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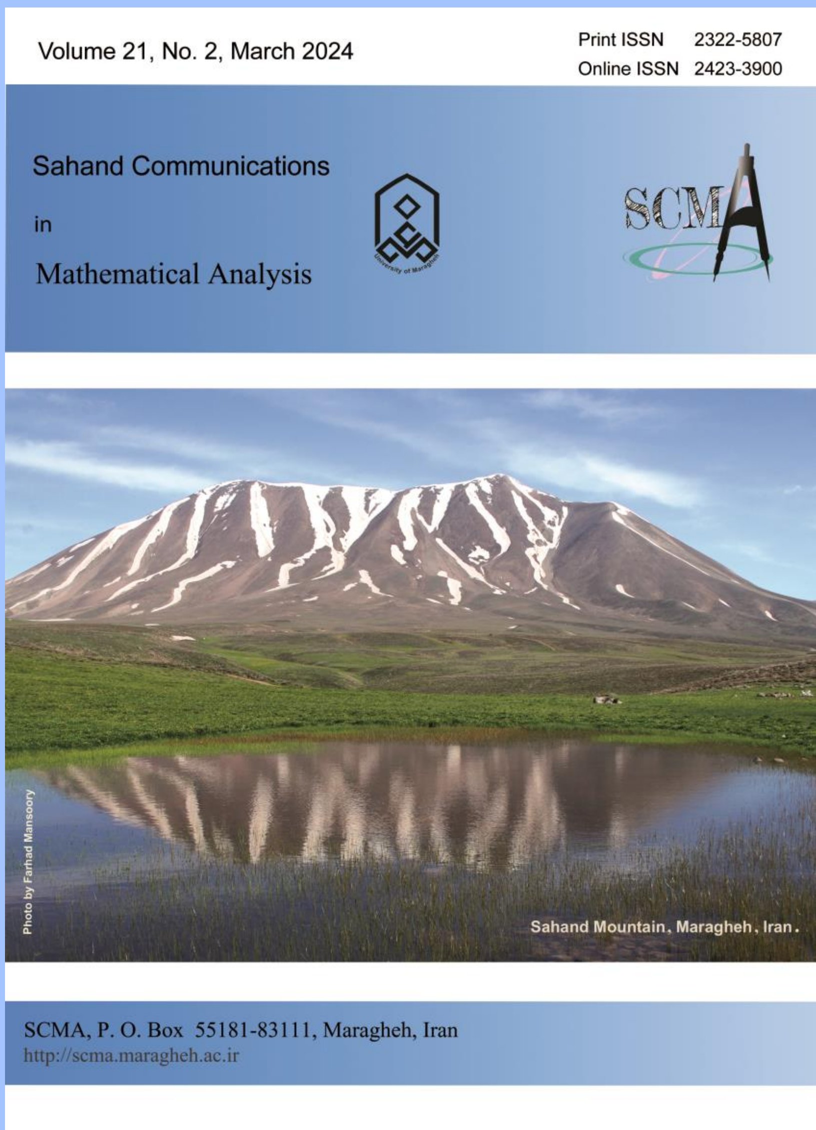

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Extensions of φ-Fixed Point Results via w-Distance

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Abstract. In this paper, we obtain a φ-fixed point result concerning w-distance. There are three illustrative examples. In a separate section, we compare of the present result with that of the corresponding results prevalent in metric spaces and indicate certain new features obtained using w-distance. One such feature is that under certain circumstances, the fixed point can be a point of discontinuity, which is impossible in the metric case. We give an application to non-linear integral equations. The paper ends with a conclusion.

1. Introduction

In this work, we derive some basic φ-fixed point results using w-distance inequalities. The w-distances are additional distance functions defined on metric spaces for various purposes in mathematical analysis. Originally, it was defined by Kada et al. [10] in 1996 to obtain nonconvex minimization results and some fixed point results. After that, it has been utilized in many works to prove new fixed point theorems for functions satisfying w-distance inequalities rather than metric inequalities. In this way, several existing results on metric fixed point theory, which has a hundred-year-old origination in the work of Banach [4] and a continued expansion through works like [2, 21], could be further extended. These new results could be applied to the classes of new family functions that did not fall under the purview of fixed point theorems existing in ordinary metric spaces. These efforts have substantially enriched fixed point theory and constitute an active research branch. Some

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recent works from this line of study are [3, 5, 12, 16, 17, 23, 25, 27]. We refer to the recent book by Rakočević [24] for a comprehensive account of the development of fixed point theory based on w-distances.

