$$

$$\begin{aligned} &= \sup\{\langle a_\alpha - a, f.u\rangle : f \in K\} \\ &= \sup\{\langle a_\alpha - a, g\rangle : g \in K_u\} \\ &= 0. \end{aligned}$$

Then $u.a_\alpha \to u.a$ in the norm-topology. Now consider $a_0 \in \mathcal{X}$ such that $\phi(a_0) = 1$ and define

$$D : \mathcal{X} \to \mathcal{X}, \quad a \to a.a_0 - \phi(a)a_0.$$

From the above argument, D is a $\mathcal{T}_M$-norm continuous derivation on $\mathcal{X}$. D attains its values in the $\mathcal{T}_M$-closed submodule $ker(\phi)$. Thus, there exists $n \in \mathcal{X}$ such that $D(a) = a.n - \phi(a).n$ for all $n \in \mathcal{X}$. Put $m = n - a_0$. We obtain an invariant ϕ-mean $m \in \mathcal{X}$. □

Let G be a locally compact group. Let $L^\infty(G)$ be the set of all locally measurable functions that are bounded except on a locally null set, modulo functions that are zero locally a.e., let $L_0^\infty(G)$ be the subspace of $L^\infty(G)$ consisting of all functions $f \in L^\infty(G)$ that vanish at infinity and let $M(G)$ be the Banach space of regular complex Borel measures on G. Write $C(G)$ for the space of all continuous complex-valued functions on G. We consider the subspace $C_b(G)$ consisting of bounded continuous functions on G. We say that a continuous function $f \in C_b(G)$ is left uniformly continuous (respectively, right uniformly continuous) if the map $x \mapsto L_x f (respectively, x \mapsto R_x f)$ is continuous in the uniform norm. We denote the collection of all left uniformly continuous functions (respectively, right uniformly continuous functions) on G by $LUC(G)(respectively, RUC(G))$. We call a function $f \in L^\infty(G)$, weakly almost periodic (respectively, almost periodic) if the set $\{L_x f : x \in G\}$ of translates of G is relatively weakly compact (respectively, compact). The spaces of these functionals are denoted by $WAP(G)$ and $AP(G)$, respecively. The spaces $Luc(G)$, $Ruc(G)$, $AP(G)$, $WAP(G)$ and $L_0^\infty(G)$ are introverted subalgebras of $L^\infty(G)$ [10].

Theorem 3.5. *Let G be a locally compact group and $\mathcal{X}$ be each of subspaces $AP(G)$, $Luc(G)$, $Ruc(G)$, $WAP(G)$, $L_0^\infty(G)$ and $\phi \in \Delta(\mathcal{X})$. Then the following are equivalent:*

(i) *$\mathcal{X}$ is ϕ-$\mathcal{T}_M$-amenable.*
(ii) *$\mathcal{X}$ is ϕ-amenable.*
(iii) *$\mathcal{X}$ is ϕ-contractible.*

Proof. This is a consequence of Theorem 3.3 and Theorem 3.4. □

Theorem 3.6. *Let G be a compact group and $\phi \in \Delta(M(G))$. Then Banach algebra $M(G)$ is ϕ-$\mathcal{T}_M$-amenable if and only if $M(G)$ is ϕ-contractible.*

Proof. Let $M(G)$ be ϕ-$\mathcal{T}_M$-amenable. Since G is compact, each element of $M(G)$ is weakly compact [2]. This means that for each $g \in M(G)$, $f \to f * g$ is weakly compact and then $f \to f * g$ is $\mathcal{T}_M$-norm-continuous. Now we consider $M(G)$ as ϕ-$\mathcal{T}_M$-bimodule whose underling space is $M(G)$ with the following action

$$g.f = \phi(f)g, \quad f.g = f * g.$$

Fix $g \in M(G)$ such that $\phi(g) = 1$ and consider $D : M(G) \to M(G)$ with $D(f) = f.g - \phi(f)g$. Note that D is a $\mathcal{T}_M$-norm-continuous derivation and so there exists $g_0 \in M(G)$ such that $D(f) = f.g_0 - \phi(f)g_0$. Put $h_0 = g - g_0$. It can be seen that for each $f \in M(G)$, $f * h_0 = \phi(f)h_0$ and then $M(G)$ is ϕ-contractible. □

