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Abhishikta Das and Tarapada Bag

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A Study on Fixed Circles in ϕ-Metric Spaces

Abhishikta Das[1] and Tarapada Bag[2*]

Abstract. Our present work is the extension of the line of research in the context of ϕ-metric spaces. We introduce the notion of fixed circle and obtain suitable conditions for the existence and uniqueness of fixed circles for self mappings. Additionally, we present some figures and examples in support of our results.

1. Introduction and Preliminaries

To generalize the structure of Euclidean geometry, the concept of metric space was formulated by Frechet[1] and Hausdorff[2].Following this concept, many mathematicians involved themselves with the structure of metric spaces and generalized it the same in different approaches. On the other hand, there is another forward movement on metric spaces through the study of metric fixed point theories followed by the famous 'Banach fixed point theorem'[3]. In the last few decades, numerous generalized metric spaces have been introduced in different approaches such as 2-metric[4], b-metric[5–8], G-metric[9, 10], S-metric[11, 12], cone-metric[13], generalized parametric metric[14, 15], etc.

Recently we introduce another generalized metric space, named as ϕ-metric space[16] by changing the 'triangle inequality' of metric axioms.

Following is the definition of ϕ-metric space.

Definition 1.1 ([16])**.** Let Ω be a nonempty set. A ϕ-metric is a function $M_\phi : \Omega \times \Omega \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

$(M_\phi 1)$ $M_\phi(\alpha, y) = 0$ if and only if $\alpha = y$;
$(M_\phi 2)$ $M_\phi(\alpha, y) = M_\phi(y, \alpha)$;

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* Corresponding author.

($M_\phi 3$) $M_\phi(\alpha, y) \leq M_\phi(\alpha, \xi) + M_\phi(\xi, y) + \phi(\alpha, y, \xi)$ for all $\alpha, y, \xi \in \Omega$ where $\phi : \Omega^3 \to \mathbb{R}_{\geq 0}$ is a function satisfying
- ($\phi 1$) $\phi(\mu_1, \mu_2, \mu_3) = 0$ if $\mu_1 = \mu_3$ or $\mu_2 = \mu_3$;
- ($\phi 2$) $\phi(\mu_1, \mu_2, \mu_3) = \phi(\mu_2, \mu_1, \mu_3)$;
- ($\phi 3$) for all $\epsilon > 0$ there exists $\delta > 0$ such that $\phi(\mu_1, \mu_2, \mu_3) < \epsilon$ whenever $M_\phi(\mu_1, \mu_3) < \delta$ or $M_\phi(\mu_2, \mu_3) < \delta$;

for all $\mu_1, \mu_2, \mu_3 \in \Omega$.

The pair (Ω, M_ϕ) is called a ϕ-metric space.

Example 1.2 ([16])**.** If M be a metric on a nonempty set Ω then M_ϕ, defined by $M_\phi(x, \eta) = (M(x, \eta))^2$ for all $x, \eta \in \Omega$, is a ϕ-metric on Ω.

We studied the structure of ϕ-metrics, worked on the notion of convergence of sequences, proved some results on compactness, completeness, and proved 'Banach type fixed point theorem' in this setting.

We recall some backgrounds on ϕ-metric spaces.

Definition 1.3 ([16])**.** Let (Ω, M_ϕ) be a ϕ-metric space.

(i) A sequence $\{\mu_n\} \subseteq \Omega$ is said to be convergent and converges to μ if for any $\epsilon > 0$ there exists a positive integer N satisfying $M_\phi(\mu_n, \mu) < \epsilon \ \forall n \geq N$. That is $M_\phi(\mu_n, \mu) \to 0$ as $n \to \infty$.

(ii) $\{\mu_n\} \subseteq \Omega$ is said to be a Cauchy sequence if for any $\epsilon > 0$, there exists a positive integer N such that $M_\phi(\mu_n, \mu_m) < \epsilon \ \forall m, n \geq N$. That is $M_\phi(\mu_n, \mu_m) \to 0$ as $m, n \to \infty$.

Proposition 1.4 ([16])**.** *In a ϕ-metric space (Ω, M_ϕ), every convergent sequence has unique limit.*

Definition 1.5 ([16])**.** A ϕ-metric space (Ω, M_ϕ) is said to be complete if every Cauchy sequence in it converges to some point in it.

Recently, the study of existence of fixed circles have been initiated to study geometric properties of fixed points set for self mappings in both metric spaces and generalized metric spaces. The fixed point set for some self mappings contains a circle which is known as fixed circle for the respective self mapping.

Ozgur and Tas, in [17] first introduced the concept of fixed circles in a metric space, as follows.

Definition 1.6 ([17])**.** Let (Y, M) be a metric space and $\mathcal{T} : Y \to Y$ be a function. A circle $\mathcal{C}_{x,t} = \{y \in Y : M(x, y) = t\}$ is called fixed-circle for $\mathcal{T}$ if $\mathcal{T}z = z \ \forall z \in \mathcal{C}_{x,t}$.