φ-fixed point is a relatively recent concept which occurs along with zeroes of an auxiliary function. After its introduction in 2014 by Jleli et al. [9], φ-fixed points have been considered in several works [1, 11, 14, 26]. Essentially, such fixed points are derived by considering an inequality involving an appropriate three-variable function and an φ-function. There are several applications of this type of fixed point theorem [6, 8, 13, 28].

In this paper, our program is to establish some φ-fixed point results using w-distance inequalities. By doing so, we have generalizations of specific existing results in which, most importantly, the continuity assumption on the control function is omitted. Furthermore, we show that our result applies to a more significant category of functions compared to that for which the previous results are applicable. We compare of different aspects of the present result obtained through w-distance inequality with the previous results, thereby indicating the leverage of the use of w-distance in fixed point theory. The paper ends with an application to a problem of a nonlinear integral equation.

2. Mathematical Preliminaries

Definition 2.1 ([10, 24])**.** Let (X, d) be a metric space. A function $\omega : X \times X \to [0, \infty)$ is called a w-distance on X if ω satisfies the following properties:

(i) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$ for all $x, y, z \in X$;
(ii) ω is lower semi-continuous in the second variable, i.e., if $x \in X$ and $y_n \to y$ in X, then $\omega(x, y) \leq \lim_{n\to\infty} \inf \omega(x, y_n)$;
(iii) for each $\varepsilon > 0$ there exists $\delta > 0$ such $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The following lemma will be utilized in the subsequent sections.

Lemma 2.2 ([10, 24])**.** *Let (X, d) be a metric space and ω be a w-distance on X.*

(i) *Let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} \omega(x_n, x) = \lim_{n\to\infty} \omega(x_n, y) = 0.$$

Then $x = y$. In particular if $\omega(z, x) = \omega(z, y) = 0$, then $x = y$.

(ii) *If $\omega(x_n, y_n) \leq \alpha_n$ and $\omega(x_n, y) \leq \beta_n$ for any $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, \infty)$ both converging to 0, then $\{y_n\}$ converges to y.*

(iii) *Let $\{x_n\}$ be a sequence in X such that* $\lim\limits_{m,n\to\infty} \omega(x_n, x_m) = 0$, *that is, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m > n > N$, $\omega(x_n, x_m) < \varepsilon$. Then $\{x_n\}$ is a Cauchy sequence.*

Several examples of w-distances are given in [24].
Let X be a non-empty set and $T : X \to X$ be a mapping. The set of all fixed point of T will be denoted by

$$F_T := \{z \in X : Tz = z\}.$$

Let $\varphi : X \to [0, \infty)$, where X is a non-empty set. The set of zeroes of φ will be denoted by

$$Z_\varphi := \{x \in X : \varphi(x) = 0\}.$$

Definition 2.3 ([9])**.** Let X be a non-empty set, $T : X \to X$ and $\varphi : X \to [0, \infty)$. An element $x \in X$ is said to be a φ-fixed point of the operator T if $x \in F_T \cap Z_\varphi$.

Definition 2.4 ([9])**.** Let (X, d) be a metric space. An operator $T : X \to X$ is said to be a φ-Picard operator if there exists $x_* \in X$ such that

(i) $F_T \cap Z_\varphi = \{x_*\}$;
(ii) the sequence $T^n x$ converges to x_* for each $x \in X$ where x_* is a φ-fixed point of T.

Definition 2.5 ([9])**.** Let (X, d) be a metric space. An operator $T : X \to X$ is said to be a weakly φ-Picard operator if

(i) $F_T \cap Z_\varphi \neq \phi$;
(ii) the sequence $T^n x$ converges for each $x \in X$ and the limit is a φ-fixed point.

3. Main Results

$\mathcal{F}_w$ denotes the set of all functions $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following condition:

(F_w) $\max\{a, b\} \leq F(a, b, c)$ for all $a, b, c \in [0, \infty)$.

Let $\mathcal{J}$ denotes the set of all functions $\Theta : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(Θ1) Θ is non-decreasing, i.e., $t_1 < t_2$ implies $\Theta(t_1) \leq \Theta(t_2)$;
(Θ2) Θ is continuous;
(Θ3) $\sum\limits_{n=0}^{\infty} \Theta^n(t) < \infty$ for all $t > 0$.

