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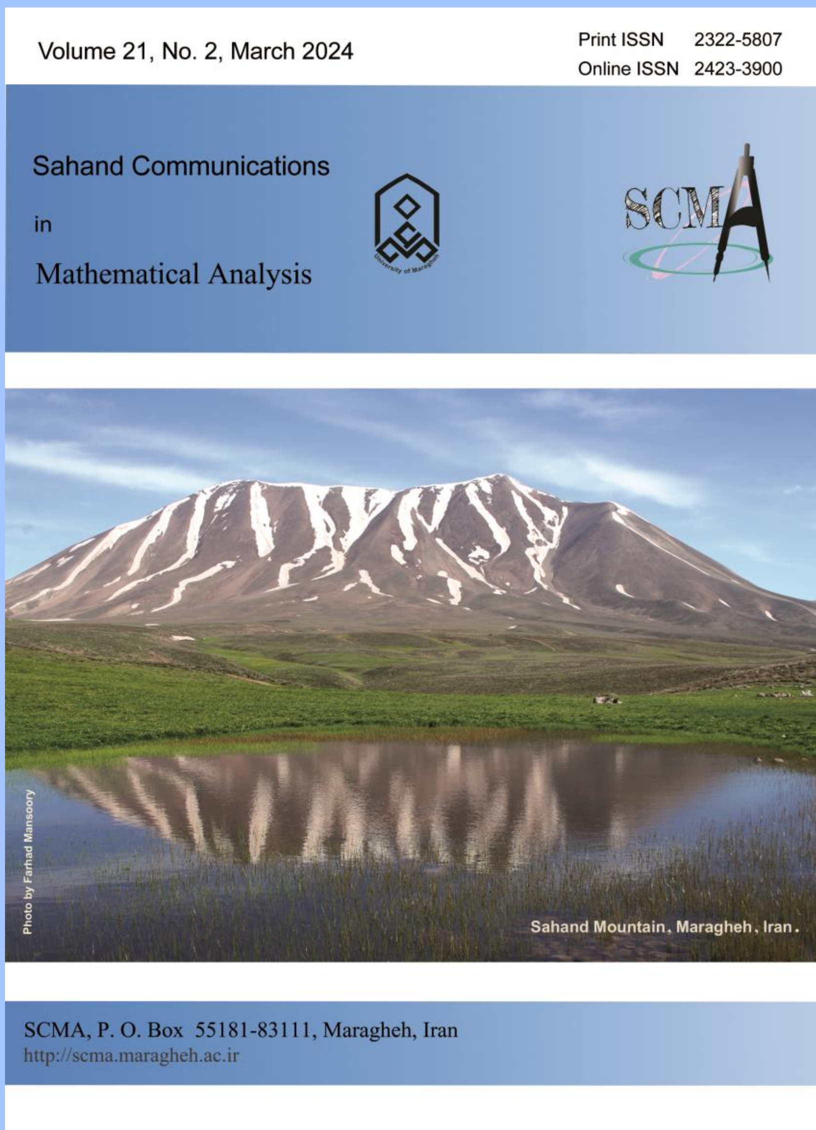

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New Results for Some Intuitionistic Fuzzy Partial Functional Differential Equations with State-Dependent Delay

Bouchra Ben Amma[1*], Said Melliani[2] and Lalla Saadia Chadli[3]

Abstract. In this research work, we investigate novel findings concerning the existence and uniqueness of intuitionistic fuzzy solutions for state-dependent delay intuitionistic fuzzy partial functional differential equations with local initial conditions in a new weighted intuitionistic fuzzy complete metric space under suitable assumptions. The main results of this paper are based on the Banach fixed point theorem. An illustrated example of our results is given with some numerical simulations for β-cuts of the intuitionistic fuzzy solutions.

1. Introduction

In 1965 Zadeh [39] introduced the concept of fuzzy sets with the purpose of modeling ambiguity, uncertainty, and vagueness in complicated systems. It can be considered an extension of the usual (crisp) set theory. It has greater flexibility to capture different aspects of incompleteness, imperfection, and uncertainty in data across many situations. The membership μ of an element of a fuzzy set is a single value within the range of 0 and 1. Therefore, in reality, it may not always be true that the degree of non-membership ν of an element in a fuzzy set is equal to 1 minus the membership degree (1-μ) because there may be some uncertainty degree. Thus, since the fuzzy set has no means to characterize the neutral state, neither support nor opposition, then Atanassov [1] included the non-membership function and defined the degree of uncertainty as 1-μ-ν. He introduced the topic of intuitionistic fuzzy sets (IFSs) as an

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extension of the standard fuzzy sets [2]. Several applications of IFS theory in diverse fields have been carried out, then they are a necessary and powerful tool for modeling imprecision, valuable applications of IFSs have been flourished in many different fields.

The concept of intuitionistic fuzzy differential equations (IFDEs) has been attracted growing interest for some time. The first attempt to treat IFDEs was made in [25]. In recent years, the authors have focused on existence-uniqueness results for intuitionistic fuzzy solutions of some types of IFDEs. They have defined the concept of intuitionistic fuzzy solutions and introduced conditions for existence and uniqueness results using different techniques. However, in [29], the authors proved the existence and uniqueness theorem of an intuitionistic fuzzy solutions for IFDEs with nonlocal initial conditions using the notion of intuitionistic fuzzy semigroup and the contraction mapping principle. In [15], the authors considered the existence of an intuitionistic fuzzy solution to the Cauchy problem for complex IFDEs. In [22–24, 30] the authors showed that intuitionistic fuzzy fractional boundary value problem has an unique intuitionistic fuzzy solution under some hypothesises. In [7] the authors established the existence and uniqueness results for the Cauchy problem of IFDEs. In [32], they studied the existence of intuitionistic fuzzy Dirichlet problem. The literature related to intuitionistic fuzzy functional differential equations with delay is very limited: to the best of our knowledge, there is only one work [8] from 2018, in which they first introduced intuitionistic fuzzy functional differential equations with delay, and they proved local and global existence and uniqueness results for intuitionistic fuzzy solutions. For the local existence and uniqueness they used the method of successive approximations, and for global existence and uniqueness they applied the contraction principle. This type of delay brings additional difficulties in proving such basic properties of solutions as uniqueness and continuous dependence on the initial condition. Moreover, the topic of numerical methods for solving intuitionistic fuzzy differential equations has been rapidly growing in recent years: see for instance [4–6, 9, 12, 31].

Banach's fixed point theorem gives a general criterion for the iteration procedure of a function to yield a fixed point. As a result, many researchers introduced several fixed point theorems for a given function under different conditions and spaces. The subject of fuzzy metric space can be considered a simple function in the study of fuzzy and intuitionistic fuzzy sets. The idea of fuzzy metric space which may be seen as a generalization of the statistical metric space, was first suggested by Kramosil and Michalek [21] in 1975 and in 2012 M. Verma and R. S. Chandel [38] established theorem for absorbing mappings in complete intuitionistic fuzzy metric space. Consequently, various authors have come up with a number of distinct definitions for this topic using diverse methods [18, 19, 33–37].

The concept of intuitionistic fuzzy partial differential equations was first initiated in 2000 [27], intuitionistic fuzzy partial differential equations with integral boundary conditions were first studied in 2019 [10], in which the authors obtained sufficient assumptions for the existence and uniqueness results and introduced a method of steps to find intuitionistic fuzzy solutions of such types of equations. As far as we know, the evolution of research on state-dependent delay problems of intuitionistic fuzzy partial functional differential equations (IFPFDEs) has been slow. The study of IFPFDEs with state-dependent delay is naturally more difficult and was initiated only recently, in 2019 [11]. In this paper, we established several existence and uniqueness results and some new notions on intuitionistic fuzzy solutions of partial differential equations with both local and nonlocal initial conditions, based on some complicated hypotheses on the domain. However, those assumptions can be reduced to be milder. Motivated by this consideration, in this work, our models are embedded into a new weighted metric space of intuitionistic fuzzy valued functions, for which some fixed point theorems for contractive mappings are valid under relaxed smoothness conditions, which may be convenient for applications depending on the Lipschitz condition on the right side of the equations, without any constraints in the data. More precisely, we investigate the well-posedness for state-dependent delay intuitionistic fuzzy partial functional differential equations with local initial conditions of the following forms:

$$\begin{cases} \frac{\partial^2 \langle \psi_1, \psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega} = F_1\left(\eta,\omega,\langle\psi_1,\psi_2\rangle_{(\eta,\omega)}\right), & (\eta,\omega)\in I_a\times I_b,\\ \langle\psi_1,\psi_2\rangle(\eta,\omega)=\xi(\eta,\omega), & (\eta,\omega)\in[-\rho,0]\times[-\rho,0], \end{cases}$$

and

$$\begin{cases} \frac{\partial^2 \langle \psi_1, \psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega} = \frac{\partial(h(\eta,\omega)\langle\psi_1,\psi_2\rangle(\eta,\omega))}{\partial\omega} + F_2\left(\eta,\omega,\langle\psi_1,\psi_2\rangle_{(\eta,\omega)}\right),\\ \langle\psi_1,\psi_2\rangle(\eta,\omega)=\xi(\eta,\omega), \end{cases}$$

where $F_1, F_2 : I_a = [0, a] \times I_b = [0, b] \times C([-\rho, 0] \times [-\rho, 0], \mathrm{IF}_n) \to \mathrm{IF}_n$ is continuous and $h \in C(I_a \times I_b, \mathbb{R})$.