Theorem 1.7 ([17])**.** *Let (Y, M) be a metric space and $\mathcal{C}_{\eta,t}$ be any circle in Y. If $\mathcal{T} : Y \to Y$ be a function satisfying*

(a) $M(z, \mathcal{T}z) \leq \psi(z) - \psi(\mathcal{T}z)$

(b) $M(\mathcal{T}z, \eta) \geq t$

for all $z \in \mathcal{C}_{\eta,t}$ *where* $\psi : Y \to [0, \infty)$ *be a function defined by* $\psi(y) = M(y, \eta)$ $\forall y \in Y$ *then* $\mathcal{C}_{\eta,t}$ *is a fixed circle for* $\mathcal{T}$*.*

Self mappings with fixed point in metric spaces are used in neural networks as activation function. Some well-known fixed point theorem (viz. Brouwer's fixed point theorem, Banach fixed point theorem) have been used for artificial neural networks[18, 19]. At present, real-valued neural networks with discontinuous activation functions have been a significant importance in practice. In $\mathbb{C}$ with respect to usual metric, the Mobius Transformation $\mathcal{T}z = \frac{c_1 z + c_2}{c_3 z + c_4}$, $c_1 c_4 - c_2 c_3 \neq 0$ has been used in such objectives. Moreover, Mobius Transformations can have more than one fixed points.

Fixed circle often arises naturally for some functions in some special metric spaces. For example, $\mathbb{C}$ with respect to usual metric, the Mobius Transformation $\mathcal{T}z = \frac{1}{z}$ $\forall z \in \mathbb{C} \setminus \{0\}$ fixes the circle $C_{0,1} = \{z \in \mathbb{C} : |z| = 1\}$. Ozdemir et al. [22] shown that there exist a special type of activation functions which fix a circle for complex valued neural network (CVNN) which grantees the existence of fixed point of complex valued Hopfield neural network (CVHNN).

In [17, 20, 21], Ozgur and Tas studied the theory of fixed circles in metric spaces and established the existence and uniqueness criteria of fixed circles for self mappings. They worked on new fixed circle theorems with applications, obtained new results on discontinuous activation function. Later the theory of fixed circle has been established in some generalized metric spaces [23–26] with various geometric interpretation.

Being motivated by the application of fixed circle theorems and the on-going study on several generalized metric spaces, we think it is very interesting to introduce the concept of 'fixed circle' and 'self-mappings with fixed circle' on ϕ-metric spaces. At first we introduce the idea of fixed circle in ϕ-metric space and then exercise on fixed circle theorems. We prove the existence of fixed circle for a self mapping along with the conditions of uniqueness using different techniques. Some examples are given to show that there exist self mappings having or not fixed circles, in another words sometimes with more than one fixed point. For some self mappings, the fixed point set contains fixed circle for the respective self mapping. We draw some circles in ϕ-metric space and verify our results through illustrative examples as well.

2. Main Result

In this section, we introduce the notion of fixed circle of a self-mapping on ϕ-metric space with some illustrative examples and study some results on the existence-uniqueness criteria of fixed circles.

Definition 2.1. In a ϕ-metric space (Ω, M_ϕ), for some $t > 0$ and $a \in \Omega$, a circle with center a and radius t is defined by the set

$$\mathcal{C}_{a,t} = \{x \in \Omega : M_\phi(x, a) = t\}.$$

The circle $\mathcal{C}_{a,t}$ is called a fixed circle of a self-mapping $\mathscr{T}$ on Ω if for every $\eta \in \mathcal{C}_{a,t},\ \mathscr{T}(\eta) = \eta$.

Example 2.2. Let $\Omega = \mathbb{R} \times \mathbb{R}$ and define $M_\phi : \Omega \times \Omega \to \mathbb{R}_{\geq 0}$ by

$$M_\phi(\alpha, \beta) = \sum_{i=1}^{2} \left(\left|{\alpha_i}^2 - {\beta_i}^2\right| + |\alpha_i - \beta_i|^2 \right),$$

$\forall\ \alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2) \in \Omega$.

We first show that M_ϕ is a ϕ-metric on Ω. Clearly M_ϕ satisfies $(M\phi 1)$ and $(M\phi 2)$ we only verify the inequality $(M\phi 3)$. For, let $\alpha = (\alpha_1, \alpha_2),\ \beta = (\beta_1, \beta_2),\ \gamma = (\gamma_1, \gamma_2) \in \Omega$. Then,

$$\begin{aligned}
M_\phi(\alpha, \beta) &= \sum_{i=1}^{2} \left(\left|{\alpha_i}^2 - {\beta_i}^2\right| + |\alpha_i - \beta_i|^2 \right) \\
&= \sum_{i=1}^{2} \left(\left|\left({\alpha_i}^2 - {\gamma_i}^2\right) + \left({\gamma_i}^2 - {\beta_i}^2\right)\right| + |(\alpha_i - \gamma_i) + (\gamma_i - \beta_i)|^2 \right) \\
&\leq \sum_{i=1}^{2} \left(\left|{\alpha_i}^2 - {\gamma_i}^2\right| + \left|{\gamma_i}^2 - {\beta_i}^2\right| + |\alpha_i - \gamma_i|^2 + |\gamma_i - \beta_i|^2 \right) \\
&\quad + 2 \sum_{i=1}^{2} |\alpha_i - \gamma_i||\gamma_i - \beta_i| \\
&= \sum_{i=1}^{2} \left(\left|{\alpha_i}^2 - {\gamma_i}^2\right| + |\alpha_i - \gamma_i|^2 \right) + \sum_{i=1}^{2} \left(\left|{\gamma_i}^2 - {\beta_i}^2\right| + |\gamma_i - \beta_i|^2 \right) \\
&\quad + 2 \sum_{i=1}^{2} |\alpha_i - \gamma_i|\, |\gamma_i - \beta_i| \\
&= M_\phi(\alpha, \gamma) + M_\phi(\gamma, \beta) + \phi(\alpha, \beta, \gamma)
\end{aligned}$$

where

$$\phi(\alpha, \beta, \gamma) = 2 \sum_{i=1}^{2} |\alpha_i - \gamma_i|\, |\gamma_i - \beta_i|,$$

$\forall \alpha = (\alpha_1, \alpha_2),\ \beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2) \in \Omega$.