Theorem 3.7. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Then the following are equivalent:*

(i) *For each ϕ-$\mathcal{T}_M$-$\mathcal{A}$-bimodule X, each $\mathcal{T}_M$-norm-continuous derivation $D : \mathcal{A} \to X$ is inner.*
(ii) *For each ϕ^{**}-$\mathcal{T}_M$-$\mathcal{A}^{**}$-bimodule X, each $\mathcal{T}_M$-norm-continuous derivation $D^{**} : \mathcal{A}^{**} \to X$ is inner.*

Proof. Let X be a ϕ^{**}-$\mathcal{T}_M$-$\mathcal{A}^{**}$-bimodule and $D : \mathcal{A}^{**} \to X$ be a $\mathcal{T}_M$-norm-continuous derivation. It is clear that X is a ϕ-$\mathcal{T}_M$-$\mathcal{A}$-bimodule and then $D^{**}|_{\mathcal{A}} = D : \mathcal{A} \to X$ is inner. Then there exists x_0 such that $D(a) = \phi(a)x_0 - x_0.a$ for each $a \in \mathcal{A}$. Now let $m \in \mathcal{A}^{**}$, then from Goldeshtain theorem, there exists a bounded net $\{m_\alpha\}$ such that $m_\alpha \to m$ in the weak*-topology. Note that D^{**} is weak*-weak-continuous [15]. This follows that $D^{**}(m_\alpha) \to D^{**}(m)$ in the weak topology. Then there exists a subnet $\{m_{\alpha'}\}$ such that $D^{**}(m_{\alpha'}) \to D^{**}(m)$ in the norm topology. Note that since X is a ϕ-$\mathcal{T}_M$-$\mathcal{A}$-bimodule, for each $x \in X$, module maps $T_x : \mathcal{A} \to X$, with $T_x(a) = x.a$ and $T_x(a) = \phi(a)x$ are $\mathcal{T}_M$-norm-continuous. This implies that the module maps $T_x^{**}(m) = x.m$ and $T_x^{**}(m) = \phi(m)x$ are weak-*-weak-continuous [15]. Then, by passing a subnet, we have $x_0.m_{\alpha'} \to x_0.m$ and $\phi(m_{\alpha'})x_0 \to \phi(m)x_0$ in the norm topology. This follows that

$$\begin{aligned} D^{**}(m) &= \lim_{\alpha'} D^{**}(m_{\alpha'}) \\ &= \lim_{\alpha'}(x_0.m_{\alpha'} - \phi(m_{\alpha'})x_0) \\ &= x_0.m - \phi(m)x_0. \end{aligned}$$

Therefore D^{**} is inner derivation.

Let X be a ϕ-$\mathcal{T}_M$-$\mathcal{A}$-bimodule and $D : \mathcal{A} \to X$ be a $\mathcal{T}_M$-norm continuous derivation. We show that X is a ϕ^{**}-$\mathcal{T}_M$-$\mathcal{A}^{**}$-bimodule. Let $F_\alpha \to F$ in the $\mathcal{T}_M$-topology of $\mathcal{A}^{**}$. Fix $x \in X$. We show that $x.F_\alpha \to x.F$ in the norm-topology. Let X_1 be unit ball in X^*. Since X is $\mathcal{T}_M$-$\mathcal{A}$-bimodule.