Over the last five decades, the majority of works are devoted to equations with state-dependent delays, which arise as models in applications from biology, population studies, mathematical epidemiology, electrodynamics, automatic and machine cutting, neural networks, remote control and economics, etc.; see [3, 14, 16, 17, 20, 26, 40] for details. They were addressed earlier in survey works on the larger area of functional differential equations, and for this reason the study of this type of equations has received considerable attention from many researchers. By combining the two aspects suggested, intuitionistic fuzzy mathematics and partial functional differential equations with state-dependent delays, we obtain intuitionistic fuzzy partial functional differential equations with state-dependent delays, which will attract an appreciable interest both in mathematics and in applications. The literature related to partial functional differential equations with state-dependent delay in the intuitionistic fuzzy case is very rare – to our knowledge, only the recent work [11] – so the study of intuitionistic fuzzy partial functional differential equations with state-dependent delay is still an untreated topic and it is amongst the main aims of our paper.

The paper is organized in this way: In Sect.2 the main properties and definitions are introduced. Sect.3 is devoted to presenting a key result for this work, which gives suitable assumptions to ensure the existence and uniqueness of intuitionistic fuzzy solutions in a new weighted metric space. In Sect.4, illustrative computational example is showed. Finally Sect.5 offers the conclusion and suggest brief future research.

2. Preliminaries

We will briefly recall some necessary preliminaries:
we denote by:

$$\mathrm{IF}_n = \left\{\langle\psi_1,\psi_2\rangle : \mathbb{R}^n \to [0,1]^2, |\forall\ \omega\in\mathbb{R}^n\ 0 \le \psi_1(\omega)+\psi_2(\omega)\le 1\right\}$$

We say $\langle\psi_1,\psi_2\rangle$ of IF_n an intuitionistic fuzzy number if it verifies the following hypothesises:

- $\langle\psi_1,\psi_2\rangle$ is normal i.e there exists $\omega_0,\omega_1\in\mathbb{R}^n$ such that $\psi_1(\omega_0)=1$ and $\psi_2(\omega_1)=1$.
- ψ_1 is fuzzy convex and ψ_2 is fuzzy concave.
- ψ_1 is upper semi-continuous and ψ_2 is lower semi-continuous.
- $supp\langle\psi_1,\psi_2\rangle = \mathrm{cl}\{\omega\in\mathbb{R}^n : |\ \psi_1(\omega)<1\}$ is bounded.

We define the upper and lower β-cuts of $\langle\psi_1,\psi_2\rangle\in\mathrm{IF}^n$, $\langle\psi_1,\psi_2\rangle$ with $\beta\in[0,1]$ by:

$$[\langle\psi_1,\psi_2\rangle]^\beta = \{\omega\in\mathbb{R}^n : \psi_2(\omega)\le 1-\beta\}$$

and

$$[\langle\psi_1,\psi_2\rangle]_\beta = \{\omega\in\mathbb{R}^n : \psi_1(\omega)\ge \beta\}.$$

We define $0_{(1,0)}\in IF_n$ as

$$0_{(1,0)}(\zeta) = \begin{cases}(1,0) & \zeta = 0\\ (0,1) & \zeta\neq 0.\end{cases}$$

Let $\zeta \in \mathbb{R}$ and $\langle \psi_1, \psi_2 \rangle$, $\langle \psi_1', \psi_2' \rangle \in \mathrm{IF}_n$, we define the following operations by:

$$\begin{aligned} &\left(\langle \psi_1, \psi_2 \rangle + \langle \psi_1', \psi_2' \rangle\right)(\zeta) \\ &\qquad = \left(\sup_{\zeta = \omega + \eta} \min\left(\psi_1(\omega), \psi_1'(\eta)\right) \inf_{\zeta = \omega + \eta} \max\left(\psi_2(\omega), \psi_2'(\eta)\right) \right) \end{aligned}$$

and

$$\zeta \langle \psi_1, \psi_2 \rangle = \begin{cases} \langle \zeta\psi_1, \zeta\psi_2 \rangle & \text{if } \zeta \neq 0 \\ 0_{(1,0)} & \text{if } \zeta = 0. \end{cases}$$

For $\zeta \in \mathbb{R}$ and $\langle \psi_1, \psi_2 \rangle$, $\langle \psi_1', \psi_2' \rangle \in \mathrm{IF}_n$, therefore:

$$\begin{aligned} &\left[\langle \psi_1, \psi_2 \rangle + \langle \psi_1', \psi_2' \rangle\right]^\beta = \left[\langle \psi_1, \psi_2 \rangle\right]^\beta + \left[\langle \psi_1', \psi_2' \rangle\right]^\beta \\ &\left[\langle \psi_1, \psi_2 \rangle + \langle \psi_1', \psi_2' \rangle\right]_\beta = \left[\langle \psi_1, \psi_2 \rangle\right]_\beta + \left[\langle \psi_1', \psi_2' \rangle\right]_\beta \\ &\left[\zeta \langle \psi_1, \psi_2 \rangle\right]^\beta = \zeta \left[\langle \psi_1, \psi_2 \rangle\right]^\beta \\ &\left[\zeta \langle \psi_1, \psi_2 \rangle\right]_\beta = \zeta \left[\langle \psi_1, \psi_2 \rangle\right]_\beta . \end{aligned}$$

Definition 2.1. *Let $\langle \psi_1, \psi_2 \rangle \in IF_n$ and $\beta \in [0,1]$, we define sets as follows:*

$$\begin{aligned} &\left[\langle \psi_1, \psi_2 \rangle\right]_l^+ (\beta) = \inf\{\omega \in \mathbb{R}^n \mid \psi_1(\omega) \geq \beta\} \\ &\left[\langle \psi_1, \psi_2 \rangle\right]_r^+ (\beta) = \sup\{\omega \in \mathbb{R}^n \mid \psi_1(\omega) \geq \beta\} \\ &\left[\langle \psi_1, \psi_2 \rangle\right]_l^- (\beta) = \inf\{\omega \in \mathbb{R}^n \mid \psi_2(\omega) \leq 1 - \beta\} \\ &\left[\langle \psi_1, \psi_2 \rangle\right]_r^- (\beta) = \sup\{\omega \in \mathbb{R}^n \mid \psi_2(\omega) \leq 1 - \beta\}. \end{aligned}$$

Remark 2.1.

$$\begin{aligned} &\left[\langle \psi_1, \psi_2 \rangle\right]_\beta = \left[\left[\langle \psi_1, \psi_2 \rangle\right]_l^+ (\beta), \left[\langle \psi_1, \psi_2 \rangle\right]_r^+ (\beta)\right] \\ &\left[\langle \psi_1, \psi_2 \rangle\right]^\beta = \left[\left[\langle \psi_1, \psi_2 \rangle\right]_l^- (\beta), \left[\langle \psi_1, \psi_2 \rangle\right]_r^- (\beta)\right]. \end{aligned}$$

Proposition 2.1 ([28])**.** *For all β_1, $\beta_2 \in [0,1]$ and $\langle \psi_1, \psi_2 \rangle \in IF_n$*

(1) $\left[\langle \psi_1, \psi_2 \rangle\right]_{\beta_1} \subset \left[\langle \psi_1, \psi_2 \rangle\right]^{\beta_1}$.
(2) $\left[\langle \psi_1, \psi_2 \rangle\right]_{\beta_1}$ *and* $\left[\langle \psi_1, \psi_2 \rangle\right]^{\beta_1}$ *are nonempty compact convex sets in $\mathbb{R}^n$.*
(3) *If $\beta_1 \leq \beta_2$ then $\left[\langle \psi_1, \psi_2 \rangle\right]_{\beta_2} \subset \left[\langle \psi_1, \psi_2 \rangle\right]_{\beta_1}$ and $\left[\langle \psi_1, \psi_2 \rangle\right]^{\beta_2} \subset \left[\langle \psi_1, \psi_2 \rangle\right]^{\beta_1}$.*
(4) *If $\beta_n \nearrow \beta_1$, then:*

$$\left[\langle \psi_1, \psi_2 \rangle\right]_{\beta_1} = \bigcap_n \left[\langle \psi_1, \psi_2 \rangle\right]_{\beta_n}$$

and

$$\left[\langle \psi_1, \psi_2 \rangle\right]^{\beta_1} = \bigcap_n \left[\langle \psi_1, \psi_2 \rangle\right]^{\beta_n}.$$

Let $\beta \in [0,1]$ and N any set, we denote by

$$N_\beta = \{\omega \in \mathbb{R}^n : \psi_1(\omega) \geq \beta\}$$

and

$$N^\beta = \{\omega \in \mathbb{R}^n : \psi_2(\omega) \leq 1-\beta\}.$$

Lemma 2.1 ([28])**.** *Let $\{N_\beta,\ \beta \in [0,1]\}$ and $\left\{N^\beta,\ \beta \in [0,1]\right\}$ two families of subsets of $\mathbb{R}^n$ verifies (1)–(3) in proposition 2.1, if ψ_1 and ψ_2 define as follows:*

$$\psi_1(\omega) = \begin{cases} 0 & \text{if } \omega \notin N_0 \\ \sup\{\beta \in [0,1] :\ \omega \in N_\beta\} & \text{if } \omega \in N_0 \end{cases}$$

$$\psi_1(\omega) = \begin{cases} 1 & \text{if } \omega \notin N_0 \\ 1-\sup\{\beta \in [0,1] :\ \omega \in N_\beta\} & \text{if } \omega \in N_0. \end{cases}$$

Therefore $\langle \psi_1, \psi_2 \rangle \in IF_n$.

