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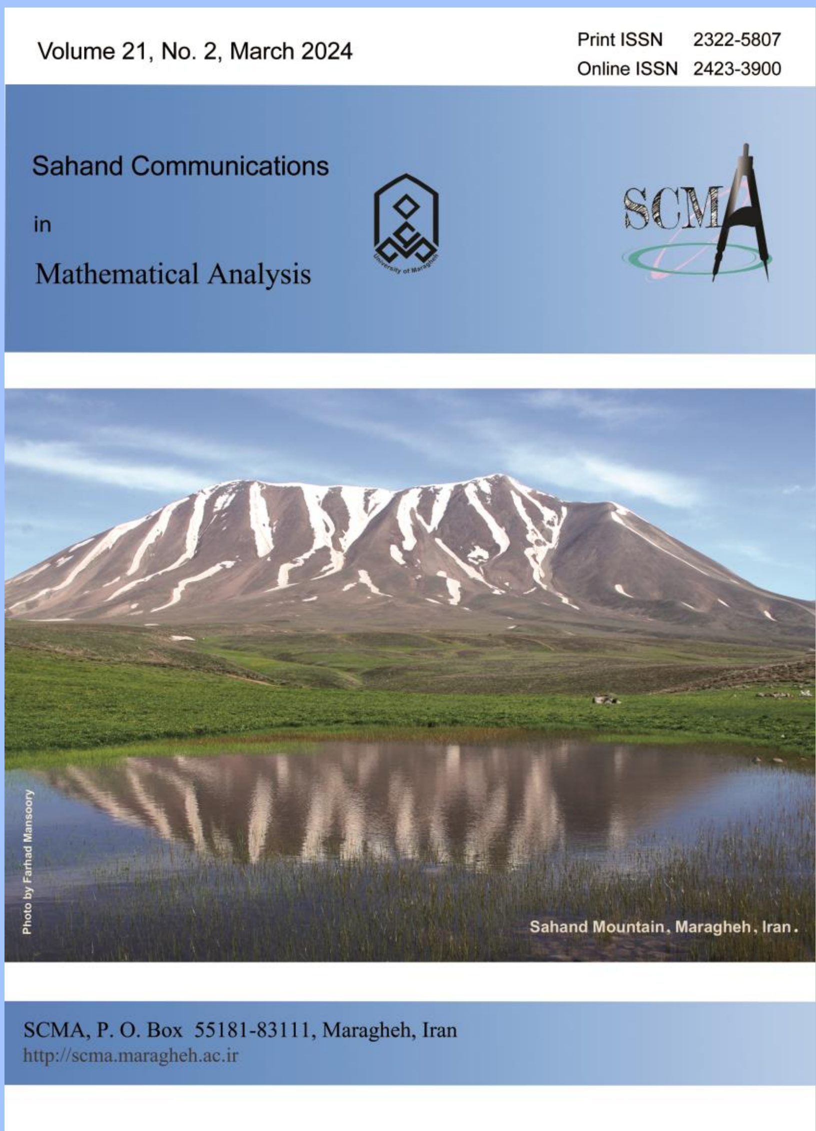
Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 2
Pages: 289-304

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.2002136.1324



Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 2 (2024), 289-304
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.2002136.1324

Companion of Ostrowski Inequality for Multiplicatively Convex Functions

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Abstract. The objective of this paper is to examine integral inequalities related to multiplicatively differentiable functions. Initially, we establish a novel identity using the two-point Newton-Cotes formula for multiplicatively differentiable functions. Using this identity, we derive Companion of Ostrowski's inequalities for multiplicatively differentiable convex mappings. The work also provides the results' applications.

1. Introduction

The concept of convexity plays a pivotal role in various fields, including mathematics, economics, optimization and game theory [32]. Convexity provides a fundamental framework for analyzing and modeling relationships between variables and it has wide-ranging applications in decision-making, resource allocation and risk management. The notion of convexity allows for the formulation of precise mathematical properties, such as monotonicity, concavity, and convex combinations, which facilitate rigorous analysis and enable the development of powerful optimization techniques. Moreover, convexity serves as a fundamental building block in the theory of inequalities, providing a basis for establishing important results and inequalities that have applications in various areas of science and engineering.

2010 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.
Key words and phrases. Non-Newtonian calculus, Companion of Ostrowski inequality, Multiplicatively convex functions.
Received: 14 May 2023, Accepted: 25 June 2023.
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It is worth noting that a function $\mathcal{F}$ is considered convex if it meets the criterion that for all $x, y \in I$ and all $t \in [0,1]$, the inequality

$$\mathcal{F}\left(tx + (1-t)\,y\right) \leq t\mathcal{F}\left(x\right) + (1-t)\,\mathcal{F}(y)$$

holds, see [32].