Now we verify the properties $(\phi 1) - (\phi 3)$ of the Definition 1.1 for the function ϕ. Clearly ϕ satisfies $(\phi 1)$ and $(\phi 2)$. For $(\phi 3)$, let $s > 0$. Then,

$\phi(\alpha,\beta,\gamma) < s$ if

$$2\sum_{i=1}^{2} |\alpha_i - \gamma_i|\,|\gamma_i - \beta_i| < s$$

where $\alpha = (\alpha_1, \alpha_2),\ \beta = (\beta_1, \beta_2),\ \gamma = (\gamma_1, \gamma_2)$.

If for some $\delta > 0$, $M_\phi(\alpha,\gamma) < \delta$ or $M_\phi(\gamma,\beta) < \delta$, then

$$\sum_{i=1}^{2} \left(\left|{\alpha_i}^2 - {\gamma_i}^2\right| + \left|\alpha_i - \gamma_i\right|^2\right) < \delta$$

or

$$\sum_{i=1}^{2} \left(\left|{\gamma_i}^2 - {\beta_i}^2\right| + |\gamma_i - \beta_i|^2\right) < \delta$$

i.e. $\left|{\alpha_i}^2 - {\gamma_i}^2\right| < \delta$ and $|\alpha_i - \gamma_i|^2 < \delta$ or $\left|{\gamma_i}^2 - {\beta_i}^2\right| < \delta$ and $|\gamma_i - \beta_i|^2 < \delta$ for $i = 1, 2$ i.e. $|\alpha_i - \gamma_i| < \sqrt{\delta}$ or $|\gamma_i - \beta_i| < \sqrt{\delta}$ for $i = 1, 2$.

Let us take $\delta = \frac{s}{4}$. Then,

$$|\alpha_i - \gamma_i| < \frac{\sqrt{s}}{2} \quad \text{and} \quad |\gamma_i - \beta_i| < \frac{\sqrt{s}}{2} \text{ for } i = 1, 2.$$

Therefore, $M_\phi(\alpha,\gamma) < \delta$ and $M_\phi(\gamma,\beta) < \delta$ implies

$$2\sum_{i=1}^{2} |\alpha_i - \gamma_i|\,|\gamma_i - \beta_i| < 2 \cdot \left(\frac{s}{4} + \frac{s}{4}\right) = s.$$

Hence for any $s > 0$, there exists $\delta \left(= \frac{s}{4}\right) > 0$ such that $\phi(\alpha,\beta,\gamma) < s$ whenever $M_\phi(\alpha,\gamma) < \delta$ and $M_\phi(\beta,\gamma) < \delta$. Thus ϕ satisfies (ϕ3). Therefore M_ϕ is a ϕ-metric on Ω. Moreover this ϕ-metric is not generated by a metric. For, if possible suppose that there exists a metric M on Ω which generates the ϕ-metric M_ϕ. Then

$$M_\phi(x,\eta) = M(x,\eta), \quad \forall x = (x_1, x_2), \eta = (\eta_1, \eta_2) \in \Omega$$

i.e.

$$\sum_{i=1}^{2} \left(\left|{x_i}^2 - {\eta_i}^2\right| + |x_i - \eta_i|^2\right) = M(x,\eta), \quad \forall x, \eta \in \Omega$$

or

$$M(x,\eta) = \sum_{i=1}^{2} \left(\left|{x_i}^2 - {\eta_i}^2\right| + |x_i - \eta_i|^2\right), \quad \forall x, \eta \in \Omega.$$

But clearly this is not a metric.

Next we choose $a = (0, 0)$ and $t = 2$. Then

$$\begin{aligned} \mathcal{C}_{a,t} &= \{x = (x_1, x_2) \in \Omega : M_\phi(x, a) = t\} \\ &= \left\{(x_1, x_2) \in \Omega : x_1^2 + x_2^2 = 1\right\}. \end{aligned}$$

Let us define a mapping $\mathscr{T} : \Omega \to \Omega$ by

$$\mathscr{T}(x_1, x_2) = \begin{cases} \left(\frac{x_1}{\sqrt{{x_1}^2+{x_2}^2}}, \frac{x_2}{\sqrt{{x_1}^2+{x_2}^2}} \right) & \text{if } x = (x_1, x_2) \neq (0,0) \\ (0,0) & \text{if } (x_1, x_2) = (0,0) \end{cases}$$

Then observe that $\mathscr{T}x = x$ if $x = (x_1, x_2) \in \mathcal{C}_{a,t}$ i.e $\mathcal{C}_{a,t}$ is a fixed circle for $\mathscr{T}$.

In the following theorem we investigate existence criteria of fixed circles for a self mapping.

Theorem 2.3. *Let (Ω, M_ϕ) be a ϕ-metric space and $\mathcal{C}_{a,t}$ be any circle on Ω. Let us define the mapping $\psi : \Omega \to [0, \infty)$ by $\psi(\eta) = M_\phi(\eta, a)$ $\forall \eta \in \Omega$. If there exists a self mapping $\mathscr{T}$ on Ω satisfying*

(i) $M_\phi(\eta, \mathscr{T}\eta) \leq \psi(\eta) - \psi(\mathscr{T}\eta)$
(ii) $M_\phi(\mathscr{T}\eta, a) \geq t$

for each $\eta \in \mathcal{C}_{a,t}$, then the circle $\mathcal{C}_{a,t}$ is a fixed circle for $\mathscr{T}$.