We consider the following metric on the space IF_n:

$$\begin{aligned} d_\infty^n\left(\langle\psi_1,\psi_2\rangle, \langle\psi_1',\psi_2'\rangle\right) = \frac{1}{4}\sup_{0<\beta\leq 1}&\left\|[\langle\psi_1,\psi_2\rangle]_r^+(\beta) - \left[\langle\psi_1',\psi_2'\rangle\right]_r^+(\beta)\right\| \\ &+ \frac{1}{4}\sup_{0<\beta\leq 1}\left\|[\langle\psi_1,\psi_2\rangle]_l^+(\beta) - \left[\langle\psi_1',\psi_2'\rangle\right]_l^+(\beta)\right\| \\ &+ \frac{1}{4}\sup_{0<\beta\leq 1}\left\|[\langle\psi_1,\psi_2\rangle]_r^-(\beta) - \left[\langle\psi_1',\psi_2'\rangle\right]_r^-(\beta)\right\| \\ &+ \frac{1}{4}\sup_{0<\beta\leq 1}\left\|[\langle\psi_1,\psi_2\rangle]_l^-(\beta) - \left[\langle\psi_1',\psi_2'\rangle\right]_l^-(\beta)\right\| \end{aligned}$$

where $\|.\|$: The usual Euclidean norm in $\mathbb{R}^n$.

Theorem 2.1 ([28])**.** *d_∞^n define a metric on IF_n.*

Theorem 2.2 ([28])**.** *The metric space (IF_n, d_∞^n) is complete.*

We denote:

- $I_a = [0,a]$, $I_b = [0,b]$, $I_0 = [-\rho,0]\times[-\rho,0]$, $I_\rho = [-\rho,a]\times[-\rho,b]$ and $\widehat{I_0} = I_\rho \setminus (0,a]\times(0,b]$.
- $\mathcal{C}(I_j,\mathrm{IF}_n)$: The space of all continuous mappings defined over I_j into IF_n, (j=a, b, 0, ρ).

It can be shown that $(\mathcal{C}(I_\rho,\mathrm{IF}_n), D_\sigma)$ is a complete metric space for any arbitrary $\sigma>0$, where the supremum weighted metric D_σ on $\mathcal{C}(I_\rho,\mathrm{IF}_n)$ is defined by:

$$\begin{aligned} &D_\sigma\left(\langle\psi_1,\psi_2\rangle,\langle\psi_1',\psi_2'\rangle\right) \\ &\quad= \sup_{(\rho_1,\rho_2)\in I_\rho}\left\{d_\infty^n(\langle\psi_1,\psi_2\rangle(\rho_1,\rho_2),\left\langle\psi_1',\psi_2'\right\rangle(\rho_1,\rho_2))e^{-\sigma(\rho_1+\rho_2)}\right\}. \end{aligned}$$

For more details see [13].

Definition 2.2 ([10]). *A mapping $F_1 : I_\rho \to IF_n$ is continuous at point $(\eta_0, \omega_0) \in I_\rho$ provided for any arbitrary $\varepsilon > 0$, there exists an $\delta(\varepsilon)$ such that*

$$d_\infty^n \left(F_1(\eta, \omega), F_1(\eta_0, \omega_0)\right) < \varepsilon$$

whenever $\max\{|\eta - \eta_0|, |\omega - \omega_0|\} < \delta(\varepsilon)$, $\forall\ (\eta, \omega) \in I_\rho$.

Definition 2.3 ([10]). *A map $F_1 : I_\rho \times IF_n \to IF_n$ is continuous at point $(\eta_0, \omega_0, \langle\psi_1, \psi_2\rangle_0) \in I_\rho \times IF_n$ provided for any arbitrary $\varepsilon > 0$, there exists an $\delta(\varepsilon)$ such that*

$$d_\infty^n \left(F_1\left(\eta, \omega, \langle\psi_1, \psi_2\rangle\right), F_1\left(\eta_0, \omega_0, \langle\psi_1, \psi_2\rangle_0\right)\right) < \varepsilon$$

whenever $\max\{|\eta - \eta_0|, |\omega - \omega_0|\} < \delta(\varepsilon)$ *and* $d_\infty^n (\langle\psi_1, \psi_2\rangle, \langle\psi_1, \psi_2\rangle_0) < \delta(\varepsilon)$ *for all* $(\eta, \omega) \in I_\rho$, $\langle\psi_1, \psi_2\rangle \in IF_n$.

Definition 2.4 ([11]). *A mapping $F_1 : I_a \times I_b \to IF_n$ is called strongly measurable if for $\forall\beta \in [0, 1]$ the set-valued mappings $F_{1\beta} : I_a \times I_b \to P_k(\mathbb{R}^n)$ defined by $F_{1\beta}(\eta, \omega) = [F_1(\eta, \omega)]_\beta$ and $F_1^\beta : I_a \times I_b \to P_k(\mathbb{R}^n)$ defined by $F_1^\beta(\eta, \omega) = [F_1(\eta, \omega)]^\beta$ are (Lebesgue) measurable.*

Definition 2.5 ([11]). *$F_1 : I_a \times I_b \to IF_n$ is integrably bounded if there exists an integrable function $g : I_a \times I_b \to \mathbb{R}^n$ such that $\|(t, s)\| \le g(\eta, \omega)$ holds for any $(t, s) \in supp\left(F_1(\eta, \omega)\right)$, $(\eta, \omega) \in I_a \times I_b$.*

Definition 2.6 ([11]). *Assume that $F_1 : I_a \times I_b \to IF_n$ is integrably bounded and strongly measurable for each $\beta \in (0, 1]$, then:*

$$\begin{aligned}\left[\int_0^a \int_0^b F_1(\eta, \omega)\, d\omega d\eta\right]_\beta &= \int_0^a \int_0^b \left[F_1(\eta, \omega)\right]_\beta d\omega d\eta \\ &= \int_0^a \int_0^b H(\eta, \omega)\, d\omega d\eta | H : I_a \times I_b \to \mathbb{R}^n\end{aligned}$$

is a measurable selection for $F_{1\beta}$.

$$\begin{aligned}\left[\int_0^a \int_0^b F_1(\eta, \omega)\, d\omega d\eta\right]^\beta &= \int_0^a \int_0^b \left[F_1(\eta, \omega)\right]^\beta d\omega d\eta \\ &= \int_0^a \int_0^b H(\eta, \omega)\, d\omega d\eta | H : I_a \times I_b \to \mathbb{R}^n\end{aligned}$$

is a measurable selection for F_1^β.

Let $F_1, F_2 : I_a \times I_b \to \mathrm{IF}_n$ be integrable and $k \in \mathbb{R}$. The integral has the elementary properties as follows:

- $$\int_0^a \int_0^b \left[F_1(\eta, \omega) + F_2(\eta, \omega)\right] d\omega d\eta$$

$$= \int_0^a \int_0^b F_1(\eta,\omega)\, d\omega d\eta + \int_0^a \int_0^b F_2(\eta,\omega)\, d\omega d\eta.$$

- $$\int_0^a \int_0^b kF_1(\eta,\omega)\, d\omega d\eta = k \int_0^a \int_0^b F_1(\eta,\omega)\, d\omega d\eta.$$

- $$d_\infty^n \left(\int_0^a \int_0^b F_1(\eta,\omega) d\omega d\eta, \int_0^a \int_0^b F_2(\eta,\omega) d\omega d\eta \right) \leq \int_0^a \int_0^b d_\infty^n (F_1(\eta,\omega), F_2(\eta,\omega)) d\omega d\eta.$$

Definition 2.7 ([10]). *Let $\langle \psi_1, \psi_2 \rangle$ and $\langle \psi_1', \psi_2' \rangle \in IF_n$, the Hukuhara difference is the intuitionistic fuzzy number $\langle \varphi, \varphi' \rangle \in IF_n$, if it exists, such that*

$$\langle \psi_1, \psi_2 \rangle - \langle \psi_1', \psi_2' \rangle = \langle \varphi, \varphi' \rangle \quad \Leftrightarrow \quad \langle \psi_1, \psi_2 \rangle = \langle \psi_1', \psi_2' \rangle + \langle \varphi, \varphi' \rangle.$$

Definition 2.8 ([11]). *Let $F_1 : [t_1, t_2] \to IF_n$ be differentiable and suppose that the derivative h' is integrable over $[t_1, t_2]$, if the Hukuhara difference $F_1(t_2) - F_1(t_1)$ exists in IF_n, then we have $\int_{t_1}^{t_2} F_1'(t)dt = F_1(t_2) - F_1(t_1)$.*