The fundamental inequality related to the notion of convexity is the Hermite-Hadamard inequality, which can be stated as follows: Let $\mathcal{F}$ be a convex function on $[c,d]$, then we have

$$\mathcal{F}\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \mathcal{F}\left(x\right)dx \leq \frac{\mathcal{F}\left(c\right)+\mathcal{F}\left(d\right)}{2}. \tag{1.1}$$

If the function $\mathcal{F}$ is concave, the inequality (1.1) is satisfied in the opposite direction, as stated in [32].

Concerning some papers dealing with inequality (1.1) see [16–19, 21, 23] and references therein.

In [13], Dragomir and Agarwal established the following inequalities connected with the inequality (1.1) known as trapezium-type inequalities.

$$\left|\frac{\mathcal{F}\left(c\right)+\mathcal{F}\left(d\right)}{2} - \frac{1}{d-c}\int_c^d \mathcal{F}\left(x\right)dx\right| \leq \frac{d-c}{8}\left(\left|\mathcal{F}'\left(c\right)\right| + \left|\mathcal{F}'\left(d\right)\right|\right).$$

In [31], Pearce and Pečarić provided another type of inequalities related to (1.1) known as midpoint-type inequalities.

$$\left|\mathcal{F}\left(\frac{c+d}{2}\right) - \frac{1}{d-c}\int_c^d \mathcal{F}\left(x\right)dx\right| \leq \frac{d-c}{8}\left(\left|\mathcal{F}'\left(c\right)\right| + \left|\mathcal{F}'\left(d\right)\right|\right).$$

Alomari et al. [3] gave the companion of Ostrowski's inequalities for differentiable convex functions as follows:

$$\begin{aligned}
&\left|\frac{\mathcal{F}\left(x\right)+\mathcal{F}\left(c+d-x\right)}{2} - \frac{1}{d-c}\int_c^d \mathcal{F}\left(t\right)dt\right| \\
&\quad\leq \frac{\left(x-c\right)^2}{6\left(d-c\right)}\left(\left|\mathcal{F}'\left(c\right)\right| + \left|\mathcal{F}'\left(d\right)\right|\right) \\
&\qquad+ \frac{8\left(x-c\right)^2+3\left(c+d-2x\right)^2}{24\left(d-c\right)}\left(\left|\mathcal{F}'\left(x\right)\right| + \left|\mathcal{F}'\left(c+d-x\right)\right|\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\mathcal{F}\left(x\right)+\mathcal{F}\left(c+d-x\right)}{2} - \frac{1}{d-c}\int_c^d \mathcal{F}\left(t\right)dt\right| \\
&\quad\leq \frac{1}{2^{\frac{1}{q}}\left(d-c\right)\left(p+1\right)^{\frac{1}{p}}}\left\{\left(x-c\right)^2\left(\left|\mathcal{F}'\left(c\right)\right|^q + \left|\mathcal{F}'\left(x\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad+ \frac{\left(c+d-2x\right)^2}{2}\left(\left|\mathcal{F}'\left(x\right)\right|^q + \left|\mathcal{F}'\left(c+d-x\right)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$+ (x-c)^2 \left(\left|\mathcal{F}'(c+d-x)\right|^q + \left|\mathcal{F}'(d)\right|^q\right)^{\frac{1}{q}} \bigg\}.$$

In 1967, Grossman and Katz introduced the first non-Newtonian computational system known as geometric calculus. In the subsequent years, they developed a vast range of non-Newtonian calculus, which significantly altered the classical calculus formulated by Newton and Leibniz in the 17th century. These modified calculi, collectively referred to as the non-Newtonian calculus or multiplicative calculus, deviate considerably from the traditional calculus of Newton and Leibniz. In this alternative approach, the ordinary product and ratio are respectively utilized as the equivalent of addition and exponential difference within the realm of positive real numbers (refer to [15]). This form of calculation proves valuable when dealing with exponentially varying functions.

The comprehensive mathematical formulation of multiplicative calculus was provided by Bashirov et al. in their work [5]. Additionally, in the literature, there is evidence of a similar calculation proposed by mathematical biologists Volterra and Hostinsky [34] in 1938, known as the Volterra calculus, which can be identified as a specific instance of multiplicative calculus.

The comprehensive mathematical explanation of multiplicative calculus provided by Bashirov in [5] has sparked significant interest in this type of calculation among researchers, owing to its potential for theoretical and practical applications. Several subsequent works have further explored various aspects of multiplicative calculus. Aniszewska [4] presented the multiplicative version of the Runge-Kutta method, employing it to solve differential equations in the multiplicative domain. Misirli and Gurefe [24] introduced the multiplicative Adams Bashforth-Moulton methods. Bhat et al. defined the multiplicative Fourier transform [9] and the multiplicative Sumudu transform [10]. Rıza et al. devised numerical solutions for multiplicative differential equations using the multiplicative finite difference methods [33]. Bashirov [6] explored double integrals within the framework of multiplicative calculus, while Bashirov and Norozpour [7] extended multiplicative integration to complex-valued functions. This notion of multiplicative calculus has applications in various other areas, as evident in [20, 25, 27, 28].