Therefore the module map $T_x : \mathcal{A} \to X$, with $T_x(a) = x.a$ is weakly compact and so $T_x^* : X^* \to \mathcal{A}^*$ with $T_x^*(x^*) = x^*.x$ is weak*-weak-continuous [15]. This means that $X_1.x = \{x^*.x : x^* \in X_1\}$ is weakly compact. Then

$$\begin{aligned} \|x.(F_\alpha - F)\| &= \sup\{|\langle x.(F_\alpha - F), x^*\rangle| : x^* \in X_1\} \\ &= \sup\{|\langle F_\alpha - F, x^*.x\rangle| : x^* \in X_1\} \\ &= 0. \end{aligned}$$

This follows that the left module action $F \to x.F$ is $\mathcal{T}_M$-norm continuous. Since ϕ is $\mathcal{T}_M$-norm continuous, ϕ^{**} is $\mathcal{T}_M$-norm continuous. This means that X is ϕ^{**}-$\mathcal{T}_M$-$\mathcal{A}^{**}$-bimodule. By [[11] Theorem 1], D^{**} is $\mathcal{T}_M$-norm continuous derivation from $\mathcal{A}^{**}$ to $\mathcal{X}$. Now by assumption $D^{**} : \mathcal{A}^{**} \to X$ is inner and hence D is inner. □

Definition 3.8. Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. An element $M \in WAP(\mathcal{A}\hat{\otimes}\mathcal{A})$ is called ϕ-$\mathcal{T}_M$-diagonal, if for every $a \in \mathcal{A}$, $M.a = \phi(a)M$ and $\langle \Delta(M), \phi\rangle = 1$.

Theorem 3.9. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Then $\mathcal{A}$ is ϕ-$\mathcal{T}_M$-amenable if and only if there exists a ϕ-$\mathcal{T}_M$-diagonal $M \in WAP(\mathcal{A}\hat{\otimes}\mathcal{A})$.*

Proof. Let $M = M_1\hat{\otimes}M_2$ be a ϕ-$\mathcal{T}_M$-diagonal for $\mathcal{A}$. We show that for each Banach ϕ-$\mathcal{T}_M$-bimodule E, every $\mathcal{T}_M$-norm continuous derivation $D : \mathcal{A} \to E$ is inner. Put $g = \langle M_1, \phi\rangle D(M_2)$. Since $M_1\hat{\otimes}M_2$ is a ϕ-$\mathcal{T}_M$-diagonal, it is easy to see that $\langle M_1, \phi\rangle D(M_2.a) = \phi(a)\langle M_1, \phi\rangle D(M_2)$ and $\langle \phi, M_1\rangle\langle \phi, M_2\rangle = 1$. Then for each $e^* \in E^*$ and $a \in \mathcal{A}$,

$$\begin{aligned} \langle e^*, g.a\rangle &= \langle a.e^*, g\rangle \\ &= \langle M_1, \phi\rangle\langle a.e^*, D(M_2)\rangle \\ &= \langle M_1, \phi\rangle\langle e^*, D(M_2).a\rangle \\ &= \langle M_1, \phi\rangle\langle e^*, D(M_2.a) - M_2.D(a)\rangle \\ &= \langle M_1, \phi\rangle\langle e^*, \phi(a)D(M_2) - M_2.D(a)\rangle \\ &= \phi(a)\langle\langle M_1, \phi\rangle e^*, D(M_2)\rangle - \langle \phi, M_2\rangle\langle \phi, M_1\rangle\langle e^*, D(a)\rangle. \end{aligned}$$

It follows that $D(a) = \phi(a)g - g.a = g.a - a.g$. Thus D is inner.

Consider the Banach $\mathcal{A}$-bimodule $WAP(\mathcal{A}\hat{\otimes}\mathcal{A})$ with the module action given by

$$\begin{aligned} (u \otimes u).a &= u \otimes u.a, \\ a.(u \otimes u) &= \phi(a)(u \otimes u). \end{aligned}$$

Then $WAP(\mathcal{A}\hat{\otimes}\mathcal{A})$ is a ϕ-$\mathcal{T}_M$-bimodule. Let $(a_0 \otimes a_0) \in WAP(\mathcal{A}\hat{\otimes}\mathcal{A})$ such that $\phi(a_0) = 1$. Define the derivation

$$D : \mathcal{A} \to WAP(\mathcal{A}\hat{\otimes}\mathcal{A}), \quad D(a) = (a_0 \otimes a_0).a - \phi(a)(a_0 \otimes a_0).$$