Definition 2.9 ([10]). *Let $F_1 : I_a \times I_b \to IF_n$. The intuitionistic fuzzy partial derivative of F_1 with respect to η at the point $(\eta_0, \omega_0) \in I_a \times I_b$ is the intuitionistic fuzzy quantity $\frac{\partial F_1(\eta_0,\omega_0)}{\partial \eta} \in IF_n$ if there exists, such that for all $\triangle k > 0$ sufficiently small, the H-difference $F_1(\eta_0 + \triangle k, \omega_0) - F_1(\eta_0, \omega_0)$ and $F_1(\eta_0, \omega_0) - F_1(\eta_0 - \triangle k, \omega_0)$ exist in IF_n and the limits*

$$\frac{\partial F_1(\eta_0,\omega_0)}{\partial \eta} = \lim_{\triangle k \to 0^+} \frac{F_1(\eta_0 + \triangle k, \omega_0) - F_1(\eta_0,\omega_0)}{\triangle k}$$

and

$$\frac{\partial F_1(\eta_0,\omega_0)}{\partial \eta} = \lim_{\triangle k \to 0^+} \frac{F_1(\eta_0,\omega_0) - F_1(\eta_0 - \triangle k, \omega_0)}{\triangle k}.$$

The intuitionistic fuzzy partial derivative of F_1 with respect to ω at the point $(\eta_0, \omega_0) \in I_a \times I_b$ and higher order of intuitionistic fuzzy partial derivative of F_1 are defined similarly.

3. The Main Results

We establish the existence-uniqueness result for the intuitionistic fuzzy partial functional differential equations with state-dependent delay in form:

$$(3.1)\quad \begin{cases} \frac{\partial^2 \langle\psi_1,\psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega} = F_1\left(\eta,\omega,\langle\psi_1,\psi_2\rangle_{(\eta,\omega)}\right), & (\eta,\omega)\in I_a\times I_b, \\ \langle\psi_1,\psi_2\rangle(\eta,\omega)=\xi(\eta,\omega), & (\eta,\omega)\in \hat{I}_\rho, \\ \langle\psi_1,\psi_2\rangle(\eta,0)=g_1(\eta), & \eta\in I_a \\ \langle\psi_1,\psi_2\rangle(0,\omega)=g_2(\omega), & \omega\in I_b \\ \xi(0,0)=g_1(0)=g_2(0) \end{cases}$$

where $F_1 : I_a \times I_b \times C(I_0, \mathrm{IF}_n) \to \mathrm{IF}_n$, $\xi \in C(I_0, \mathrm{IF}_n)$, $g_1 \in \mathcal{C}(I_a, \mathrm{IF}_n)$, $g_2 \in \mathcal{C}(I_b, \mathrm{IF}_n)$ are given functions and we define the state-dependent delays $\langle\psi_1,\psi_2\rangle_{(\eta,\omega)}(\delta,\tau)$ by:

$$\langle\psi_1,\psi_2\rangle_{(\eta,\omega)}(\delta,\tau)=\langle\psi_1,\psi_2\rangle(\eta+\delta,\omega+\delta),\quad (\delta,\tau)\in I_0.$$

Definition 3.1. *A function $\langle\psi_1,\psi_2\rangle \in C(I_\rho, IF_n)$ can be an intuitionistic fuzzy solution of the model (3.1) if it verifies*

$$\langle\psi_1,\psi_2\rangle = g_1(\eta)+g_2(\omega)-\xi(0,0)+\int_0^\eta\int_0^\omega F_1(\delta,\tau,\langle\psi_1,\psi_2\rangle_{(\delta,\tau)})d\delta d\delta$$

if $(\eta,\omega)\in I_a\times I_b$, and

$$\langle\psi_1,\psi_2\rangle=\xi(\eta,\omega),\quad \textit{if}\quad (\eta,\omega)\in\widehat{I}_\rho.$$

Theorem 3.1. *Assume that*

(i) *$F_1 : I_\rho \times C(I_0, IF_n) \to IF_n$ is continuous.*

(ii) *For $(\eta,\omega)\in I_a\times I_b$, $\langle\psi_1,\psi_2\rangle, \langle\psi_1',\psi_2'\rangle \in C(I_\rho, IF_n)$ and $\rho_1,\rho_2 \in C(I_0, IF_n)$, we have*

$$d_\infty^n\left(F_1\left(\eta,\omega,\langle\psi_1,\psi_2\rangle_{(\eta,\omega)}\right), F_1\left(\eta,\omega,\left\langle\psi_1',\psi_2'\right\rangle_{(\eta,\omega)}\right)\right) \le L d_\infty^n\left(\langle\psi_1,\psi_2\rangle(\eta+\rho_1,\omega+\rho_2), \left\langle\psi_1',\psi_2'\right\rangle(\eta+\rho_1,\omega+\rho_2)\right)$$

where $L > 0$ is a given constant.

Therefore, for all $\sigma > \sqrt{L}$, the problem (3.1) has a unique intuitionistic fuzzy solution in $C(I_\rho, IF_n)$ with the metric D_σ.

Proof. Consider the operator $M_1 : \mathcal{C}(I_\rho, \mathrm{IF}_n) \longrightarrow \mathcal{C}(I_\rho, \mathrm{IF}_n)$ defined by:

$$M_1(\langle\psi_1,\psi_2\rangle)(\eta,\omega) := \begin{cases} \xi(\eta,\omega), & (\eta,\omega)\in\hat{I}_\rho, \\ g_1(\eta)+g_2(\omega)-\xi(0,0) + \int_0^\eta\int_0^\omega F_1\left(\delta,\tau,\langle\psi_1,\psi_2\rangle_{(\delta,\tau)}\right)d\tau d\delta. & (\eta,\omega)\in I_a\times I_b, \end{cases}$$

We prove that M_1 is a contraction operator.
Indeed, let $\langle\psi_1,\psi_2\rangle, \langle\psi_1',\psi_2'\rangle \in \mathcal{C}(I_\rho, \mathrm{IF}_n)$ and $(\eta,\omega)\in I_a\times I_b$, then

$$M_1(\langle\psi_1,\psi_2\rangle)(\eta,\omega) := g_1(\eta)+g_2(\omega)-\xi(0,0) + \int_0^\eta\int_0^\omega F_1\left(\delta,\tau,\langle\psi_1,\psi_2\rangle_{(\delta,\tau)}\right)d\tau d\delta,$$

and

$$M_1\left(\left\langle\psi_1',\psi_2'\right\rangle\right)(\eta,\omega) := g_1(\eta)+g_2(\omega)-\xi(0,0) + \int_0^\eta\int_0^\omega F_1\left(\delta,\tau,\left\langle\psi_1',\psi_2'\right\rangle_{(\delta,\tau)}\right)d\tau d\delta.$$

Then,

(3.2)

$$\begin{aligned}
&d_\infty^n\left(M_1\left(\langle\psi_1,\psi_2\rangle\right)(\eta,\omega), M_1\left(\left\langle\psi_1',\psi_2'\right\rangle\right)(\eta,\omega)\right)\\
&= d_\infty^n\Bigg(g_1(\eta)+g_2(\omega)-\xi(0,0)+\int_0^\eta\int_0^\omega F_1\left(\delta,\tau,\langle\psi_1,\psi_2\rangle_{(\delta,\tau)}\right)d\tau d\delta,\\
&\quad g_1(\eta)+g_2(\omega)-\xi(0,0)\int_0^\eta\int_0^\omega F_1\left(\delta,\tau,\left\langle\psi_1',\psi_2'\right\rangle_{(\delta,\tau)}\right)d\tau d\delta\Bigg)\\
&= d_\infty^n\Bigg(g_1(\eta)+g_2(\omega)-\xi(0,0)\\
&\quad+\int_0^\eta\int_0^\omega F_1(\delta,\tau,\langle\psi_1,\psi_2\rangle(\delta+\rho_1,\tau+\rho_2))d\tau d\delta, g_1(\eta)+g_2(\omega)\\
&\quad-\xi(0,0)+\int_0^\eta\int_0^\omega F_1\left(\delta,\tau,\left\langle\psi_1',\psi_2'\right\rangle(\delta+\rho_1,\tau+\rho_2)\right)d\tau d\delta\Bigg)\\
&\le \int_0^\eta\int_0^\omega d_\infty^n\Big(F_1(\delta,\tau,\psi_1,\psi_2\rangle\langle(\delta+\rho_1,\tau+\rho_2)),\\
&\quad F_1(\delta,\tau,\left\langle\psi_1',\psi_2'\right\rangle(\delta+\rho_1,\tau+\rho_2))\Big)d\tau d\delta\\
&\le L\int_0^\eta\int_0^\omega d_\infty^n(\langle\psi_1,\psi_2\rangle(\delta+\rho_1,\tau+\rho_2),\left\langle\psi_1',\psi_2'\right\rangle(\delta+\rho_1,\tau+\rho_2))d\tau d\delta\\
&\le LD_\sigma\left(\langle\psi_1,\psi_2\rangle,\left\langle\psi_1',\psi_2'\right\rangle\right)\int_0^\eta\int_0^\omega e^{\sigma(\rho_1+\rho_2+\delta+\tau)}d\tau d\delta\\
&\le LD_\sigma\left(\langle\psi_1,\psi_2\rangle,\left\langle\psi_1',\psi_2'\right\rangle\right)\frac{1}{\sigma}e^{\sigma(\rho_1+\rho_2)}\int_0^\eta\left(e^{\sigma(\delta+\omega)}-e^{\omega\delta}\right)d\delta\\
&\le \frac{L}{\sigma^2}D_\sigma\left(\langle\psi_1,\psi_2\rangle,\left\langle\psi_1',\psi_2'\right\rangle\right)e^{\sigma(\rho_1+\rho_2)}(e^{\sigma\eta}-1)(e^{\sigma\omega}-1).
\end{aligned}$$