Proof. We have to prove that $\mathscr{T}\eta = \eta$ for all $\eta \in \mathcal{C}_{a,t}$.
Let $y \in \mathcal{C}_{a,t}$. Then by (i),

$$\begin{aligned} M_\phi(y, \mathscr{T}y) &\leq \psi(y) - \psi(\mathscr{T}y) \\ &= M_\phi(y, a) - M_\phi(\mathscr{T}y, a) \\ &= t - M_\phi(\mathscr{T}y, a) \end{aligned}$$

By (ii), we have $M_\phi(\mathscr{T}y, a) \geq t$. If $M_\phi(\mathscr{T}y, a) > t$ then $M_\phi(y, \mathscr{T}y) < 0$ - a contradiction. Therefore $M_\phi(\mathscr{T}y, a)$ must be equals to t and hence $M_\phi(y, \mathscr{T}y) = 0$ i.e. $\mathscr{T}y = y$. Hence $\mathscr{T}y = y$ whenever $y \in \mathcal{C}_{a,t}$ i.e. $\mathcal{C}_{a,t}$ is a fixed circle for the self mapping $\mathscr{T}$. □

Remark 2.4. If in particular case ϕ-metric becomes a metric, then the Theorem 2.3 is reduced to the corresponding results of metric space.

Remark 2.5. The condition (i) ensures that for each $y \in \mathcal{C}_{a,t}$, $\mathscr{T}y$ is not in the exterior of the circle $\mathcal{C}_{a,t}$. Again the condition (ii) ensures that for each $y \in \mathcal{C}_{a,t}$, $\mathscr{T}y$ is not in the interior of $\mathcal{C}_{a,t}$. Hence $\mathscr{T}y \in \mathcal{C}_{a,t}$ for each $y \in \mathcal{C}_{a,t}$ and consequently $\mathscr{T}(\mathcal{C}_{a,t}) \subset \mathcal{C}_{a,t}$.

The Figure 1 ((a)-(c)) illustrate this observation geometrically.

Now we give some illustrative examples.

Example 2.6. Let $b \in \mathbb{R}$ and M_ϕ be a ϕ-metric over $\mathbb{R}$ such that $M_\phi(a, b) > t$ for some $a \in \mathbb{R}$ and $t > 0$.

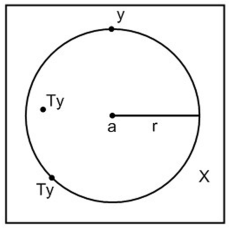
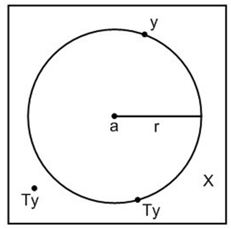
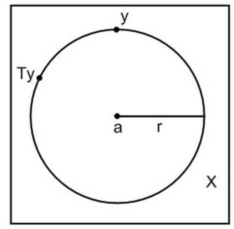
(a)  (b)  (c) 

FIGURE 1. (a) View of the condition (i) (b) View of the condition (ii) (c) View of the condition combining (i) and (ii)

Next we consider the circle $\mathcal{C}_{a,t}$ and define a mapping $\mathscr{T} : \mathbb{R} \to \mathbb{R}$ by

$$\mathscr{T}(\eta) = \begin{cases} \eta & \text{if } M_\phi(\eta, a) \leq t \\ \\ b & \text{otherwise} \end{cases}$$

Then observe that for any $y \in \mathcal{C}_{a,t}$,

$$\begin{aligned} &M_\phi(y, \mathscr{T}y) = M_\phi(y, y) = 0 \\ &\psi(y) - \psi(\mathscr{T}y) = M_\phi(y, a) - M_\phi(\mathscr{T}y, a) = 0 \\ &M_\phi(\mathscr{T}y, a) = M_\phi(y, a) = t \end{aligned}$$

implies.

Therefore, $\mathscr{T}$ satisfies the conditions (i) and (ii) of Theorem 2.3 for the circle $\mathcal{C}_{a,t}$. Consequently $\mathcal{C}_{a,t}$ is a fixed circle for $\mathscr{T}$.

In the following example we show that there exists a self mapping having more than one fixed circles.

Example 2.7. We consider a ϕ-metric space (Ω, M_ϕ) where $\Omega = \mathbb{R}$ and $M_\phi(\eta_1, \eta_2) = |\eta_1 - \eta_2|^2$ for all $\eta_1, \eta_2 \in \Omega$.

Next we consider the circles $\mathcal{C}_{0,1}$ and $\mathcal{C}_{3,4}$ where

$$\begin{aligned} \mathcal{C}_{0,1} &= \{\eta \in \mathbb{R} : M_\phi(\eta, 0) = 1\} \\ &= \left\{\eta \in \mathbb{R} : |\eta|^2 = 1\right\} \\ &= \{+1, -1\} \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{3,4} &= \{\eta \in \mathbb{R} : M_\phi(\eta, 3) = 4\} \\ &= \left\{\eta \in \mathbb{R} : |\eta - 3|^2 = 4\right\} \\ &= \{+1, +5\}. \end{aligned}$$

Now we define a mapping $\mathscr{T} : \mathbb{R} \to \mathbb{R}$ by

$$\mathscr{T}(\eta) = \begin{cases} \eta & \text{if } -\infty < \eta \leq 5 \\ \\ 2\eta & \text{if } \quad 5 < \eta < \infty \end{cases}$$

In this example $\mathscr{T}$ satisfies the conditions (i) and (ii) for both the circles $\mathcal{C}_{0,1}$ and $\mathcal{C}_{3,4}$ and hence both are fixed circles for $\mathscr{T}$.
In other words, in this example the mapping $\mathscr{T}$ has more than one fixed points in Ω and the set of fixed points contain fixed circle for $\mathscr{T}$.