Lemma 3.1 ([13])**.** *If $\Theta \in \mathcal{J}$ then $\Theta(t) < t$ for all $t > 0$ and $\Theta(0) = 0$.*

Theorem 3.2. *Let (X, d) be a complete metric space, ω be a w-distance defined on X, $\varphi : X \to [0, \infty)$. Let $F \in \mathcal{F}_w$, $\Theta \in \mathcal{J}$. Let $T : X \to X$ be a mapping satisfying the following two conditions:*

$$F\left(\omega(Tx, Ty), \varphi(Tx), \varphi(Ty)\right) \leq \Theta\left(F\left(\omega(x, y), \varphi(x), \varphi(y)\right)\right) \tag{3.1}$$

for all $x, y \in X$ and for all $x \in X$,

$$\inf\{\omega(x, y) + \omega(x, Tx) : x \in X\} > 0 \tag{3.2}$$

for every $y \in X$ with $Ty \neq y$. Then

(i) $F_T \subseteq Z_\varphi$;
(ii) *T is a φ-Picard operator;*
(iii) *$\omega(x_*, x_*) = 0$ where $\{x_*\} = F_T \cap Z_\varphi$.*

Proof. Let $x_* \in F_T$, that is, $Tx_* = x_*$. Applying (3.1) with $x = y = x_*$, we get

$$F\left(\omega(x_*, x_*), \varphi(x_*), \varphi(x_*)\right) \leq \Theta\left(F\left(\omega(x_*, x_*), \varphi(x_*), \varphi(x_*)\right)\right).$$

Then from Lemma 3.1, $F\left(\omega(x_*, x_*), \varphi(x_*), \varphi(x_*)\right) = 0$. Using (F_w),

$$\max\{\omega(x_*, x_*), \varphi(x_*)\} \leq F\left(\omega(x_*, x_*), \varphi(x_*), \varphi(x_*)\right) = 0.$$

Therefore,

$$\omega(x_*, x_*) = \varphi(x_*) = 0. \tag{3.3}$$

That is, $x_* \in Z_\varphi$. Hence

$$F_T \subseteq Z_\varphi. \tag{3.4}$$

Let $x_0 \in X$ and $\{x_n\}$ be the Picard sequence defined by $x_{n+1} = Tx_n$ for all $n \geq 0$.

$$\begin{aligned}
&F\left(\omega(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})\right) \\
&\quad = F\left(\omega(Tx_{n-1}, Tx_n), \varphi(Tx_{n-1}), \varphi(Tx_n)\right) \\
&\quad \leq \Theta\left(F\left(\omega(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n)\right)\right) \\
&\quad \leq \Theta^2\left(F\left(\omega(x_{n-2}, x_{n-1}), \varphi(x_{n-2}), \varphi(x_{n-1})\right)\right) \\
&\quad\quad \vdots \\
&\quad \leq \Theta^n\left(F\left(\omega(x_0, x_1), \varphi(x_0), \varphi(x_1)\right)\right).
\end{aligned}$$

Then using (F_w),

$$\omega(x_n, x_{n+1}) \leq \Theta^n\left(F\left(\omega(x_0, x_1), \varphi(x_0), \varphi(x_1)\right)\right).$$

Therefore, by $(\Theta 3)$

$$\lim_{n\to\infty} \omega(x_n, x_{n+1}) = 0. \tag{3.5}$$

For $m > n$,

$$\omega(x_n, x_m) \leq \omega(x_n, x_{n+1}) + \omega(x_{n+1}, x_{n+2}) + \cdots + \omega(x_{m-1}, x_m)$$

$$\leq \sum_{k=n}^{m-1} \Theta^k \left(F\left(\omega(x_0,x_1),\varphi(x_0),\varphi(x_1)\right)\right)$$
$$\leq \sum_{k=n}^{\infty} \Theta^k \left(F\left(\omega(x_0,x_1),\varphi(x_0),\varphi(x_1)\right)\right).$$

Then from Lemma 2.2 (iii) and $(\Theta 3)$, we get that $\{x_n\}$ is a Cauchy sequence. Since (X,d) is complete, there exists $x_* \in X$ such that $x_n \to x_*$. Since $\omega(x_n,.)$ is lower semi-continuous,

$$\begin{aligned}\omega(x_n,x_*) &\leq \liminf_{m\to\infty} \omega(x_n,x_m)\\ &\leq \sum_{k=n}^{\infty} \Theta^k \left(F\left(\omega(x_0,x_1),\varphi(x_0),\varphi(x_1)\right)\right).\end{aligned}$$