Multiplying (3.2) by $e^{-\sigma(\eta+\omega)}$, and taking supremum metric, we obtain:

$$
\begin{aligned}
&e^{-\sigma(\eta+\omega)} d_\infty^n \left(M_1(\langle \psi_1, \psi_2 \rangle)(\eta, \omega), M_1\left(\langle \psi_1', \psi_2' \rangle\right)(\eta, \omega)\right) \\
&\quad \leq \frac{L}{\sigma^2} D_\sigma \left(\langle \psi_1, \psi_2 \rangle, \langle \psi_1', \psi_2' \rangle \right) e^{\sigma(\rho_1+\rho_2)} (e^{\sigma\eta} - 1)(e^{\sigma\omega} - 1) e^{-\sigma(\eta+\omega)} \\
&\quad \leq \frac{L}{\sigma^2} D_\sigma \left(\langle \psi_1, \psi_2 \rangle, \langle \psi_1', \psi_2' \rangle \right) e^{\sigma(\rho_1+\rho_2)} e^{\sigma\eta} e^{\sigma\omega} e^{-\sigma(\eta+\omega)} \\
&\quad \leq \frac{L}{\sigma^2} D_\sigma \left(\langle \psi_1, \psi_2 \rangle, \langle \psi_1', \psi_2' \rangle \right).
\end{aligned}
$$

If $(\eta, \omega) \in \widehat{I}_\rho$, then

$$
d_\infty^n \left(M_1 \left(\langle \psi_1, \psi_2 \rangle \right)(\eta, \omega), M_1 \left(\langle \psi_1', \psi_2' \rangle \right)(\eta, \omega) \right) = 0
$$

and so for each $(\eta, \omega) \in I_\rho$

$$
D_\sigma \left(M_1 \left(\langle \psi_1, \psi_2 \rangle \right), M_1 \left(\langle \psi_1', \psi_2' \rangle \right) \right) \leq \frac{L}{\sigma^2} D_\sigma \left(\langle \psi_1, \psi_2 \rangle, \langle \psi_1', \psi_2' \rangle \right). \quad \square
$$

It is easy to see that for all $\sigma > 0$ satisfying $\sigma > \sqrt{L}$, therefore M_1 is a contraction operator. Then by Banach fixed point theorem, M_1 has a unique fixed point, which is the solution of (3.1).

In second part of this section, we establish the existence of intuitionistic fuzzy solutions for the following intuitionistic fuzzy partial functional differential equations with state-dependent delay in general form:

$$
(3.3) \quad \begin{cases}
\frac{\partial^2 \langle \psi_1, \psi_2 \rangle(\eta, \omega)}{\partial\eta\partial\omega} = \frac{\partial(h(\eta,\omega)\langle \psi_1, \psi_2 \rangle(\eta,\omega))}{\partial\omega} & (\eta, \omega) \in I_a \times I_b \\
\quad + F_2 \left(\eta, \omega, \langle \psi_1, \psi_2 \rangle_{(\eta,\omega)} \right), & \\
\langle \psi_1, \psi_2 \rangle(\eta, \omega) = \xi(\eta, \omega), & (\eta, \omega) \in \hat{I}_\rho \\
\langle \psi_1, \psi_2 \rangle(\eta, 0) = g_1(\eta), & \eta \in I_a \\
\langle \psi_1, \psi_2 \rangle(0, \omega) = g_2(\omega), & \omega \in I_b \\
\xi(0, 0) = g_1(0) = g_2(0) &
\end{cases}
$$

where $F_2 : I_a \times I_b \times C(I_0, \mathrm{IF}_n) \to \mathrm{IF}_n$, ξ, g_1, g_2 are as in the problem (3.1) and $h \in C(I_a \times I_b, \mathbb{R})$.

Definition 3.2. *A function $\langle \psi_1, \psi_2 \rangle \in C(I_\rho, IF_n)$ is can be a solution to the problem (3.3) if it satisfies*

$$
\begin{aligned}
\langle \psi_1, \psi_2 \rangle = p(\eta, \omega) + \int_0^\eta h(\delta, \omega) \langle \psi_1, \psi_2 \rangle(\delta, \omega) d\delta \\
+ \int_0^\eta \int_0^\omega F_2(\delta, \tau, \langle \psi_1, \psi_2 \rangle_{(\delta,\tau)}) d\tau d\delta,
\end{aligned}
$$

where

$$p(\eta,\omega)=g_1(\eta)+g_2(\omega)-\xi(0,0)-\int_0^\eta h(\delta,0)g_1(\delta)d\delta,\quad (\eta,\omega)\in I_a\times I_b$$

and

$$\langle\psi_1,\psi_2\rangle=\xi(\eta,\omega),\quad \textit{if } (\eta,\omega)\in\widehat{I}_\rho.$$

Theorem 3.2. *Assume that*

(i) $F_2: I_\rho\times C(I_0, IF_n)\to IF_n$ *is continuous.*

(ii) *For* $(\eta,\omega)\in I_a\times I_b$, $\langle\psi_1,\psi_2\rangle,\langle\psi_1',\psi_2'\rangle\in C(I_\rho,IF_n)$ *and* $\rho_1,\rho_2\in C(I_0,IF_n)$, *we have*

$$d_\infty^n\left(F_2\left(\eta,\omega,\langle\psi_1,\psi_2\rangle_{(\eta,\omega)}\right),F_2\left(\eta,\omega,\left\langle\psi_1',\psi_2'\right\rangle_{(\eta,\omega)}\right)\right)$$
$$\leq Ld_\infty^n\left(\langle\psi_1,\psi_2\rangle(\eta+\rho_1,\omega+\rho_2),\left\langle\psi_1',\psi_2'\right\rangle(\eta+\rho_1,\omega+\rho_2)\right),$$

where $L>0$ *is a given constant. Moreover, for all* $\sigma>0$ *satisfying*

$$\sigma^2-\sigma\sup_{(\delta,\tau)\in I_a\times I_b}|h(\delta,\tau)|-L>0. \tag{3.4}$$

Therefore the problem (3.3) *has a unique intuitionistic fuzzy solution in* $C(I_\rho,IF_n)$ *with the metric* D_σ.

Proof. We will prove that the intuitionistic fuzzy solution of the problem (3.3) is fixed point of the operator $M_2:\mathcal{C}(I_\rho,\mathrm{IF}_n)\longrightarrow\mathcal{C}(I_\rho,\mathrm{IF}_n)$ defined as follows:

$$M_2(\langle\psi_1,\psi_2\rangle)(\eta,\omega)$$
$$:=\begin{cases}\xi(\eta,\omega), & (\eta,\omega)\in\hat{I}_\rho,\\ \\ p(\eta,\omega)+\int_0^\eta h(\delta,\omega)\langle\psi_1,\psi_2\rangle(\delta,\omega)d\delta & \\ \quad+\int_0^\eta\int_0^\omega F_2\left(\delta,\tau,\langle\psi_1,\psi_2\rangle_{(\delta,\tau)}\right)d\tau d\delta, & (\eta,\omega)\in I_a\times I_b\end{cases}$$

where $p(\eta,\omega)=g_1(\eta)+g_2(\omega)-\xi(0,0)-\int_0^\eta h(\delta,0)g_1(\delta)d\delta$.