Example 2.8. Let $\Omega = [1,4] \times [1,4]$ and define a function M_ϕ on Ω by

$$M_\phi(x,y) = \left|\log\left(\frac{x_1}{y_1}\right)\right|^2 + \left|\log\left(\frac{x_2}{y_2}\right)\right|^2, \quad \forall\, x = (x_1,x_2),\ y = (y_1,y_2) \in \Omega.$$

Clearly M_ϕ satisfies $(M_\phi 1)$ and $(M_\phi 2)$. For $(M_\phi 3)$, take $x = (x_1, x_2)$, $y = (y_1, y_2),\ z = (z_1, z_2) \in \Omega$. Then,

$$\begin{aligned} M_\phi(x,y) &= \left|\log\left(\frac{x_1}{y_1}\right)\right|^2 + \left|\log\left(\frac{x_2}{y_2}\right)\right|^2 \\ &= \sum_{i=1}^{2}\left(|\log x_i - \log z_i + \log z_i - \log y_i|^2\right) \\ &\leq \sum_{i=1}^{2}\left(|\log x_i - \log z_i|^2 + |\log z_i - \log y_i|^2\right) \\ &\quad + 2\sum_{i=1}^{2}|\log x_i - \log z_i|\,|\log z_i - \log y_i| \\ &= \sum_{i=1}^{2}\left(\left|\log\left(\frac{x_i}{z_i}\right)\right|^2 + \left|\log\left(\frac{z_i}{y_i}\right)\right|^2\right) \\ &\quad + 2\sum_{i=1}^{2}\left|\log\left(\frac{x_i}{z_i}\right)\right|\left|\log\left(\frac{z_i}{y_i}\right)\right| \\ &= M_\phi(x,z) + M_\phi(z,y) + \phi(x,y,z) \end{aligned}$$

where

$$\phi(x,y,z) = 2\sum_{i=1}^{2}\left|\log\left(\frac{x_i}{z_i}\right)\right|\left|\log\left(\frac{z_i}{y_i}\right)\right|,$$

$\forall x = (x_1, x_2),\ y = (y_1, y_2),\ z = (z_1, z_2) \in X$. Now we verify the properties $(\phi 1) - (\phi 3)$. Clearly ϕ satisfies $(\phi 1)$ and $(\phi 2)$. For $(\phi 3)$, let $\alpha > 0$.

Then, $\phi(x,y,z) < \alpha$ if

$$2\sum_{i=1}^{2}\left|\log\left(\frac{x_i}{z_i}\right)\right|\left|\log\left(\frac{z_i}{y_i}\right)\right| < \alpha$$

if

$$\left|\log\left(\frac{x_1}{z_1}\right)\right|\left|\log\left(\frac{z_1}{y_1}\right)\right| + \left|\log\left(\frac{x_2}{z_2}\right)\right|\left|\log\left(\frac{z_2}{y_2}\right)\right| < \frac{\alpha}{2}.$$

If for some $\beta > 0$, $M_\phi(x,z) < \beta$ or $M_\phi(z,y) < \beta$, then

$$\sum_{i=1}^{2}\left|\log\left(\frac{x_i}{z_i}\right)\right|^2 < \beta \quad \text{or} \quad \sum_{i=1}^{2}\left|\log\left(\frac{z_i}{y_i}\right)\right|^2 < \beta$$

i.e.

$$\left|\log\left(\frac{x_i}{z_i}\right)\right|^2 < \beta \quad \text{or} \quad \left|\log\left(\frac{z_i}{y_i}\right)\right|^2 < \beta \text{ for } i = 1,2$$

i.e.

$$\left|\log\left(\frac{x_i}{z_i}\right)\right| < \sqrt{\beta} \quad \text{or} \quad \left|\log\left(\frac{z_i}{y_i}\right)\right| < \sqrt{\beta} \text{ for } i = 1,2.$$

Let us take $\beta = \dfrac{\alpha}{4}$. Then, $M_\phi(x,z) < \beta$ and $M_\phi(z,y) < \beta$ i.e.

$$\left|\log\left(\frac{x_i}{z_i}\right)\right| < \frac{\sqrt{\alpha}}{2} \quad \text{and} \quad \left|\log\left(\frac{z_i}{y_i}\right)\right| < \frac{\sqrt{\alpha}}{2} \text{ for } i = 1,2.$$

implies $2\sum_{i=1}^{2}\left|\log\left(\dfrac{x_i}{z_i}\right)\right|\left|\log\left(\dfrac{z_i}{y_i}\right)\right| < 2(\frac{\alpha}{4}+\frac{\alpha}{4}) = \alpha$. Thus for any $\alpha > 0$, there exists $\beta(=\dfrac{\alpha}{4}) > 0$ such that $\phi(x,y,z) < \alpha$ whenever $M_\phi(x,z) < \beta$ and $M_\phi(z,y) < \beta$. So ϕ satisfies ($\phi3$). Hence M_ϕ is a ϕ-metric on Ω. Moreover by the construction of M_ϕ it is clear that this ϕ-metric is not generated by a metric. Next we choose $a = (1,1)$, $b = (e,e)$ and $r = s = 1$. Then the circles $\mathcal{C}_{a,r}$ and $\mathcal{C}_{b,s}$ are defined by

$$\begin{aligned}\mathcal{C}_{a,r} &= \{x = (x_1,x_2) \in \Omega : M_\phi(x,a) = r\}\\ &= \left\{(x_1,x_2) \in \Omega : |\log x_1|^2 + |\log x_2|^2 = 1\right\}\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{b,s} &= \{x = (x_1,x_2) \in \Omega : M_\phi(x,b) = s\}\\ &= \left\{(x_1,x_2) : |\log x_1 - 1|^2 + |\log x_2 - 1|^2 = 1\right\}.\end{aligned}$$