Suppose $Tx_* \neq x_*$. Then, by (3.2), (3.5), $(\Theta 3)$ and the above inequality, we have

$$\begin{aligned}0 &< \inf\{\omega(x,x_*) + \omega(x,Tx) : x \in X\}\\ &\leq \inf\{\omega(x_n,x_*) + \omega(x_n,Tx_n) : n \in \mathbb{N}\}\\ &= \inf\{\omega(x_n,x_*) + \omega(x_n,x_{n+1}) : n \in \mathbb{N}\}\\ &\leq \inf\left\{\sum_{k=n}^{\infty} \Theta^k \left(F\left(\omega(x_0,x_1),\varphi(x_0),\varphi(x_1)\right)\right) + \omega(x_n,x_{n+1}) : n \in \mathbb{N}\right\}\\ &= 0.\end{aligned}$$

which is a contradiction. Therefore $Tx_* = x_*$, that is $x_* \in F_T$ and consequently $x_* \in F_T \cap Z_\varphi$. Let $y_* \in F_T \cap Z_\varphi$. Then

$$\begin{aligned}F\left(\omega(x_*,y_*),0,0\right) &= F\left(\omega(Tx_*,Ty_*),\varphi(Tx_*),\varphi(Ty_*)\right)\\ &\leq \Theta\left(F\left(\omega(x_*,y_*),\varphi(x_*),\varphi(y_*)\right)\right)\\ &= \Theta\left(F\left(\omega(x_*,y_*),0,0\right)\right).\end{aligned}$$

From Lemma 3.1, $F\left(\omega(x_*,y_*),0,0\right) = 0$. Then using (F_w), $\omega(x_*,y_*) = 0$.

Also from (3.3), $\omega(x_*,x_*) = 0$. Therefore from Lemma 2.2, we get $x_* = y_*$. Therefore $\{x_*\} = F_T \cap Z_\varphi$. This proves that T is a φ-Picard operator. □

Example 3.1. Consider the complete metric space (X,d) where $X = [0,2]$ and d is the usual metric on X. Let ω be a w-distance defined by $\omega(x,y) = y$ for all $x,y \in X$. Consider $F \in \mathcal{F}_w$ defined by $F(a,b,c) = a+b$. Let T be a self-map on X defined by

$$Tx = \begin{cases} 0 & \text{if } 0 \leq x < 1\\ \frac{\sqrt{x^2+1}}{10} & \text{if } 1 \leq x \leq 2.\end{cases}$$

Let $\varphi : X \to [0,\infty)$ be defined by $\varphi(x) = x$ for all $x \in X$. Clearly $\inf\{\omega(x,y) + \omega(x,Tx) : x \in X\} > 0$ for all $y > 0$.

$$F\left(\omega(Tx,Ty),\varphi(Tx),\varphi(Ty)\right) = Ty + Tx.$$

and

$$F\left(\omega(x,y),\varphi(x),\varphi(y)\right) = x + y.$$

If both $x, y \in [0,1)$, then inequality (3.1) is trivially satisfied. If both $x, y \in [1,2]$, then

$$\sup_{x,y\in[1,2]} \frac{Tx+Ty}{x+y} = \frac{1}{\sqrt{5}}.$$

If at least one of $x, y \in [1,2]$, then

$$\sup_{x\wedge y\in[1,2]} \frac{Tx+Ty}{x+y} = \frac{1}{2\sqrt{5}}.$$

Therefore inequality (3.1) is satisfied with $\Theta(t) = \frac{1}{2\sqrt{5}}t$. Then all the conditions of Theorem 3.2 are satisfied. Here $\{x_* = 0\} = F_T \cap Z_\varphi$.

Note 3.1. In Example 3.1, if we take φ to be identically zero, then putting $x = 1, y = 1 - \frac{\sqrt{2}}{10}$, we get

$$\begin{aligned} F\left(d(Tx,Ty),\varphi(Tx),\varphi(Ty)\right) &= F\left(d(x,y),\varphi(x),\varphi(y)\right) \\ &= F\left(\frac{\sqrt{2}}{10},0,0\right). \end{aligned}$$

Therefore, satisfaction of inequality (3.1) with the metric distance d will imply $\Theta\left(F\left(\frac{\sqrt{2}}{10},0,0\right)\right) \geq F\left(\frac{\sqrt{2}}{10},0,0\right)$, which is not possible due to Lemma 3.1.

In the following, we establish a theorem with a more general inequality for which we prove that the mapping is a weakly φ-Picard operator. It generalizes a theorem of Kada et al. [10].