To show that M_2 a contraction operator, we let $\langle\psi_1,\psi_2\rangle,\langle\psi_1',\psi_2'\rangle\in\mathcal{C}(I_\rho,\mathrm{IF}_n)$ and $(\eta,\omega)\in I_a\times I_b$, then

$$M_2(\langle\psi_1,\psi_2\rangle)(\eta,\omega):=p(\eta,\omega)+\int_0^\eta h(\delta,\omega)\langle\psi_1,\psi_2\rangle(\delta,\omega)d\delta$$
$$+\int_0^\eta\int_0^\omega F_2\left(\delta,\tau,\langle\psi_1,\psi_2\rangle_{(\delta,\tau)}\right)d\tau d\delta$$

and

$$M_2\left(\left\langle\psi_1',\psi_2'\right\rangle\right)(\eta,\omega):=p(\eta,\omega)+\int_0^\eta h(\delta,\omega)\left\langle\psi_1',\psi_2'\right\rangle(\delta,\omega)d\delta$$
$$+\int_0^\eta\int_0^\omega F_2\left(\delta,\tau,\left\langle\psi_1',\psi_2'\right\rangle_{(\delta,\tau)}\right)d\tau d\delta.$$

Then

(3.5)

$$
\begin{aligned}
&d_\infty^n(M_2(\langle\psi_1,\psi_2\rangle)(\eta,\omega), M_2\left(\langle\psi_1',\psi_2'\rangle\right)(\eta,\omega))\\
&\quad= d_\infty^n\Bigg(p(\eta,\omega)+\int_0^\eta h(\delta,\omega)\langle\psi_1,\psi_2\rangle(\delta,\omega)d\delta\\
&\qquad+\int_0^\eta\int_0^\omega F_2\left(\delta,\tau,\langle\psi_1,\psi_2\rangle_{(\delta,\tau)}\right)d\tau d\delta,\\
&\qquad+\int_0^\eta\int_0^\omega F_2\left(\delta,\tau,\langle\psi_1,\psi_2\rangle_{(\delta,\tau)}\right)d\tau d\delta, p(\eta,\omega)\\
&\qquad+\int_0^\eta h(\delta,\omega)\left\langle\psi_1',\psi_2'\right\rangle(\delta,\omega)d\delta+\int_0^\eta\int_0^\omega F_2\left(\delta,\tau,\left\langle\psi_1',\psi_2'\right\rangle_{(\delta,\tau)}\right)d\tau d\delta\Bigg)\\
&\quad\le d_\infty^n\Bigg(\int_0^\eta h(\delta,\omega)\langle\psi_1,\psi_2\rangle(\delta,\omega)d\delta+\int_0^\eta\int_0^\omega F_2\left(\delta,\tau,\langle\psi_1,\psi_2\rangle_{(\delta,\tau)}\right)d\tau d\delta,\\
&\qquad\int_0^\eta h(\delta,\omega)\left\langle\psi_1',\psi_2'\right\rangle(\delta,\omega)d\delta+\int_0^\eta\int_0^\omega F_2\left(\delta,\tau,\left\langle\psi_1',\psi_2'\right\rangle_{(\delta,\tau)}\right)d\tau d\delta\Bigg)\\
&\quad\le \sup_{(\delta,\tau)\in I_a\times I_b}|h(\delta,\tau)|\int_0^\eta d_\infty^n(\langle\psi_1,\psi_2\rangle(\delta,\omega),\left\langle\psi_1',\psi_2'\right\rangle(\delta,\omega))d\delta\\
&\qquad+\int_0^\eta\int_0^\omega F_2\left(\delta,\tau,\left\langle\psi_1',\psi_2'\right\rangle_{(\delta,\tau)}\right)d\tau d\delta.
\end{aligned}
$$

Set $q=\sup\limits_{(\delta,\tau)\in I_a\times I_b}|h(\delta,\tau)|$. One gets following assessment

$$
\begin{aligned}
\sup_{(\delta,\tau)\in I_a\times I_b}|h(\delta,\tau)|&\int_0^\eta d_\infty^n(\langle\psi_1,\psi_2\rangle(\delta,\omega),\left\langle\psi_1',\psi_2'\right\rangle(\delta,\omega))d\delta\\
&= q\int_0^\eta d_\infty^n(\langle\psi_1,\psi_2\rangle(\delta,\omega),\left\langle\psi_1',\psi_2'\right\rangle(\delta,\omega))d\delta\\
&\le qD_\sigma\left(\langle\psi_1,\psi_2\rangle,\left\langle\psi_1',\psi_2'\right\rangle\right)\int_0^\eta e^{\sigma(\delta+\omega)}d\delta\\
&\le qD_\sigma\left(\langle\psi_1,\psi_2\rangle,\left\langle\psi_1',\psi_2'\right\rangle\right)\frac{1}{\sigma}\left(e^{\sigma(\eta+\omega)}-e^{\sigma\eta}\right).
\end{aligned}
$$

Multiplying both sides of (3.5) by $e^{-\sigma(\eta+\omega)}$, then we have:

(3.6)

$$
e^{-\sigma(\eta+\omega)}\sup_{(\delta,\tau)\in I_a\times I_b}|h(\delta,\tau)|\int_0^\eta d_\infty^n(\langle\psi_1\psi_2\rangle(\delta,\omega),\left\langle\psi_1',\psi_2'\right\rangle(\delta,\omega))d\delta
$$

$$\leq qD_\sigma\left(\langle\psi_1,\psi_2\rangle,\left\langle\psi_1',\psi_2'\right\rangle\right)\frac{1}{\sigma}\left(e^{\sigma(\eta+\omega)}-e^{\sigma\eta}\right)e^{-\sigma(\eta+\omega)}$$
$$\leq \frac{q}{\sigma}D_\sigma\left(\langle\psi_1,\psi_2\rangle,\left\langle\psi_1',\psi_2'\right\rangle\right)$$

and

$$e^{-\sigma(\eta+\omega)}d_\infty^n\left(\int_0^\eta\int_0^\omega F_2(\delta,\tau,\langle\psi_1,\psi_2\rangle(\delta+\rho_1,\tau+\rho_2)),\right.$$
$$\left.\int_0^\eta\int_0^\omega F_2\left(\delta,\tau,\left\langle\psi_1',\psi_2'\right\rangle(\delta+\rho_1,\tau+\rho_2)\right)\right)d\tau d\delta$$
$$\leq\frac{L}{\sigma^2}D_\sigma\left(\langle\psi_1,\psi_2\rangle,\left\langle\psi_1',\psi_2'\right\rangle\right). \tag{3.7}$$

From (3.6) and (3.7), we obtain:

$$e^{-\sigma(\eta+\omega)}d_\infty^n(M_2(\langle\psi_1,\psi_2\rangle)(\eta,\omega),M_2\left(\left\langle\psi_1',\psi_2'\right\rangle\right)(\eta,\omega))$$
$$\leq\left(\frac{L}{\sigma^2}+\frac{q}{\sigma}\right)D_\sigma\left(\langle\psi_1,\psi_2\rangle,\left\langle\psi_1',\psi_2'\right\rangle\right).$$

If $(\eta,\omega)\in\widehat{I}_\rho$,

$$d_\infty^n(M_2(\langle\psi_1,\psi_2\rangle)(\eta,\omega),M_2\left(\left\langle\psi_1',\psi_2'\right\rangle\right)(\eta,\omega))=0.$$

Hence, for each $(\eta,\omega)\in I_\rho$, we get:

$$D_\sigma\left(M_2\left(\langle\psi_1,\psi_2\rangle\right),M_2\left(\left\langle\psi_1',\psi_2'\right\rangle\right)\right)\leq\left(\frac{L}{\sigma^2}+\frac{q}{\sigma}\right)D_\sigma\left(\langle\psi_1,\psi_2\rangle,\left\langle\psi_1',\psi_2'\right\rangle\right).$$

□

It is easy to see that for all $\sigma>0$ satisfying (3.4) we have $\left(\frac{L}{\sigma^2}+\frac{q}{\sigma}\right)<1$. Therefore, M_2 is a contraction operator. Then, M_2 has a unique fixed point, which is the solution of (3.3).

4. Application

Consider the following state-dependent delay intuitionistic fuzzy partial functional differential equation:

$$\frac{\partial^2\langle\psi_1,\psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega}=-e^{\frac{1}{2}}\langle\psi_1,\psi_2\rangle\left(\eta-\frac{1}{2},\omega-\frac{1}{2}\right)+\lambda_1\left(\eta+\frac{1}{2}\right)e^{\omega}+\lambda_2e^{\frac{1}{2}}, \tag{4.1}$$

for $(\eta,\omega)\in[0,1]\times[0,1]$, with the local initial conditions

$$\langle\psi_1,\psi_2\rangle(0,0)=\lambda_2 \tag{4.2}$$
$$\langle\psi_1,\psi_2\rangle(\eta,0)=\lambda_1\eta+\lambda_2 \tag{4.3}$$
$$\langle\psi_1,\psi_2\rangle(0,\omega)=\lambda_2 \tag{4.4}$$

where λ_1 and $\lambda_2 \in IF_1$ are triangular intuitionistic fuzzy numbers and

$$\langle \psi_1, \psi_2 \rangle (\eta, \omega) = \lambda_1 \eta \omega + \lambda_2 \omega$$

for $(\eta, \omega) \in \left[-\frac{1}{2}, 1\right] \times \left[-\frac{1}{2}, 1\right] \setminus (0,1] \times (0,1]$.
From (4.1), we have the function

$$F_1 : [0,1] \times [0,1] \times C\left(\left[-\frac{1}{2}, 0\right] \times \left[-\frac{1}{2}, 0\right], \mathrm{IF}_1\right) \to \mathrm{IF}_1$$

defined as follows:

$$\begin{aligned} &F_1\left(\eta, \omega, \langle \psi_1, \psi_2 \rangle_{(\eta,\omega)}\right) \\ &\quad = -e^{\frac{1}{2}} \langle \psi_1, \psi_2 \rangle \left(\eta - \frac{1}{2}, \omega - \frac{1}{2}\right) + \lambda_1 \left(\eta + \frac{1}{2}\right) e^{\omega} + \lambda_2 e^{\frac{1}{2}} \end{aligned}$$

verifies hypothesises (1) and (2) of the Theorem 3.1.
Indeed: It is easy to see that F_1 is continuous and

$$\begin{aligned} &d_\infty \left(F_1\left(\eta, \omega, \langle \psi_1, \psi_2 \rangle_{(\eta,\omega)}\right), F_1\left(\eta, \omega, \left\langle \psi_1', \psi_2' \right\rangle_{(\eta,\omega)}\right)\right) \\ &\quad \leq e^{\frac{1}{2}} d_\infty \left(\langle \psi_1, \psi_2 \rangle \left(\eta - \frac{1}{2}, \omega - \frac{1}{2}\right), \langle \psi_1', \psi_2' \rangle \left(\eta - \frac{1}{2}, \omega - \frac{1}{2}\right)\right). \end{aligned}$$

If we choose the positive weighted number σ verifying $\sigma > \sqrt{L}$ with $L = e^{\frac{1}{2}}$, so $\sigma > \sqrt{e^{\frac{1}{2}}}$. This leads to the verification of all the assumptions of Theorem 3.1; therefore, we have a unique intuitionistic fuzzy solution of (4.1)-(4.4).