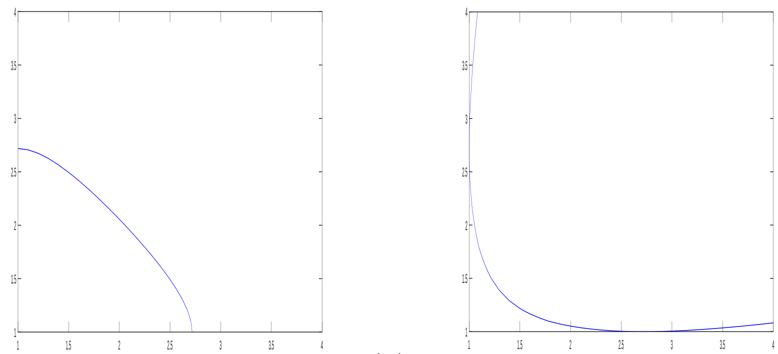
Define a mapping $\mathscr{T} : \Omega \to \Omega$ by

$$\mathscr{T}(\eta) = \begin{cases} \frac{\eta}{|\log \eta_1|^2 + |\log \eta_2|^2} & \text{if } \eta = (\eta_1,\eta_2) \neq (1,1) \\ \\ (1,1) & \text{if } \eta = (1,1) \end{cases}$$

for all $\eta = (\eta_1, \eta_2) \in \Omega$.

Then clearly $\mathscr{T}$ satisfies the conditions (i)-(ii) for the circle $\mathcal{C}_{a,r}$ and hence $\mathcal{C}_{a,r}$ is a fixed circle for $\mathscr{T}$.

Observe that the Figure 2 ((a) and (b)) shows that how the shape of fixed circles are changed according to the center.



(a) (b)

FIGURE 2. (a) $\mathcal{C}_{a,r} = \{(x_1, x_2) \quad : |\log x_1|^2 + |\log x_2|^2 = 1\}$ (b) $\mathcal{C}_{b,s} = \{(x_1, x_2) \in \Omega : |\log x_1 - 1|^2 + |\log x_2 - 1|^2 = 1\}$

Example 2.9. Consider the mapping $M_\phi : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by $M_\phi(x, \eta) = |e^x - e^\eta|^2 \ \forall x, \eta \in \mathbb{R}$. Since $M(x, \eta) = |e^x - e^\eta| \ \forall x, \eta \in \mathbb{R}$ is a metric on $\mathbb{R}$, so M_ϕ is ϕ-metric on $\mathbb{R}$ with $\phi(x, \eta, z) = 2\sqrt{M_\phi(x, z) M_\phi(z, \eta)} \ \forall x, \eta, z \in \mathbb{R}$(for details please see [16]).

Define a self-mapping $\mathscr{T}$ on $\mathbb{R}$ by $\mathscr{T}\mu = \frac{\mu + ln3}{2} \ \forall \mu \in \mathbb{R}$. Now we consider the circle $\mathcal{C}_{0,4}$ in $\mathbb{R}$ where

$$\begin{aligned} \mathcal{C}_{0,4} &= \{\eta \in \mathbb{R} : M_\phi(\eta, 0) = 4\} \\ &= \{\eta \in \mathbb{R} : |e^\eta - 1|^2 = 4\} \\ &= \{\ln 3\}. \end{aligned}$$

It is clear that $\mathscr{T}$ satisfies the conditions (i) and (ii) of the Theorem 2.3 for the circle $\mathcal{C}_{0,4}$. So $\mathcal{C}_{0,4}$ is a fixed circle of $\mathscr{T}$.

In the following examples, we construct self-mappings which satisfy one of the conditions (i) and (ii) of the Theorem 2.3 and verify the existence of fixed circles.

Example 2.10. Let (Ω, M_ϕ) be a ϕ-metric space and consider a circle $\mathcal{C}_{a,t}$ in Ω. If we define $\mathscr{T}x = a \ \forall x \in \Omega$ then it is clear that $\mathscr{T}$ satisfies the condition (i) of the Theorem 2.3 but does not satisfies (ii) for $\mathcal{C}_{a,t}$. Moreover $\mathcal{C}_{a,t}$ is not a fixed circle of $\mathscr{T}$.

Example 2.11. Consider the ϕ-metric space of Example 2.7. Here we consider the circle $\mathcal{C}_{2,9}$ which is defined by the set $\{\eta \in \mathbb{R} : |\eta - 2|^2 = 9\} = \{-1, 5\}$ and define a self mapping by

$$\mathscr{T}\eta = \begin{cases} 6 & \text{if } |\eta| \leq 5 \\ \\ \eta & \text{otherwise} \end{cases}$$

Then $\mathscr{T}$ satisfies (ii) but not (i) for $\mathcal{C}_{2,9}$ and it is clear that $\mathcal{C}_{2,9}$ is not a fixed circle of $\mathscr{T}$.