Definition 1.1 ([32])**.** We say that a positive function $\mathcal{F} : I \to \mathbb{R}^+$ is logarithmically convex or multiplicatively convex, if

$$\mathcal{F}(tx + (1-t)y) \;\leq [\mathcal{F}(x)]^t \,[\mathcal{F}(y)]^{1-t}$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

In a recent study, Ali et al. [1] introduced the Hermite-Hadamard inequality in relation to multiplicatively convex functions.

Theorem 1.2. *For a positive multiplicatively convex function $\mathcal{F}$ on $[c,d]$, the following inequalities hold*

$$\mathcal{F}\left(\frac{c+d}{2}\right) \leq \left(\int_c^d \mathcal{F}(x)^{dx}\right)^{\frac{1}{d-c}} \leq \sqrt{\mathcal{F}(c)\,\mathcal{F}(d)}. \tag{1.2}$$

In the same paper, the authors proved the inequalities for the product and quotient of two multiplicatively convex functions. Özcan established the Hermite-Hadamard type inequalities for multiplicatively s-convex functions in [29] and for multiplicatively h-convex functions in [30]. Ali et al. investigated Ostrowski as well as Simpson type inequalities for multiplicatively convex functions in [2], while the Dual Simpson inequalities were established by Meftah et al. in [22]. In [11], Budak et al. extended the Hermite-Hadamard inequality for multiplicative convex functions to the fractional framework and the fractional analogues of Simpson and Bullen-type inequalities for multiplicatively convex functions were established in [26] and [12], respectively.

In [8], Berehail et al. established the Midpoint- and Trapezoid-type inequalities involving multiplicative differentiable convex functions as follows:

$$\left|\mathcal{F}\left(\frac{c+d}{2}\right)\left(\int_c^d \mathcal{F}(u)^{du}\right)^{\frac{1}{c-d}}\right| \leq \left(\mathcal{F}^*(c)\left(\mathcal{F}^*\left(\frac{c+d}{2}\right)\right)^4 \mathcal{F}^*(d)\right)^{\frac{d-c}{24}}$$

and

$$\left|\sqrt{\mathcal{F}(c)\,.\mathcal{F}(d)}\left(\int_c^d \mathcal{F}(u)^{du}\right)^{\frac{1}{c-d}}\right| \leq \left(\mathcal{F}^*(c)\mathcal{F}^*(d)\right)^{\frac{d-c}{8}}.$$

Inspired by the aforementioned papers, this study aims to propose a new identity for multiplicatively differentiable functions. Building upon this identity, we establish a Companion of Ostrowski inequality specifically tailored for multiplicatively convex functions. Additionally, we provide practical applications showcasing the implications of our obtained results.

2. Preliminaries

In this section, we will start by revisiting and summarizing key definitions, properties and concepts related to derivation and multiplicative integration. This foundation will serve as a basis for our subsequent discussions and analyses.

Definition 2.1 ([5])**.** Let $\mathcal{F}:\mathbb{R}\rightarrow\mathbb{R}^{+}$ be a positive function. The definition of multiplicative derivative of $\mathcal{F}$ noted $\mathcal{F}^{*}$ is as follows

$$\frac{d^{*}\mathcal{F}}{dt}=\mathcal{F}^{*}\left(t\right)=\lim_{h\rightarrow 0}\left(\frac{\mathcal{F}\left(t+h\right)}{\mathcal{F}\left(t\right)}\right)^{\frac{1}{h}}.$$

Remark 2.2. If $\mathcal{F}$ is a positive-valued function that is differentiable at t, then the multiplicative derivative $\mathcal{F}^{*}$ also exists and we have the following relation between $\mathcal{F}^{*}$ and the ordinary derivative $\mathcal{F}'$.

$$\mathcal{F}^{*}\left(t\right)=e^{\left(\ln\mathcal{F}(t)\right)'}=e^{\frac{\mathcal{F}'(t)}{\mathcal{F}(t)}}.$$

Now we give some properties of the multiplicative derivatives.

Theorem 2.3 ([5])**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be multiplicatively differentiable functions and α be an arbitrary constant. Then the functions $\alpha\mathcal{F}$, $\mathcal{F}\mathcal{G}$, $\mathcal{F}+\mathcal{G}$, $\mathcal{F}/\mathcal{G}$ and $\mathcal{F}^{\mathcal{G}}$ are also *-differentiable and their *-derivatives can be determined as follows:*