Now we apply the method of steps proposed in [10, 11] to find the solution of the problem (4.1)-(4.4).

The classical solution of the crisp equation is:

$$v = c_1 \eta e^{\omega} + c_2.$$

We assume that the parametric forms of the intuitionistic fuzzy numbers $\lambda_i (i = 1, 2)$ are

$$\begin{aligned} [\lambda_i]_\beta &= [\lambda_{il}^+(\beta), \lambda_{ir}^+(\beta)] \\ [\lambda_i]^\beta &= [\lambda_{il}^-(\beta), \lambda_{ir}^-(\beta)] \end{aligned}$$

where they satisfy the assumptions of Lemma (2.1).
Additionally, the parametric forms of the function F_1 are defined as follows:

$$\begin{aligned} [F_1]_\beta &= \left[F_{1l}^+(\beta), F_{1r}^+(\beta)\right] \\ [F_1]^\beta &= \left[F_{1l}^-(\beta), F_{1r}^-(\beta)\right] \end{aligned}$$

where

$$F_{1l}^{+}(\beta) = -e^{\frac{1}{2}}\left[\langle\psi_1,\psi_2\rangle\left(\eta-\frac{1}{2},\omega-\frac{1}{2}\right)\right]_l^{+}(\beta) + \lambda_{1l}^{+}(\beta)\left(\eta+\frac{1}{2}\right)e^{\omega} + \lambda_{2l}^{+}(\beta)e^{\frac{1}{2}}$$

$$F_{1r}^{+}(\beta) = -e^{\frac{1}{2}}\left[\langle\psi_1,\psi_2\rangle\left(\eta-\frac{1}{2},\omega-\frac{1}{2}\right)\right]_r^{+}(\beta) + \lambda_{1r}^{+}(\beta)\left(\eta+\frac{1}{2}\right)e^{\omega} + \lambda_{2r}^{+}(\beta)e^{\frac{1}{2}}$$

$$F_{1l}^{-}(\beta) = -e^{\frac{1}{2}}\left[\langle\psi_1,\psi_2\rangle\left(\eta-\frac{1}{2},\omega-\frac{1}{2}\right)\right]_l^{-}(\beta) + \lambda_{1l}^{-}(\beta)\left(\eta+\frac{1}{2}\right)e^{\omega} + \lambda_{2l}^{-}(\beta)e^{\frac{1}{2}}$$

$$F_{1r}^{-}(\beta) = -e^{\frac{1}{2}}\left[\langle\psi_1,\psi_2\rangle\left(\eta-\frac{1}{2},\omega-\frac{1}{2}\right)\right]_r^{-}(\beta) + \lambda_{1r}^{-}(\beta)\left(\eta+\frac{1}{2}\right)e^{\omega} + \lambda_{2r}^{-}(\beta)e^{\frac{1}{2}}.$$

If

$$[\langle\psi_1,\psi_2\rangle(\eta,\omega)]_{\beta} = \left[[\langle\psi_1,\psi_2\rangle(\eta,\omega)]_l^{+}(\beta), [\langle\psi_1,\psi_2\rangle(\eta,\omega)]_r^{+}(\beta)\right]$$
$$[\langle\psi_1,\psi_2\rangle(\eta,\omega)]^{\beta} = \left[[\langle\psi_1,\psi_2\rangle(\eta,\omega)]_l^{-}(\beta), [\langle\psi_1,\psi_2\rangle(\eta,\omega)]_r^{-}(\beta)\right].$$

Then,

$$\left[\frac{\partial^2\langle\psi_1,\psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega}\right]_{\beta} = \left[\left[\frac{\partial^2\langle\psi_1,\psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega}\right]_l^{+}(\beta), \left[\frac{\partial^2\langle\psi_1,\psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega}\right]_r^{+}(\beta)\right]$$
$$\left[\frac{\partial^2\langle\psi_1,\psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega}\right]^{\beta} = \left[\left[\frac{\partial^2\langle\psi_1,\psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega}\right]_l^{-}(\beta), \left[\frac{\partial^2\langle\psi_1,\psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega}\right]_r^{-}(\beta)\right].$$

Therefore, we solve the following state-dependent delay partial functional differential equations:

$$\begin{cases} \left[\frac{\partial^2\langle\psi_1,\psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega}\right]_l^+(\beta) = -e^{\frac{1}{2}}\left[\langle\psi_1,\psi_2\rangle\left(\eta-\frac{1}{2},\omega-\frac{1}{2}\right)\right]_l^+(\beta) \\ \qquad +\lambda_{1l}^+(\beta)\left(\eta+\frac{1}{2}\right)e^{\omega}+\lambda_{2l}^+(\beta)e^{\frac{1}{2}}, \quad (\eta,\omega)\in[0,1]\times[0,1] \\ \left[\frac{\partial^2\langle\psi_1,\psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega}\right]_r^+(\beta) = -e^{\frac{1}{2}}\left[\langle\psi_1,\psi_2\rangle\left(\eta-\frac{1}{2},\omega-\frac{1}{2}\right)\right]_r^+(\beta) \\ \qquad +\lambda_{1r}^+(\beta)\left(\eta+\frac{1}{2}\right)e^{\omega}+\lambda_{2r}^+(\beta)e^{\frac{1}{2}}, \quad (\eta,\omega)\in[0,1]\times[0,1] \\ \left[\frac{\partial^2\langle\psi_1,\psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega}\right]_l^-(\beta) = -e^{\frac{1}{2}}\left[\langle\psi_1,\psi_2\rangle\left(\eta-\frac{1}{2},\omega-\frac{1}{2}\right)\right]_l^-(\beta) \\ \qquad +\lambda_{1l}^-(\beta)\left(\eta+\frac{1}{2}\right)e^{\omega}+\lambda_{2l}^-(\beta)e^{\frac{1}{2}}, \quad (\eta,\omega)\in[0,1]\times[0,1] \\ \left[\frac{\partial^2\langle\psi_1,\psi_2\rangle(\eta,\omega)}{\partial\eta\partial\omega}\right]_r^-(\beta) = -e^{\frac{1}{2}}\left[\langle\psi_1,\psi_2\rangle\left(\eta-\frac{1}{2},\omega-\frac{1}{2}\right)\right]_r^-(\beta) \\ \qquad +\lambda_{1r}^-(\beta)\left(\eta+\frac{1}{2}\right)e^{\omega}+\lambda_{2r}^-(\beta)e^{\frac{1}{2}}, \quad (\eta,\omega)\in[0,1]\times[0,1]. \end{cases}$$

With initial conditions for $(\eta,\omega)\in\left[-\frac{1}{2},1\right]\times\left[-\frac{1}{2},1\right]\setminus(0,1]\times(0,1]$ we have:

$$(4.5)\qquad \begin{cases} [\langle\psi_1,\psi_2\rangle(\eta,\omega)]_l^+(\beta) = \eta\omega\lambda_{1l}^+(\beta)+\omega\lambda_{2l}^+(\beta)(\beta) \\ [\langle\psi_1,\psi_2\rangle(\eta,\omega)]_r^+(\beta) = \eta\omega\lambda_{1r}^+(\beta)+\omega\lambda_{2r}^+(\beta) \\ [\langle\psi_1,\psi_2\rangle(\eta,\omega)]_l^-(\beta) = \eta\omega\lambda_{1l}^-(\beta)+\omega\lambda_{2l}^-(\beta) \\ [\langle\psi_1,\psi_2\rangle(\eta,\omega)]_r^-(\beta) = \eta\omega\lambda_{1r}^-(\beta)+\omega\lambda_{2r}^-(\beta) \end{cases}$$

and for $(\eta,\omega)\in[0,1]\times[0,1]$

$$
(4.6)\qquad \begin{cases}
[\langle\psi_1,\psi_2\rangle(0,0)]_l^+(\beta)=\lambda_{2l}^+(\beta)\\
[\langle\psi_1,\psi_2\rangle(0,0)]_r^+(\beta)=\lambda_{2r}^+(\beta)\\
[\langle\psi_1,\psi_2\rangle(0,0)]_l^-(\beta)=\lambda_{2l}^-(\beta)\\
[\langle\psi_1,\psi_2\rangle(0,0)]_r^-(\beta)=\lambda_{2r}^-(\beta)\\
[\langle\psi_1,\psi_2\rangle(\eta,0)]_l^+(\beta)=\eta\lambda_{1l}^+(\beta)+\lambda_{2l}^+(\beta)\\
[\langle\psi_1,\psi_2\rangle(\eta,0)]_r^+(\beta)=\eta\lambda_{1r}^+(\beta)+\lambda_{2r}^+(\beta)\\
[\langle\psi_1,\psi_2\rangle(\eta,0)]_l^-(\beta)=\eta\lambda_{1l}^-(\beta)+\lambda_{2l}^-(\beta)\\
[\langle\psi_1,\psi_2\rangle(\eta,0)]_r^-(\beta)=\eta\lambda_{1r}^-(\beta)+\lambda_{2r}^-(\beta)\\
[\langle\psi_1,\psi_2\rangle(0,\omega)]_l^+(\beta)=\lambda_{2l}^+(\beta)\\
[\langle\psi_1,\psi_2\rangle(0,\omega)]_r^+(\beta)=\lambda_{2r}^+(\beta)\\
[\langle\psi_1,\psi_2\rangle(0,\omega)]_l^-(\beta)=\lambda_{2l}^-(\beta)\\
[\langle\psi_1,\psi_2\rangle(0,\omega)]_r^-(\beta)=\lambda_{2r}^-(\beta).
\end{cases}
$$