Following is another theorem for the existence of fixed circle for a self mapping.

Theorem 2.12. *Let (Ω, M_ϕ) be a ϕ-metric space and $\mathcal{C}_{a,t}$ be a circle on Ω. Define a mapping $\psi_t : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by*

$$\psi(\eta) = \begin{cases} \eta - t & \text{if } \eta > 0 \\ \\ 0 & \text{if } \eta = 0 \end{cases}$$

for all $\eta \in \mathbb{R}_{\geq 0}$. If there exists a function $\mathscr{T} : \Omega \to \Omega$ satisfying

(i) $M_\phi(\mathscr{T}\eta, a) = t$ for each $\eta \in \mathcal{C}_{a,t}$,

(ii) $M_\phi(\mathscr{T}\eta, \mathscr{T}\zeta) > t$ for each $\eta, \zeta \in \mathcal{C}_{a,t}$ with $\eta \neq \zeta$,

(iii) $M_\phi(\mathscr{T}\eta, \mathscr{T}\zeta) \leq M_\phi(\eta, \zeta) - \psi_t(M_\phi(\eta, \mathscr{T}\eta))$ for each $\eta, \zeta \in \mathcal{C}_{a,t}$

then $\mathcal{C}_{a,t}$ is a fixed circle for $\mathscr{T}$.

Proof. Let $\mu \in \mathcal{C}_{a,t}$ be an arbitrary element. The condition (i) implies that $\mathscr{T}\mu \in \mathcal{C}_{a,t}$. Next we prove that μ is a fixed point for $\mathscr{T}$. If not, then by (ii),

$$M_\phi\left(\mathscr{T}\mu, \mathscr{T}^2\mu\right) = M_\phi(\mathscr{T}\mu, \mathscr{T}(\mathscr{T}\mu)) > t. \tag{2.1}$$

Again the condition (iii) implies

$$M_\phi\left(\mathscr{T}\mu, \mathscr{T}^2\mu\right) \leq M_\phi(\mu, \mathscr{T}\mu) - \psi_t(M_\phi(\mu, \mathscr{T}\mu)) = t.$$

Which is a contradiction to (2.1). Therefore μ must be a fixed point for $\mathscr{T}$. Consequently $\mathcal{C}_{a,t}$ is a fixed circle for $\mathscr{T}$. □

Example 2.13. Let $\Omega = \mathbb{R}$ and M_ϕ be a ϕ-metric on Ω defined by

$$M_\phi(\eta, \mu) = \begin{cases} 0 & \text{if } \eta = \mu \\ \\ (|\eta| + |\mu|)^2 & \text{if } \eta \neq \mu \end{cases}$$

for all $\eta, \mu \in \Omega$ with $\phi(\eta, \mu, z) = 2\sqrt{M_\phi(\eta, z) \cdot M_\phi(z, \mu)}$ $\forall \eta, \mu, z \in \Omega$. Next we consider the circle $\mathcal{C}_{1,9}$ on Ω where

$$\mathcal{C}_{1,9} = \left\{\eta \in \mathbb{R} : M_\phi(\eta, 1) = (|\eta| + 1)^2 = 9\right\}$$

$$= \{+2, -2\}$$

and a mapping $\mathscr{T} : \Omega \to \Omega$ by

$$\mathscr{T}(\eta) = \begin{cases} 5 & \text{if } \eta = 1 \\ \\ \eta & \text{otherwise} \end{cases}$$

for all $\eta \in \Omega$.

Then $\mathscr{T}$ satisfies the conditions (i)-(iii) of the Theorem 2.12 for the circle $\mathcal{C}_{1,9}$ and thus $\mathscr{T}$ fixes the circle $\mathcal{C}_{1,9}$.

Observe that in this case $\mathscr{T}$ has more than one fixed point and the fixed point set contains a fixed circle for $\mathscr{T}$.

Our next proposition shows that for any collection of circles on a ϕ-metric space there exists a self mapping $\mathscr{T}$ such that each of them are fixed circle of $\mathscr{T}$.

Proposition 2.14. *In a ϕ-metric space (Ω, M_ϕ), for any given circles $\mathcal{C}_{a_1,r_1}$, $\mathcal{C}_{a_2,r_2}$, $\mathcal{C}_{a_m,r_m}$, there exists at least one self-mapping $\mathscr{T}$ on Ω such a way that $\{\mathcal{C}_{a_i,r_i} : i = 1, 2, \cdots, m\}$ are fixed circles of $\mathscr{T}$.*

Proof. Let us define $\mathscr{T} : \Omega \to \Omega$ by

$$\mathscr{T}(\eta) = \begin{cases} \eta & \text{if } \eta \in \cup_{i=1}^{m} C_{a_i,r_i} \\ \mu & \text{otherwise} \end{cases}$$

where $\mu \in \Omega$ be such that $M_\phi(a_i, \mu) \neq r_i$ for all $i \in \{1, 2, \cdots, m\}$.

Then by defining $\psi(\eta) = M_\phi(\eta, a_i)$ for each $i \in \{1, 2, \cdots m\}$, it can be easily verified that $\mathscr{T}$ satisfies the conditions (i) and (ii) of the Theorem 2.3 for each circle is a fixed circle $\mathcal{C}_{a_i,r_i}$, $i \in \{1, 2, \cdots, m\}$ and consequently each of them are fixed circles of $\mathscr{T}$. $\square$

Remark 2.15. Note that in Example 2.7 and 2.8 we see that a self-mapping can have more than one fixed circles. So fixed circle for a self-mapping is not necessarily unique. Further note that in those examples the circles have non-empty intersection. The circles $\{\mathcal{C}_{a_i,r_i} : i = 1, 2, \cdots, m\}$ in Proposition 2.14 may not be disjoint.