Theorem 3.3. *Let (X,d) be a complete metric space, ω be a w-distance defined on $X, \varphi : X \to [0,\infty)$, $F \in \mathcal{F}_w$ and $\Theta \in \mathcal{J}$. Let $T : X \to X$ be an operator satisfying*

$$F\left(\omega(Tx,T^2x),\varphi(Tx),\varphi(T^2x)\right) \leq \Theta\left(F\left(\omega(x,Tx),\varphi(x),\varphi(Tx)\right)\right) \tag{3.6}$$

for each $x \in X$. Suppose for every $x \in X$

$$\inf\{\omega(x,y) + \omega(x,Tx) : x \in X\} > 0$$

for every $y \in X$ with $Ty \neq y$. Then

(i) $F_T \subseteq Z_\varphi$;
(ii) *T is a weakly φ-Picard operator.*
(iii) *$\omega(x_*,x_*) = 0$ where $x_* \in F_T \cap Z_\varphi$.*

Proof. Let $x_* \in F_T$. Applying (3.6) with $x = x_*$ and proceeding similarly as in Theorem 3.2, we obtain $\varphi(x_*) = \omega(x_*, x_*) = 0$. Therefore $F_T \subseteq Z_\varphi$. Next let $x_0 \in X$ and consider the sequence $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Applying (3.6) with $x = x_{n-1}$, we obtain

$$\begin{aligned} & F\left(\omega(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})\right) \\ & \quad = F\left(\omega(Tx_{n-1}, T^2x_{n-1}), \varphi(Tx_{n-1}), \varphi(T^2x_{n-1})\right) \\ & \quad \leq \Theta\left(F\left(\omega(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n)\right)\right) \\ & \quad \leq \Theta^2\left(F\left(\omega(x_{n-2}, x_{n-1}), \varphi(x_{n-2}), \varphi(x_{n-1})\right)\right) \\ & \qquad \vdots \\ & \quad \leq \Theta^n\left(F\left(\omega(x_0, x_1), \varphi(x_0), \varphi(x_1)\right)\right). \end{aligned}$$

Then, by proceeding in the same way as in the proof of Theorem 3.2, we obtain that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x_* \in X$ such that $x_n \to x_*$. Then, similar to Theorem 3.2, we can show that $x_* \in F_T \cap Z_\varphi$. □

The main fixed point result of Kada et al. [10] is a Corollary of the above theorem.

Corollary 3.1. Let (X, d) be a complete metric space, ω be a w-distance on X and $T : X \to X$ be a mapping. Suppose there exists $k \in [0, 1)$ such that

$$\omega(Tx, T^2x) \leq k\omega(x, Tx)$$

for every $x \in X$ and that

$$\inf\{\omega(x, y) + \omega(x, Tx) : x \in X\} > 0$$

for every $y \in X$ with $Ty \neq y$. Then there exists $z \in X$ such that $Tz = z$. Moreover if $Tv = v$, then $\omega(v, v) = 0$.

Proof. Consider the function $F \in \mathcal{F}_w$ defined by $F(a, b, c) = a + b + c$, $\Theta(t) = kt$ and let φ be identically zero on X. Then applying Theorem 3.3 with $x \in X$,

$$F(\omega(Tx, T^2x), 0, 0) \leq kF(\omega(x, Tx), 0, 0).$$

Which implies

$$\omega(Tx, T^2x) \leq k\omega(x, Tx).$$

Hence the result. □

Example 3.2. Let $X = [0, 3]$ and d be the usual distance on X. Then (X, d) is a complete metric space. Let ω be the w-distance on X defined by $\omega(x,y)=y$. Let T be a self-map on X given by

$$Tx=\begin{cases}\frac{k}{3}\left[\frac{x}{2}\right] & \text{if } 0\le x<\frac{5}{2}\\ \frac{k}{2}\ln\frac{x}{2} & \text{if } \frac{5}{2}\le x\le 3.\end{cases}$$

Let $F:[0,\infty)^3\to[0,\infty)$ defined by $F(a,b,c)=a+b+c$ and $\varphi:X\to[0,\infty)$ defined by $\varphi(x)=x$. Assume $\Theta:[0,\infty)\to[0,\infty)$ given by

$$\Theta(t)=\begin{cases}0 & \text{if } 0\le t<1\\ k\ln t & \text{if } t\ge 1.\end{cases}$$

where $k\in(0,1)$.