- $\left(\alpha\mathcal{F}\right)^{*}\left(t\right)=\mathcal{F}^{*}\left(t\right),$
- $\left(\mathcal{F}\mathcal{G}\right)^{*}\left(t\right)=\mathcal{F}^{*}\left(t\right)\mathcal{G}^{*}\left(t\right),$
- $\left(\mathcal{F}+\mathcal{G}\right)^{*}\left(t\right)=\mathcal{F}^{*}\left(t\right)^{\frac{\mathcal{F}(t)}{\mathcal{F}(t)+\mathcal{G}(t)}}\mathcal{G}^{*}\left(t\right)^{\frac{\mathcal{G}(t)}{\mathcal{F}(t)+\mathcal{G}(t)}},$
- $\left(\frac{\mathcal{F}}{\mathcal{G}}\right)^{*}\left(t\right)=\frac{\mathcal{F}^{*}(t)}{\mathcal{G}^{*}(t)},$
- $\left(\mathcal{F}^{\mathcal{G}}\right)^{*}\left(t\right)=\mathcal{F}^{*}\left(t\right)^{\mathcal{G}(t)}\mathcal{F}\left(t\right)^{\mathcal{G}'(t)}.$

In their paper [5], Bashirov et al. introduced the multiplicative integral, denoted by $\int_{c}^{d}\left(\mathcal{F}\left(t\right)\right)^{dt}$, which offers an alternative approach to integration compared to the traditional Riemann integral. The relation between these two types of integrals can be described as follows:

Proposition 2.4 ([5])**.** *Every Riemann integrable function on $[c,d]$ is multiplicative integrable on $[c,d]$ and we have:*

$$\int_{c}^{d}\left(\mathcal{F}\left(t\right)\right)^{dt}=\exp\left(\int_{c}^{d}\ln\left(\mathcal{F}\left(t\right)\right)dt\right).$$

The properties of the multiplicative integral are as follows:

Theorem 2.5 ([5])**.** *Let $\mathcal{F}$ be a positive and Riemann integrable function on $[c,d]$, then $\mathcal{F}$ is multiplicative integrable on $[c,d]$ and*

- $\int_{c}^{d}\left(\left(\mathcal{F}\left(t\right)\right)^{p}\right)^{dt}=\left(\int_{c}^{d}\left(\mathcal{F}\left(t\right)\right)^{dt}\right)^{p},$
- $\int_{c}^{d}\left(\mathcal{F}\left(t\right)\mathcal{G}\left(t\right)\right)^{dt}=\int_{c}^{d}\left(\mathcal{F}\left(t\right)\right)^{dt}\int_{c}^{d}\left(\mathcal{G}\left(t\right)\right)^{dt},$
- $\int_{c}^{d}\left(\frac{\mathcal{F}(t)}{\mathcal{G}(t)}\right)^{dt}=\frac{\int_{c}^{d}(\mathcal{F}(t))^{dt}}{\int_{c}^{d}(\mathcal{G}(t))^{dt}},$
- $\int_{c}^{d}\left(\mathcal{F}\left(t\right)\right)^{dt}=\int_{c}^{k}\left(\mathcal{F}\left(t\right)\right)^{dt}\int_{k}^{d}\left(\mathcal{F}\left(t\right)\right)^{dt}, c<k<d,$
- $\int_{c}^{c}\left(\mathcal{F}\left(t\right)\right)^{dt}=1$ *and* $\int_{c}^{d}\left(\mathcal{F}\left(t\right)\right)^{dt}=\left(\int_{d}^{c}\left(\mathcal{F}\left(t\right)\right)^{dt}\right)^{-1}.$

Theorem 2.6 (Multiplicative Integration by Parts [5])**.** *Let $\mathcal{F} : [c,d] \to \mathbb{R}$ be multiplicatively differentiable and let $\mathcal{G} : [c,d] \to \mathbb{R}$ be differentiable. Then, the function $\mathcal{F}^{\mathcal{G}}$ is multiplicative integrable on $[c,d]$ and satisfies the following relationship:*

$$\int_c^d \left(\mathcal{F}^* (t)^{\mathcal{G}(t)}\right)^{dt} = \frac{\mathcal{F}(d)^{\mathcal{G}(d)}}{\mathcal{F}(c)^{\mathcal{G}(c)}} \times \frac{1}{\int_c^d \left(\mathcal{F}(t)^{\mathcal{G}'(t)}\right)^{dt}}.$$

Lemma 2.7 ([3])**.** *Let $\mathcal{F} : [c,d] \to \mathbb{R}$ be multiplicative differentiable, let $h : [c,d] \to \mathbb{R}$ and let $\mathcal{G} : J \subset \mathbb{R} \to \mathbb{R}$ be two differentiable functions. Then we have*

$$\int_c^d \left(\mathcal{F}^* (h(t))^{h'(t)\mathcal{G}(t)}\right)^{dt} = \frac{\mathcal{F}(h(d))^{\mathcal{G}(d)}}{\mathcal{F}(h(c))^{\mathcal{G}(c)}} \times \frac{1}{\int_c^d \left(\mathcal{F}(h(t))^{\mathcal{G}'(t)}\right)^{dt}}.$$

The multiplicative derivative and integral are highly useful in various scientific fields. In biology, they can be used to model population growth and interaction, incorporating multiplicative factors such as reproduction and mortality rates. In physics, multiplicative integrals can be applied to describe complex systems where interactions between components occur in a multiplicative manner, such as in neural networks or self-organizing phenomena. In economics, they can be employed to analyze economic processes where multiplicative effects, such as externalities, are significant. Thus, multiplicative integrals provide a mathematical framework that enhances our understanding and modeling capabilities across diverse scientific disciplines.