We obtain,

$$
\begin{cases}
[\langle\psi_1,\psi_2\rangle(\eta,\omega)]_l^+(\beta)=\eta e^{\omega}\lambda_{1l}^+(\beta)+\lambda_{2l}^+(\beta), & (\eta,\omega)\in[0,1]\times[0,1]\\
[\langle\psi_1,\psi_2\rangle(\eta,\omega)]_r^+(\beta)=\eta e^{\omega}\lambda_{1r}^+(\beta)+\lambda_{2r}^+(\beta), & (\eta,\omega)\in[0,1]\times[0,1]\\
[\langle\psi_1,\psi_2\rangle(\eta,\omega)]_l^-(\beta)=\eta e^{\omega}\lambda_{1l}^-(\beta)+\lambda_{2l}^-(\beta), & (\eta,\omega)\in[0,1]\times[0,1]\\
[\langle\psi_1,\psi_2\rangle(\eta,\omega)]_r^-(\beta)=\eta e^{\omega}\lambda_{1r}^-(\beta)+\lambda_{2r}^-(\beta), & (\eta,\omega)\in[0,1]\times[0,1].
\end{cases}
$$

Then,

$$
[\langle\psi_1,\psi_2\rangle(\eta,\omega)]_\beta=\left[\eta e^{\omega}\lambda_{1l}^+(\beta)+\lambda_{2l}^+(\beta),\eta e^{\omega}\lambda_{1r}^+(\beta)+\lambda_{2r}^+(\beta)\right]
$$
$$
[\langle\psi_1,\psi_2\rangle(\eta,\omega)]^\beta=\left[\eta e^{\omega}\lambda_{1l}^-(\beta)+\lambda_{2l}^-(\beta),\eta e^{\omega}\lambda_{1r}^-(\beta)+\lambda_{2r}^-(\beta)\right].
$$

We denote the families N_β and N^β as follows:

$$\left[\eta e^\omega \lambda_{1l}^+(\beta) + \lambda_{2l}^+(\beta), \eta e^\omega \lambda_{1r}^+(\beta) + \lambda_{2r}^+(\beta)\right] = N_\beta$$
$$\left[\eta e^\omega \lambda_{1l}^-(\beta) + \lambda_{2l}^-(\beta), \eta e^\omega \lambda_{1r}^-(\beta) + \lambda_{2r}^-(\beta)\right] = N^\beta$$

and the families N'_β and N'^β by:

$$\left[e^\omega \lambda_{1l}^+(\beta), e^\omega \lambda_{1r}^+(\beta)\right] = N'_\beta$$
$$\left[e^\omega \lambda_{1l}^-(\beta), e^\omega \lambda_{1r}^-(\beta)\right] = N'^\beta.$$

It easy to see that (N_β, N^β) and (N'_β, N'^β) satisfy the conditions of proposition 2.1; by applying the Lemma 2.1, for every $\beta \in [0,1]$ we construct the intuitionistic fuzzy solution $\langle\psi_1,\psi_2\rangle(\eta,\omega) \in IF_1$ of (4.1)-(4.4) as follows:

$$[\langle\psi_1,\psi_2\rangle(\eta,\omega)]_\beta = \left[\eta e^\omega \lambda_{1l}^+(\beta) + \lambda_{2l}^+(\beta), \eta e^\omega \lambda_{1r}^+(\beta) + \lambda_{2r}^+(\beta)\right]$$
$$[\langle\psi_1,\psi_2\rangle(\eta,\omega)]^\beta = \left[\eta e^\omega \lambda_{1l}^-(\beta) + \lambda_{2l}^-(\beta), \eta e^\omega \lambda_{1r}^-(\beta) + \lambda_{2r}^-(\beta)\right].$$

Thus, $\langle\psi_1,\psi_2\rangle(\eta,\omega)$ is an intuitionistic fuzzy solution which verifies the local initial conditions (4.2)-(4.4) and be written as:

$$\langle\psi_1,\psi_2\rangle(\eta,\omega) = \lambda_1 \eta e^\omega + \lambda_2.$$

In Figure 1 and 2, we show the graphical representation of the membership and non-membership functions of triangular intuitionistic fuzzy numbers $\lambda_1 = (-2,0,2;-2.75,0,2.75)$ and $\lambda_2 = (-1,0,1;-0.75,0,0.75)$ and the simulation of β-cuts of the intuitionistic fuzzy solution $\langle\psi_1,\psi_2\rangle(\eta,\omega)$ at some values of (η,ω).

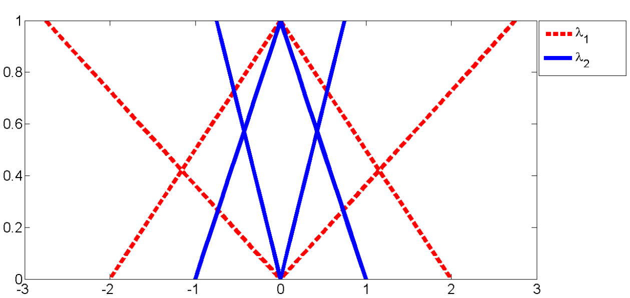

FIGURE 1. $\lambda_1 = (-2,0,2;-2.75,0,2.75)$ and $\lambda_2 = (-1,0,1;-0.75,0,0.75)$

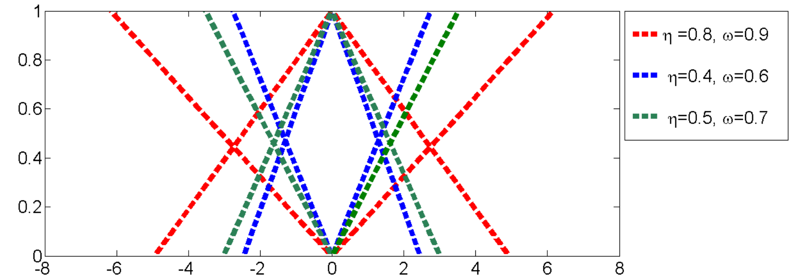

FIGURE 2. The curve of intuitionistic fuzzy solutions $\langle\psi_1, \psi_2\rangle(\eta, \omega)$ at some values of (η, ω)

In Figure 3, we show the β-cuts of the surface of intuitionistic fuzzy solution with triangular intuitionistic fuzzy numbers $\lambda_1 = (-2, 0, 2; -2.75, 0, 2.75)$ and $\lambda_2 = (-1, 0, 1; -0.75, 0, 0.75)$.

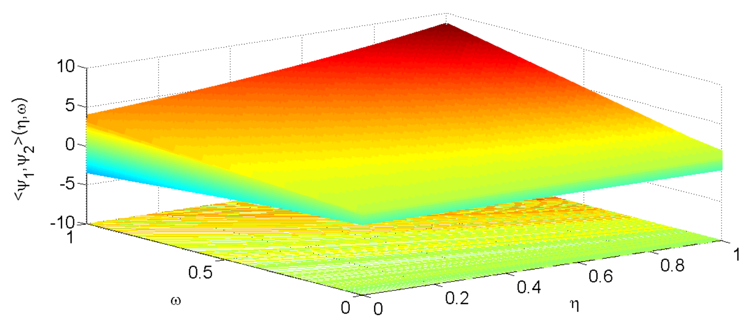

FIGURE 3. The surface of intuitionistic fuzzy solution

5. CONCLUSION

The main aims of this work have been to establish the intuitionistic fuzzy solutions of state-dependent delay partial functional differential equations with local initial conditions. By employing the Banach fixed point theorem, we have proved some new results on the existence-uniqueness of intuitionistic fuzzy solutions for some state-dependent delay partial functional differential equations with local conditions via suitable assumptions in a new weighted complete metric space. These results are illustrated by a computational example. In the next step of our future works, we can adapt these concept for studying nonlocal intuitionistic fuzzy state-dependent delay partial functional differential equations under some assumptions which are weaker than the Lipschitz condition.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Statements and Declarations

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