Consider the case when $\frac{5}{2}\le x,y\le 3$.

$$\begin{aligned}F\left(\omega(Tx,Ty),\varphi(Tx),\varphi(Ty)\right)&=Tx+2Ty\\&=\frac{k}{2}\ln\frac{x}{2}+k\ln\frac{y}{2}\\&\le k\ln(x+2y)\\&=\Theta\left(F\left(\omega(x,y),\varphi(x),\varphi(y)\right)\right).\end{aligned}$$

When $\frac{5}{2}\le x\le 3$ and $0\le y\le\frac{5}{2}$.

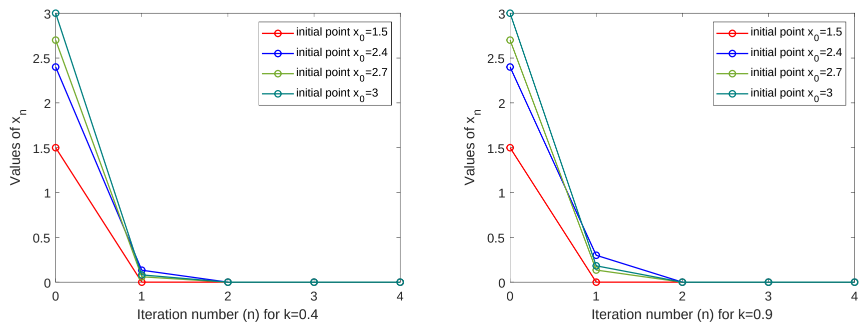
$$\begin{aligned}F\left(\omega(Tx,Ty),\varphi(Tx),\varphi(Ty)\right)&=Tx+2Ty\\&=\frac{k}{2}\ln\frac{x}{2}+\frac{2k}{3}\left[\frac{y}{2}\right]\\&\le k\ln(x+2y)\\&=\Theta\left(F\left(\omega(x,y),\varphi(x),\varphi(y)\right)\right).\end{aligned}$$

The other two cases also follow similarly. Therefore T satisfies all the conditions of Theorem 3.2. Hence by Theorem 3.2, T has a φ-fixed point $x_*=0$.

The convergence behaviors of the φ-fixed point of Example 3.2 for two different values of k are shown in Figure 1.

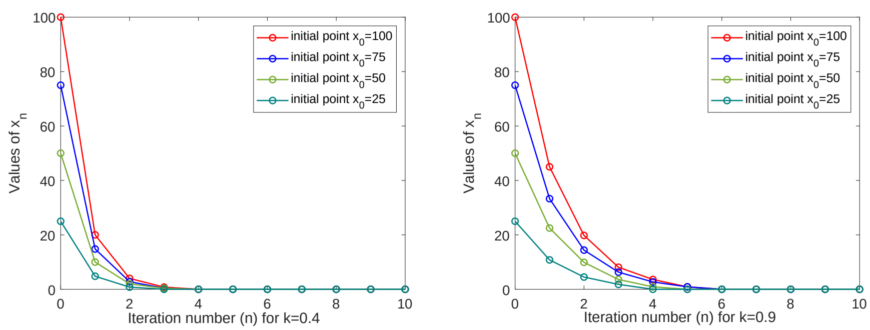
Example 3.3. Consider the complete metric space (X,d) where $X=\mathbb{R}$ and d be the usual distance. Let T be an operator on X defined by $Tx=k\left[\frac{x}{2}\right]$, where $k\in(0,1)$. Let $F\in\mathcal{F}_w$, $\Theta\in\mathcal{J}$ be the same as in Example 3.2. Let $\varphi:X\to[0,\infty)$ be defined by $\varphi(x)=|x|$ and ω be the w-distance on X defined by $\omega(x,y)=|y|$. Then T satisfies all the conditions of Theorem 3.2 and 0 is the φ-fixed point of T.

The convergence behaviors of the φ-fixed point of Example 3.3 for two different values of k are shown in Figure 2.



(A) (B)

FIGURE 1.
Convergence behaviour of Example 3.2.



(A) (B)

FIGURE 2.
Convergence behaviour of Example 3.3.

Remark 3.1. *Our main theorems are valid without any specific assumption on the function φ. They are also valid naturally for φ with T-orbitally lower semi-continuity properties [19]. The lower semi-continuity property of φ has been used in certain theorems proved in metric spaces like those discussed in the next section. We have formulated our problem concerning w-distances, for which there is no need for such a requirement.*

4. Comparison with Existing φ-Fixed Point Results

In this section, by comparing φ-fixed point results in metric spaces with those deduced in the previous section, we show that our results imply generalizations of certain other results by relaxing some requirements on the control functions. The most remarkable point we note is that in the theorems of the previous section, the φ-fixed point of the operator may be a point of discontinuity as well when φ is continuous. In contrast, in the corresponding results in metric spaces, the continuity of φ implies that such points are necessarily points of continuity. These features are also supported with examples.