In this connection our next theorem for the existence of unique fixed circle.

In the following theorem we give a condition for existence of unique fixed circle by using 'Banach type fixed point theorem' in ϕ-metric spaces.

Theorem 2.16. *Let (Ω, M_ϕ) be a ϕ-metric space and $\mathscr{T}$ be a self-mapping on Ω satisfying the conditions (i) and (ii) of the Theorem 2.3* *for the circle $\mathcal{C}_{a,t}$ on Ω. If for all $\eta_1 \in \mathcal{C}_{a,t}$ and $\eta_2 \in \Omega \setminus \mathcal{C}_{a,t}$, $\mathscr{T}$ satisfies the contraction condition*

$$M_\phi(\mathscr{T}\eta_1, \mathscr{T}\eta_2) \leq kM_\phi(\eta_1, \eta_2) \tag{2.2}$$

where $k \in [0, 1)$ then $\mathcal{C}_{a,t}$ is a unique fixed circle of $\mathscr{T}$.

Proof. If possible suppose that there exist two fixed circles $\mathcal{C}_{a,t}$ and $\mathcal{C}_{b,s}$ of the self-mapping $\mathscr{T}$.
Let μ and ξ be two arbitrary points in $\mathcal{C}_{a,t}$ and $\mathcal{C}_{b,s}$ respectively with $\mu \neq \xi$.
Then using the condition (2.2), we have

$$\begin{aligned}
&M_\phi(\mu, \xi) = M_\phi(\mathscr{T}\mu, \mathscr{T}\xi) \leq kM_\phi(\mu, \xi)\\
&M_\phi(\mu, \xi) = 0 \text{ (since } k < 1)\\
&\mu = \xi.
\end{aligned}$$

Hence it follows that $\mathcal{C}_{a,t}$ & $\mathcal{C}_{b,s}$ must be identical i.e. $\mathcal{C}_{a,t}$ is unique fixed circle of $\mathscr{T}$. □

Remark 2.17. The conditions of Theorem 2.16 are sufficient but not necessary for the existence of fixed circle.

For example, consider the self-mapping $\mathscr{T}$ and it's fixed circle $\mathcal{C}_{a,t}$ of the Example 2.2. Then for some $\eta = (\eta_1, \eta_2) \in \mathcal{C}_{a,t}$ and $\zeta = (0, 0) \in \Omega \setminus \mathcal{C}_{a,t}$,

$$\begin{aligned}
M_\phi(\mathscr{T}\eta, \mathscr{T}\zeta) &= \sum_{i=1}^{2} \left\{ \left|{\eta_i}^2\right| + |\eta_i|^2 \right\}\\
&= M_\phi(\eta, \zeta).
\end{aligned}$$

Therefore $\mathscr{T}$ does not satisfy the condition of Theorem 2.16, still $\mathcal{C}_{a,t}$ is a fixed circle for $\mathscr{T}$.

Remark 2.18. Let us consider the ϕ-metric space (Ω, M_ϕ) and the defined self-mapping $\mathscr{T}$ in Example 2.7.

Now let $\eta \in \mathcal{C}_{0,1}$ and $\zeta \in \mathcal{C}_{3,4}$ with $\eta \neq \zeta$. Then, $M_\phi(\mathscr{T}\eta, \mathscr{T}\zeta) = M_\phi(\eta, \zeta)$ shows that there does not exist any $k \in [0, 1)$ satisfying the condition (2.2).
Hence $\mathscr{T}$ does not satisfy the condition of Theorem 2.16 and thus we can not ensure the existence of unique fixed circle for $\mathscr{T}$.

Remark 2.19. In Theorem 2.16, we establish a sufficient criterion with the help of 'Banach type contraction condition' for the existence of unique fixed circles. So it is an interesting problem for the readers to find other conditions to ensure the existence-uniqueness criterion for fixed circles of self-mappings in this new settings.

Remark 2.20. The fixed circle theorems and examples show several characteristics of the corresponding self mappings. Observe that in Example 2.7, $\mathscr{T}$ is continuous on its fixed circle $\mathcal{C}_{0,1}$ but it is discontinuous on another fixed circle $\mathcal{C}_{3,4}$ because of its discontinuity at $x = 5$. The results often ensures the existence of self mappings with more than one fixed point. But there is no exact statement on its continuity criteria.

In this connection we propose two open problems.

Open Problems:

(a) Find necessary and sufficient conditions so that a self-mapping on a ϕ-metric space can have exactly one fixed circle.
(b) Also find the conditions so that a self mapping be continuous in its fixed circle.

Conclusion:

In this paper, concept of fixed circle in ϕ-metric space is introduced and obtain some fixed circle results. We exercise on the existence and uniqueness criteria of fixed circle for a self mapping and explain the conditions geometrically. There are some figures to show that how the shapes of circles can be changed with respect to center and radius. Some characterizations of fixed circle theorems are discussed. We think that the results of this manuscript will be helpful for the researchers for further development of fixed circle theorems and their properties.

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[1] Department of Mathematics, Siksha-Bhavana, Visva-Bharati, Santiniketan-731235, West Bengal, India.

Email address: abhishikta.math@gmail.com

[2] Department of Mathematics, Siksha-Bhavana, Visva-Bharati, Santiniketan-731235, West Bengal, India.

Email address: tarapadavb@gmail.com