It is easy to see that D is a $\mathcal{T}_M$-norm continuous derivation and attains its values in the $\mathcal{T}_M$-closed submodule $ker(\phi)\otimes ker(\phi)$. Hence there exists $(g_1\otimes g_2)\in ker(\phi)\otimes ker(\phi)$ such that $D(a)=(g_1\otimes g_2).a-\phi(a)(g_1\otimes g_2)$ for all $a\in\mathcal{A}$. Let $M=(a_0\otimes a_0)-(g_1\otimes g_2)$. We have $\phi(a)M=M.a$ and $\langle\Delta(M),\phi\rangle=1$ for each $a\in\mathcal{A}$. □

Theorem 3.10. *Let $\mathcal{A}$ be a unital Banach algebra and $\phi\in\Delta(\mathcal{A})$. Then*

(i) *$\mathcal{A}$ is ϕ-$\mathcal{T}_M$-amenable if and only if $ker(\phi)$ has an identity.*

(ii) *Let I be a unital closed two-sided ideal of $\mathcal{A}$ that $\phi|_I\neq 0$. If $\mathcal{A}$ is ϕ-$\mathcal{T}_M$-amenable, then for each unital ϕ-$\mathcal{T}_M$-bimodule E, each $\mathcal{T}_M$-norm-derivation $D:I\to E$ is inner.*

(iii) *Let I be a closed two-sided ideal of $\mathcal{A}$ such that $\phi|_I\neq 0$. If I is $\phi|_I$-$\mathcal{T}_M$-amenable , then $\mathcal{A}$ is ϕ-$\mathcal{T}_M$-amenable.*

Proof. (i) Let $\mathcal{A}$ be a unital ϕ-$\mathcal{T}_M$-amenable Banach algebra with identity $e_{\mathcal{A}}$ and let M be a ϕ-$\mathcal{T}_M$-diagonal in $WAP(\mathcal{A}\hat{\otimes}\mathcal{A})$. It is easy to see that $e_{\mathcal{A}}-\pi(M)$ is an identity for $ker(\phi)$. For the convers from [Proposition 2.1 in [8]] $\mathcal{A}$ is ϕ-amenable. Therefore Theorem 3.3 follows that $\mathcal{A}$ is ϕ-$\mathcal{T}_M$-amenable.

(ii) Let E be a unital $\phi|_I$-$\mathcal{T}_M$-bimodule and let $D:I\to E$ be a bounded $\mathcal{T}_M$-norm-continuous derivation. We show that E is a ϕ-$\mathcal{T}_M$-$\mathcal{A}$-bimodule. For this we show that for each $x\in E$, the module map $a\to x.a$ from $\mathcal{A}$ to E is $\mathcal{T}_M$-norm-continuous. Indeed, let $\{a_\alpha\}$ be a net in $\mathcal{A}$ such that $a_\alpha\to a$ in the $\mathcal{T}_M$-topology of $\mathcal{A}$, and let $x\in E$. Since E is pseudo-unital, there are $e_I\in I$ and $x\in E$ such that $x=x.e_I$. From [[11] Lemma12] the module map $a\to e_I.a$ is $\mathcal{T}_M$-$\mathcal{T}_M$-continuous if and only if norm-norm-continuous, then $e_I.a_\alpha\to e_I.a$ in the $\mathcal{T}_M$-topology. It follows that $x.a_\alpha=x.e_I.a_\alpha\to x.e_I.a=x.a$ in the norm-topology, because E is a $\mathcal{T}_M$-$\phi|_I$-bimodule. To extend D, let