Jleli et al. [9] introduced the notion of φ-fixed point.

$\mathcal{F}$ is the set of all functions $F : [0,\infty)^3 \to [0,\infty)$ satisfying the following conditions:

$(F1)$ $\max\{a,b\} \leq F(a,b,c)$ for all $a,b,c \in [0,\infty)$;
$(F2)$ $F(0,0,0) = 0$;
$(F3)$ F is continuous.

They established the following result.

Theorem 4.1 ([9]). *Let (X,d) be a complete metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}$. Suppose $T : X \to X$ satisfies the following condition:*

$$F\left(d(Tx,Ty),\varphi(Tx),\varphi(Ty)\right) \leq kF\left(d(x,y),\varphi(x),\varphi(y)\right), \tag{4.1}$$

for all $x, y \in X$ and φ is lower semi-continuous. Then,

(i) *$F_T \subseteq Z_\varphi$;*
(ii) *T is a φ-Picard operator.*

Following the path of Jleli et al., Kumord et al. [13] further generalized the φ-fixed point results with the help of the class of control functions $\mathcal{J}$.

Theorem 4.2 ([13]). *Let (X,d) be a complete metric space, $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\Theta \in \mathcal{J}$. Suppose $T : X \to X$ satisfies the following condition:*

$$F\left(d(Tx,Ty),\varphi(Tx),\varphi(Ty)\right) \leq \Theta\left(F\left(d(x,y),\varphi(x),\varphi(y)\right)\right), \tag{4.2}$$

for all $x, y \in X$ and φ is lower semi-continuous. Then,

(i) *$F_T \subseteq Z_\varphi$;*
(ii) *T is a φ-Picard operator.*

Theorem 4.3. *In Theorem 4.2, if the function φ is assumed to be continuous then the φ-fixed point is a point of continuity of T.*

Proof. Suppose z be a φ-fixed point and let $\{x_n\}$ be a sequence such that $x_n \to z$. Then applying (4.2) with $x = x_n, y = z$

$$F(d(Tx_n,Tz),\varphi(Tx_n),\varphi(Tz)) \leq \Theta(F(d(x_n,z),\varphi(x_n),\varphi(z))).$$

Taking limit as $n \to \infty$, by Lemma 3.1, we get

$$\lim_{n\to\infty} F(d(Tx_n,Tz),\varphi(Tx_n),\varphi(Tz)) = 0.$$

Again from $(F1)$,

$$d(Tx_n, Tz) \leq F(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)).$$

Hence $\lim_{n\to\infty} d(Tx_n, Tz) = 0$, that is, T is continuous at z. □

Theorem 3.2, with the special choice of the metric d for the w-distance, generalizes the above result in metric spaces by waiving the last two conditions on the family $\mathcal{F}$ and the condition of lower semi-continuity on φ. $\mathcal{F}_w \subset \mathcal{F}$, and the inclusion is proper. These points are discussed in the following

- The condition $F(0,0,0) = 0$ is not required for establishing a φ-fixed point with w-distance inequalities, which can be immediately verified by Example 3.1, as Example 3.1 remains valid with the choice of $F(a,b,c) = a + b + 1$.
- It is also noted that Theorem 3.2 does not require the continuity of the function $F \in \mathcal{F}_w$. One can immediately verify it as Example 3.3 holds good with the discontinuous function $F(a,b,c) = a + b + [c]$.
- φ need to be a lower semi-continuous function in Theorem 4.2 which is not the case if we use w-distance in place of metric function as in Theorem 3.2. For example define a function $\varphi_* : [0,2] \to [0,\infty)$ by the rule

$$\varphi_*(x) = \begin{cases} 0 & \text{if } x = 0; \\ 1 & \text{if } 0 < x < 1; \\ 2 & \text{otherwise.} \end{cases}$$

  Clearly, φ_* is not lower semi-continuous. But Example 3.1 remains true with φ_* in place of φ. Given of the above-mentioned points, we have the corresponding generalizations of the metric space results by choosing ω as the metric d.
- The φ-fixed point can be a point of discontinuity if w-distance is used instead of metric function even if φ is continuous. In Example 3.2, $x_* = 0$ is a φ-fixed point which is a point of discontinuity, although φ is continuous. But this is impossible in Theorem 4.2 as is evident from Theorem 4.3.