The following example illustrates the utility of the multiplicative derivative and integral concepts.

Example 2.8. Let's consider the following example of a radioactive decay phenomenon that can be modeled by a differential equation:

$$\mathcal{F}'(t) = \mathcal{K}(t)\ \mathcal{F}(t), \tag{2.1}$$

where $\mathcal{F}(t)$ represents the amount of the radioactive substance at time t and $\mathcal{K}(t)$ is a time-variable decay coefficient. It is important to note that the time variable, t, is always positive and the radioactive decay process cannot be negative. A radioactive substance's quantity decreases over time, but it cannot attain negative values. Equation (2.1) states that the rate of change of a substance's quantity is proportional to the quantity itself, with the decay factor providing the proportionality coefficient. Using multiplicative calculus, equation (2.1) can be reformulated as $e^{(\ln(\mathcal{F}(t)))'} = e^{\mathcal{K}(t)}$, or $\mathcal{F}^*(t) = e^{\mathcal{K}(t)}$, whose solution is given in the form of a multiplicative integral, as follows:

$$\mathcal{F}(t) = \alpha \int_{t_0}^{t} \left(e^{\mathcal{K}(t)}\right)^{dt}, \quad \text{with } \alpha = \mathcal{F}\left(t_0\right).$$

Thus, we were able to evaluate the utility and relationship between the concepts of multiplicative differentiation and integration and differential equations through this example.

3. Main Results

To demonstrate our findings, it is necessary to utilize the following lemma.

Lemma 3.1. *Suppose $\mathcal{F} : [c,d] \to \mathbb{R}^+$ is a multiplicative differentiable function on the interval $[c,d]$. If $\mathcal{F}^*$ is multiplicative integrable on $[c,d]$, then the following identity holds:*

$$\begin{aligned} &G\left(\mathcal{F}\left(x\right), \mathcal{F}\left(c+d-x\right)\right) . \left(\int_c^d \mathcal{F}\left(u\right)^{du}\right)^{\frac{1}{c-d}} \\ &= \left(\int_0^1 \left(\mathcal{F}^*\left(\left(1-t\right)c + tx\right)^t\right)^{dt}\right)^{\frac{(x-c)^2}{d-c}} \\ &\times \left(\int_0^1 \left(\mathcal{F}^*\left(\left(1-t\right)x + t\left(c+d-x\right)\right)^{\left(t-\frac{1}{2}\right)}\right)^{dt}\right)^{\frac{(c+d-2x)^2}{d-c}} \\ &\times \left(\int_0^1 \left(\mathcal{F}^*\left(\left(1-t\right)\left(c+d-x\right) + td\right)^{(t-1)}\right)^{dt}\right)^{\frac{(x-c)^2}{d-c}}, \end{aligned}$$

where $x \in \left[c, \frac{c+d}{2}\right]$ and $G\left(M,N\right) = \sqrt{MN}$ is the geometric mean.

Proof. Let

$$\begin{aligned} I_1 &= \left(\int_0^1 \left(\mathcal{F}^*\left(\left(1-t\right)c + tx\right)^t\right)^{dt}\right)^{\frac{(x-c)^2}{d-c}}, \\ I_2 &= \left(\int_0^1 \left(\mathcal{F}^*\left(\left(1-t\right)x + t\left(c+d-x\right)\right)^{\left(t-\frac{1}{2}\right)}\right)^{dt}\right)^{\frac{(c+d-2x)^2}{d-c}}, \\ I_3 &= \left(\int_0^1 \left(\mathcal{F}^*\left(\left(1-t\right)\left(c+d-x\right) + td\right)^{(t-1)}\right)^{dt}\right)^{\frac{(x-c)^2}{d-c}}. \end{aligned}$$

Using multiplicative integration by parts, from I_1 we have

$$I_1 = \left(\int_0^1 \left(\mathcal{F}^*\left(\left(1-t\right)c + tx\right)^t\right)^{dt}\right)^{\frac{(x-c)^2}{d-c}}$$