$$\tilde{D}:\mathcal{A}\to E,\quad a\mapsto D(ae_I)-a.D(e_I).$$

It is clear that $\tilde{D}$ is a derivation. Infact,

$$\begin{aligned}\tilde{D}(ab)&=D(abe_I)-ab.D(e_I)\\&=D(ae_I).be_I+ae_I.D(be_I)-ab.D(e_I)\\&=D(ae_I).b+a.D(be_I)-ab.D(e_I)\\&=D(ae_I).e_Ib+a.D(be_I)-ab.D(e_I)\\&=(D(ae_Ie_I)-ae_ID(e_I)).b+a.D(be_I)-ab.D(e_I)\\&=(D(ae_I)-a.D(e_I)).b+a.D(be_I)-ab.D(e_I)\\&=\tilde{D}(a).b+a.\tilde{D}(b).\end{aligned}$$

To see that $\tilde{D}$ is $\mathcal{T}_M$-norm-continuous, again let $\{a_\alpha\}$ be a net in $\mathcal{A}$ such that $a_\alpha \to a$ in the $\mathcal{T}_M$-topology of $\mathcal{A}$. We have

$$\tilde{D}(a_\alpha) = D(a_\alpha e_I) - a_\alpha.D(e_I) \to D(ae_I) - a.D(e_I) = \tilde{D}(a)$$

because D is $\mathcal{T}_M$-norm-continuous, the module map $a \to e_I.a$ is $\mathcal{T}_M$-$\mathcal{T}_M$-continuous and E is $\mathcal{T}_M$-norm-continuous $\mathcal{A}$-bimodule. From the ϕ-$\mathcal{T}_M$-amenability of $\mathcal{A}$ we conclude that $\tilde{D}$ and hence D is inner.

(iii) Let E be a $\mathcal{T}_M$-$\mathcal{A}$-bimodule such that $a.x = \phi(a)x$ for all $a \in \mathcal{A}$ and $x \in E$, and let $D : \mathcal{A} \to E$ be a $\mathcal{T}_M$-norm-continuous derivation. Clearly, $D|_I$ is a $\mathcal{T}_M$-norm-continuous derivation. Since I is $\phi|_I$-$\mathcal{T}_M$-amenable, there exists $x_0 \in E$ such that $D(i) = \phi(i)x_0 - x_0.i$ for all $i \in I$. Choose $i_0 \in I$ such that $\phi(i_0) = 1$ and put $x = x_0.i_0$. For $a \in \mathcal{A}$,

$$\begin{aligned}
\phi(a)x - x.a &= \phi(a)x_0.i_0 - \phi(a)\phi(i_0)x_0 + \phi(a)\phi(i_0)x_0 - x_0.i_0a \\
&= D|_I(i_0a) - \phi(a)D|_I(i_0) \\
&= D(a) + D|_I(i_0).a - \phi(a)D|_I(i_0) \\
&= D(a) + (x_0 - x_0.i_0).a - \phi(a)(x_0 - x_0.i_0).
\end{aligned}$$

This shows that $D(a) = \phi(a)x_0 - x_0.a$ and hence D is inner. □

Note that in the proof of above theorem we consider the class of unital ϕ-$\mathcal{T}_M$-bimodules. It is implicit from Theorem 2.5 and Theorem 3.6 this statment is not true in general. We showed that there exists compact infinite group G such that group algebra $M(G)$ is ϕ-$\mathcal{T}_M$-amenable whereas $L^1(G)$ is not ϕ-$\mathcal{T}_M$-amenable.

Theorem 3.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras such that $WAP(\mathcal{B}) = \mathcal{B}$, $WAP(\mathcal{A}) = \mathcal{A}$ and $\phi \in \Delta(\mathcal{A})$. Let $\Theta : \mathcal{A} \to \mathcal{B}$ be a Banach algebra homomorphism. Then $\mathcal{B}$ is a Banach ϕ-$\mathcal{T}_M$-$\mathcal{A}$-bimodule with the following module action:*

$$a.b = \phi(a)b \quad b.a = b.\Theta(a), \quad (a \in \mathcal{A}, b \in \mathcal{B}),$$

and the following statements are equivalent:

(i) *$\mathcal{A}$ is ϕ-$\mathcal{T}_M$-amenable*
(ii) *Every $\mathcal{T}_M$-norm continuous derivation $D : \mathcal{A} \to B$ is inner.*

Proof. First, note that since Θ is a norm-norm continuous homomorphism, Θ is $\mathcal{T}_M$-$\mathcal{T}_M$-continuous [11, Lemma 12] and so it is easy to see that $\mathcal{B}$ is a Banach ϕ-$\mathcal{T}_M$-$\mathcal{A}$-bimodule. Now, suppose that $\mathcal{A}$ is ϕ-$\mathcal{T}_M$-amenable. This implies that every $\mathcal{T}_M$-norm continuous derivation $D : \mathcal{A} \to B$ is inner.

Now let $\mathcal{X}$ be a Banach ϕ-$\mathcal{T}_M$-bimodule and D be a $\mathcal{T}_M$-norm continuous derivation from $\mathcal{A}$ to $\mathcal{X}$. Consider the module extention Banach algebra $\mathcal{A} \oplus \mathcal{X}$ with the product

$$(a, x).(b, y) = (ab, a.y + x.b), \quad (a, b \in \mathcal{A}, x, y \in X)$$

and the norm $\|(a,x)\| = \|a\| + \|x\|$. Now consider continuous homomorphism Θ from $\mathcal{A}$ to $\mathcal{A} \oplus \mathcal{X}$ with $\Theta(a) = (a, 0)$. We show that for every element (a_0, x_0), the module maps $a \to (a_0, x_0).\Theta(a)$ is $\mathcal{T}_M$-norm continuous. Let $\{a_\alpha\}$ be a net in $\mathcal{A}$ that converges to a in the $\mathcal{T}_M$-topology. Since $WAP(\mathcal{A}) = \mathcal{A}$, we have $a_0.a_\alpha \to a_0.a$ in the norm topology, also $\mathcal{X}$ is ϕ-$\mathcal{T}_M$-bimodule and so $x_0.a_\alpha \to x_0.a$ in the norm topology. Therefore

$$\begin{aligned} \|(a_0, x_0).\Theta(a_\alpha)\| &= \|x_0.a_\alpha\| + \|a_0.a_\alpha\| \\ &\to \|x_0.a\| + \|a_0.a\| \\ &= \|(a_0, x_0).\Theta(a)\|. \end{aligned}$$

By $\mathcal{T}_M$-norm continuity of ϕ, we can see $a \to a.(a_0, x_0)$ is $\mathcal{T}_M$-norm continuous. Now define $D_0 : \mathcal{A} \to \mathcal{A} \oplus \mathcal{X}$ by $D_0(a) = (0, D(a))$ for $a \in \mathcal{A}$. Then D_0 is a derivation from $\mathcal{A}$ to $\mathcal{A} \oplus \mathcal{X}$.

$$\begin{aligned} D_0(ab) &= (0, D(ab)) \\ &= (0, \phi(a)D(b) + D(a).b) \\ &= \phi(a)(0, D(b)) + (0, D(a))(b, 0) \\ &= \phi(a)D_0(b) + D_0(a)\Theta(b). \end{aligned}$$

Since D is $\mathcal{T}_M$-norm-continuous, D_0 is a $\mathcal{T}_M$-norm continuous derivation. Thus D_0 is inner and there exist $(a_0, x_0) \in \mathcal{A} \oplus \mathcal{X}$ such that

$$D_0(a) = \phi(a)(a_0, x_0) - (a_0, x_0).\Theta(a).$$

Then

$$\begin{aligned} D_0(a) &= \phi(a).(a_0, x_0) - (a_0, x_0)(a, 0) \\ &= \phi(a).(a_0, x_0) - (a_0.a, x_0.a) \\ &= (\phi(a)a_0 - a_0a, \phi(a)x_0 - x_0.a) \end{aligned}$$

and so $D(a) = \phi(a)x_0 - x_0.a$. This means that $\mathcal{A}$ is ϕ-$\mathcal{T}_M$-amenable. □

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