The above is remarkable since it gives us a new feature in the case where we use w-distances.

5. An Application to Nonlinear Integral Equation

Fixed point theory is considered to be important in mathematics for its various applications [7, 15, 20, 22]. In the following we give an application of our main result in solving a nonlinear integral equation.

Let $X = C[a,b]$ be the metric space of all real-valued continuous functions defined over $[a,b]$ with the metric $d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$. Consider the integral equation

$$x(t) = g(t) + \int_a^b K(t,s,x(s))ds \tag{5.1}$$

where $x \in C[a,b]$, $a,b \in \mathbb{R}$ with $a < b$ and $g : [a,b] \to \mathbb{R}, K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous.

Theorem 5.1. *If there exists some $\Theta \in \mathcal{J}$ such that K satisfies the condition*

$$|K(t,s,x(s))| + |K(t,s,y(s))| \le \frac{\Theta\left(|x(t)+y(t)|\right) - 2|g(t)|}{b-a} \tag{5.2}$$

for all $t,s \in [a,b]$ and for all $x,y \in C[a,b]$ then equation (5.1) has a unique solution.

Proof. Consider the operator $T : C[a,b] \to C[a,b]$ defined by

$$(Tx)(t) = g(t) + \int_a^b K(t,s,x(s))ds. \tag{5.3}$$

Let $\omega : X \times X \to [0,\infty)$ be defined by

$$\omega(x,y) = \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |y(t)|.$$

Then ω is a w-distance on X and (X,d) is complete. Also define a function $\varphi : X \to [0,\infty)$ by $\varphi(x) = 0$ for all $x \in X$ and $F \in \mathcal{F}_w$ given by $F(a,b,c) = a+b+c$. Then for $x,y \in C[a,b]$.

$$\begin{aligned}
&|(Tx)(t)| + |(Ty)(t)| \\
&\quad = \left|g(t) + \int_a^b K(t,s,x(s))ds\right| + \left|g(t) + \int_a^b K(t,s,y(s))ds\right| \\
&\quad \le |g(t)| + |g(t)| + \left|\int_a^b K(t,s,x(s))ds\right| + \left|\int_a^b K(t,s,y(s))ds\right| \\
&\quad \le 2|g(t)| + \int_a^b |K(t,s,x(s))|\,ds + \int_a^b |K(t,s,y(s))|\,ds \\
&\quad = 2|g(t)| + \int_a^b \left(|K(t,s,x(s))| + |K(t,s,y(s))|\right) ds \\
&\quad \le 2|g(t)| + \int_a^b \frac{\Theta\left(|x(t)+y(t)|\right) - 2|g(t)|}{b-a} ds \\
&\quad = \Theta\left(|x(t)+y(t)|\right).
\end{aligned}$$

From the above inequality

$$\begin{aligned}
&\omega(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\\
&\quad=\sup_{t\in[a,b]}|(Tx)(t)|+\sup_{t\in[a,b]}|(Ty)(t)|+\varphi(Tx)+\varphi(Ty)\\
&\quad=\sup_{t\in[a,b]}\Theta\left(|x(t)+y(t)|+\varphi(x)+\varphi(y)\right)\\
&\quad\leq\Theta\left(\sup_{t\in[a,b]}|x(t)|+\sup_{t\in[a,b]}|y(t)|+\varphi(x)+\varphi(y)\right).
\end{aligned}$$

Therefore, $F\left(\omega(Tx,Ty),\varphi(Tx),\varphi(Ty)\right)\ \leq\ \Theta\left(F\left(\omega(x,y),\varphi(x),\varphi(y)\right)\right).$ Thus from Theorem 3.2, T has a unique φ-fixed point, which is the solution of the integral equation (5.1). □

6. Conclusion

In this paper, we have initiated the study of φ-fixed point problems concerning w-distances. There are two-fold benefits to it. One is that the w-distance inequalities, being more general, widen the scopes of the corresponding results obtained concerning metric inequalities. The other is that the φ-fixed points confined to zeroes of φ-functions are supposed to be easier to search. The above considerations extend the scope of their applications. We have already given an application of our result to a problem of nonlinear integral equations. Particularly interesting applications for future considerations can be found in matrix equations [18]. Lastly, we note an open problem of whether the present theorem can be proved under relaxation of any of the axioms defining the $\mathcal{J}$ family of functions.

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