$$
\begin{aligned}
&= \int_0^1 \left(\mathcal{F}^* \left((1-t)\,c + tx \right)^{(x-c)\left(\frac{x-c}{d-c}t\right)} \right)^{dt} \\
&= \frac{\left(\mathcal{F}(x)\right)^{\frac{x-c}{d-c}}}{1} \cdot \frac{1}{\int_0^1 \left(\left(\mathcal{F}\left((1-t)\,c + tx \right) \right)^{\left(\frac{x-c}{d-c}\right)} \right)^{dt}} \\
&= \frac{\left(\mathcal{F}(x)\right)^{\frac{x-c}{d-c}}}{\int_0^1 \left(\left(\mathcal{F}\left((1-t)\,c + tx \right) \right)^{(x-c)\frac{1}{d-c}} \right)^{dt}} \\
&= \left(\mathcal{F}(x)\right)^{\frac{x-c}{d-c}} \left(\int_c^x \mathcal{F}(u)^{du} \right)^{\frac{1}{c-d}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
I_2 &= \left(\int_0^1 \left(\mathcal{F}^* \left((1-t)\,x + t\,(c+d-x) \right)^{\left(t-\frac{1}{2}\right)} \right)^{dt} \right)^{\frac{(c+d-2x)^2}{d-c}} \\
&= \int_0^1 \left(\mathcal{F}^* \left((1-t)\,x + t\,(c+d-x) \right)^{\frac{(c+d-2x)^2}{d-c}\left(t-\frac{1}{2}\right)} \right)^{dt} \\
&= \int_0^1 \left(\mathcal{F}^* \left((1-t)\,x + t\,(c+d-x) \right)^{(c+d-2x)\left(\frac{c+d-2x}{d-c}t - \frac{c+d-2x}{2(d-c)}\right)} \right)^{dt} \\
&= \frac{\left(\mathcal{F}^*(c+d-x)\right)^{\frac{c+d-2x}{2(d-c)}}}{\left(\mathcal{F}^*(x)\right)^{-\frac{c+d-2x}{2(d-c)}}} \cdot \frac{1}{\int_0^1 \left(\left(\mathcal{F}\left((1-t)\,x + t\,(c+d-x) \right) \right)^{\frac{c+d-2x}{d-c}} \right)^{dt}} \\
&= \frac{\left(\mathcal{F}^*(c+d-x)\right)^{\frac{c+d-2x}{2(d-c)}} \left(\mathcal{F}^*(x)\right)^{\frac{c+d-2x}{2(d-c)}}}{\int_0^1 \left(\left(\mathcal{F}\left((1-t)\,x + t\,(c+d-x) \right) \right)^{\frac{c+d-2x}{d-c}} \right)^{dt}} \\
&= \left(\mathcal{F}^*(c+d-x)\right)^{\frac{c+d-2x}{2(d-c)}} \left(\mathcal{F}^*(x)\right)^{\frac{c+d-2x}{2(d-c)}} . \left(\int_x^{c+d-x} \mathcal{F}(u)^{du} \right)^{\frac{1}{c-d}}
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \left(\int_0^1 \left(\mathcal{F}^* \left((1-t)\,(c+d-x) + td \right)^{(t-1)} \right)^{dt} \right)^{\frac{(x-c)^2}{d-c}} \\
&= \int_0^1 \left(\mathcal{F}^* \left((1-t)\,(c+d-x) + td \right)^{\frac{(x-c)^2}{d-c}(t-1)} \right)^{dt} \\
&= \int_0^1 \left(\mathcal{F}^* \left((1-t)\,(c+d-x) + td \right)^{(x-c)\left(\frac{x-c}{d-c}t - \frac{x-c}{d-c}\right)} \right)^{dt} \\
&= \frac{1}{\mathcal{F}^*(c+d-x)^{-\frac{x-c}{d-c}}} \cdot \frac{1}{\int_0^1 \left(\left(\mathcal{F}\left((1-t)\,(c+d-x) + td \right) \right)^{\frac{x-c}{d-c}} \right)^{dt}}
\end{aligned}
$$

$$=\frac{\left(\mathcal{F}\left(c+d-x\right)\right)^{\frac{x-c}{d-c}}}{\int\limits_0^1\left(\left(\mathcal{F}\left(\left(1-t\right)\left(c+d-x\right)+td\right)\right)^{\frac{x-c}{d-c}}\right)^{dt}}$$

$$=\left(\mathcal{F}\left(c+d-x\right)\right)^{\frac{x-c}{d-c}}.\left(\int_{c+d-x}^{d}\mathcal{F}\left(u\right)^{du}\right)^{\frac{1}{c-d}}.$$

Multiplying above equalities we get

$$I_1\times I_2\times I_3$$

$$=\left(\mathcal{F}\left(x\right)\right)^{\frac{x-c}{d-c}}\left(\int_c^x\mathcal{F}\left(u\right)^{du}\right)^{\frac{1}{c-d}}\left(\mathcal{F}^*\left(c+d-x\right)\right)^{\frac{c+d-2x}{2(d-c)}}\left(\mathcal{F}^*\left(x\right)\right)^{\frac{c+d-2x}{2(d-c)}}$$

$$\times\left(\int_x^{c+d-x}\mathcal{F}\left(u\right)^{du}\right)^{\frac{1}{c-d}}\left(\mathcal{F}\left(c+d-x\right)\right)^{\frac{x-c}{d-c}}.\left(\int_{c+d-x}^{d}\mathcal{F}\left(u\right)^{du}\right)^{\frac{1}{c-d}}$$

$$=\left(\mathcal{F}^*\left(c+d-x\right)\right)^{\frac{x-c}{d-c}+\frac{c+d-2x}{2(d-c)}}\left(\mathcal{F}^*\left(x\right)\right)^{\frac{x-c}{d-c}+\frac{c+d-2x}{2(d-c)}}$$

$$\times\left(\int_c^x\mathcal{F}\left(u\right)^{du}\int_x^{c+d-x}\mathcal{F}\left(u\right)^{du}\int_{c+d-x}^{d}\mathcal{F}\left(u\right)^{du}\right)^{\frac{1}{c-d}}$$

$$=\left(\mathcal{F}^*\left(c+d-x\right)\right)^{\frac{1}{2}}\left(\mathcal{F}^*\left(x\right)\right)^{\frac{1}{2}}\left(\int_c^d\mathcal{F}\left(u\right)^{du}\right)^{\frac{1}{c-d}}.$$

This completes the proof. □

Theorem 3.2. *Consider $\mathcal{F}:[c,d]\to\mathbb{R}^+$ as a multiplicative differentiable function on the interval $[c,d]$. If $\mathcal{F}^*$ is multiplicative convex on $[c,d]$, then for any $x\in\left[c,\frac{c+d}{2}\right]$, we have the following inequality:*

$$\left|G\left(\mathcal{F}\left(x\right),\mathcal{F}\left(c+d-x\right)\right).\left(\int_c^d\mathcal{F}\left(u\right)^{du}\right)^{\frac{1}{c-d}}\right|$$
$$\leq\left(\mathcal{F}^*\left(c\right)\right)^{\frac{(x-c)^2}{6(d-c)}}\left(\mathcal{F}^*\left(x\right)\right)^{\frac{(x-c)^2}{3(d-c)}+\frac{(c+d-2x)^2}{8(d-c)}}$$
$$\times\left(\mathcal{F}^*\left(c+d-x\right)\right)^{\frac{(x-c)^2}{3(d-c)}+\frac{(c+d-2x)^2}{8(d-c)}}\left(\mathcal{F}^*\left(d\right)\right)^{\frac{(x-c)^2}{6(d-c)}}.$$

Proof. Based on Lemma 3.1, the properties of multiplicative integration and the multiplicative convexity of $\mathcal{F}^*$, we have

$$\left|G\left(\mathcal{F}\left(x\right),\mathcal{F}\left(c+d-x\right)\right).\left(\int_c^d\mathcal{F}\left(u\right)^{du}\right)^{\frac{1}{c-d}}\right|$$
$$\leq\left(\exp\frac{(x-c)^2}{d-c}\int_0^1\left|\ln\left(\mathcal{F}^*\left(\left(1-t\right)c+tx\right)^t\right)\right|dt\right)$$

$$\times\left(\exp\frac{(c+d-2x)^2}{d-c}\int_0^1\left|\ln\left(\mathcal{F}^*\left((1-t)\,x+t\,(c+d-x)\right)^{\left(t-\frac{1}{2}\right)}\right)\right|dt\right)$$

$$\times\left(\exp\frac{(x-c)^2}{d-c}\int_0^1\left|\ln\left(\mathcal{F}^*\left((1-t)\,(c+d-x)+td\right)^{t-1}\right)\right|dt\right)$$

$$=\left(\exp\frac{(x-c)^2}{d-c}\int_0^1 t\left|\ln\left(\mathcal{F}^*\left((1-t)\,c+tx\right)\right)\right|dt\right)$$

$$\times\left(\exp\frac{(c+d-2x)^2}{d-c}\int_0^1\left|t-\tfrac{1}{2}\right|\left|\ln\left(\mathcal{F}^*\left((1-t)\,x+t\,(c+d-x)\right)\right)\right|dt\right)$$

$$\times\left(\exp\frac{(x-c)^2}{d-c}\int_0^1(1-t)\left|\ln\left(\mathcal{F}^*\left((1-t)\,(c+d-x)+td\right)\right)\right|dt\right)$$

$$\leq\left(\exp\frac{(x-c)^2}{d-c}\left(\ln\left(\mathcal{F}^*\left(c\right)\right)\int_0^1 t\,(1-t)+\ln\left(\mathcal{F}^*\left(x\right)\right)\int_0^1 t^2dt\right)\right)$$

$$\times\left(\exp\frac{(c+d-2x)^2}{d-c}\left(\ln\left(\mathcal{F}^*\left(x\right)\right)\int_0^1(1-t)\left|t-\tfrac{1}{2}\right|dt\right.\right.$$

$$\left.\left.+\ln\left(\mathcal{F}^*\left(c+d-x\right)\right)\int_0^1 t\left|t-\tfrac{1}{2}\right|dt\right)\right)$$

$$\times\left(\exp\frac{(x-c)^2}{d-c}\left(\ln\mathcal{F}^*\left(c+d-x\right)\int_0^1(1-t)^2\,dt\right.\right.$$

$$\left.\left.+\ln\mathcal{F}^*\left(d\right)\int_0^1 t\,(1-t)\,dt\right)\right)$$

$$=\left(\exp\frac{(x-c)^2}{d-c}\left(\tfrac{1}{6}\ln\left(\mathcal{F}^*\left(c\right)\right)+\tfrac{1}{3}\ln\left(\mathcal{F}^*\left(x\right)\right)\right)\right)$$

$$\times\left(\exp\frac{(c+d-2x)^2}{d-c}\left(\tfrac{1}{8}\ln\left(\mathcal{F}^*\left(x\right)\right)+\tfrac{1}{8}\ln\left(\mathcal{F}^*\left(c+d-x\right)\right)\right)\right)$$

$$\times\left(\exp\frac{(x-c)^2}{d-c}\left(\tfrac{1}{3}\ln\mathcal{F}^*\left(c+d-x\right)+\tfrac{1}{6}\ln\mathcal{F}^*\left(d\right)\right)\right)$$

$$=\left(\exp\frac{(x-c)^2}{d-c}\left(\ln\left(\mathcal{F}^*\left(c\right)\right)^{\frac{1}{6}}\left(\mathcal{F}^*\left(x\right)\right)^{\frac{1}{3}}\right)\right)$$

$$\times\left(\exp\frac{(c+d-2x)^2}{d-c}\left(\ln\left(\mathcal{F}^*\left(x\right)\right)^{\frac{1}{8}}\left(\mathcal{F}^*\left(c+d-x\right)\right)^{\frac{1}{8}}\right)\right)$$

$$\times\left(\exp\frac{(x-c)^2}{d-c}\left(\ln\left(\mathcal{F}^*\left(c+d-x\right)\right)^{\frac{1}{3}}\left(\mathcal{F}^*\left(d\right)\right)^{\frac{1}{6}}\right)\right)$$

$$=\left(\mathcal{F}^*\left(c\right)\right)^{\frac{(x-c)^2}{6(d-c)}}\left(\mathcal{F}^*\left(x\right)\right)^{\frac{(x-c)^2}{3(d-c)}}\left(\mathcal{F}^*\left(x\right)\right)^{\frac{(c+d-2x)^2}{8(d-c)}}$$

$$\times\left(\mathcal{F}^*\left(c+d-x\right)\right)^{\frac{(c+d-2x)^2}{8(d-c)}}\left(\mathcal{F}^*\left(c+d-x\right)\right)^{\frac{(x-c)^2}{3(d-c)}}\left(\mathcal{F}^*\left(d\right)\right)^{\frac{(x-c)^2}{6(d-c)}}$$

$$=\left(\mathcal{F}^*\left(c\right)\right)^{\frac{(x-c)^2}{6(d-c)}}\left(\mathcal{F}^*\left(x\right)\right)^{\frac{(x-c)^2}{3(d-c)}+\frac{(c+d-2x)^2}{8(d-c)}}$$

$$\times\left(\mathcal{F}^*\left(c+d-x\right)\right)^{\frac{(x-c)^2}{3(d-c)}+\frac{(c+d-2x)^2}{8(d-c)}}\left(\mathcal{F}^*\left(d\right)\right)^{\frac{(x-c)^2}{6(d-c)}}.$$

This completes the proof. ☐

Corollary 3.3. *In Theorem 3.2, if we make the assumption that $\mathcal{F}^* \leq M$, we obtain the following result:*

$$\left| G\left(\mathcal{F}\left(x\right), \mathcal{F}\left(c+d-x\right)\right) . \left(\int_c^d \mathcal{F}\left(u\right)^{du}\right)^{\frac{1}{c-d}} \right| \leq M^{\frac{(x-c)^2+\left(\frac{c+d}{2}-x\right)^2}{d-c}}.$$

Remark 3.4. Theorem 3.2 will be reduced to:

- Theorem 3.3 from [8], for $x = \frac{c+d}{2}$,
- Theorem 3.6 from [8], for $x = c$.

4. Example and Applications

In this section, we provide a numerical example along with a graphical representation to confirm the accuracy of the established results, as well as some applications to special means.

4.1. **Illustrative Example.**

Let's go back to Example 2.8 and consider the case of a decay coefficient represented by the function $\mathcal{K}(t) = 2t$ with $c = t_0 = 0$, $d = 1$ and an initial radioactivity $\alpha = 1$. We want to approximate the solution of this problem using a two-point Newton-Cotes formula. For this purpose, we consider the function $\mathcal{F}$ given by $\mathcal{F}(t) = e^{t^2}$. Indeed, the considered function satisfies the conditions of this work since $\mathcal{F}^*\left(t\right) = e^{2t}$ is multiplicatively convex. By utilizing the aforementioned insights and Theorem 3.2, we can now showcase numerical results and accompanying graphical representations that serve as evidence for the validity of the derived findings presented in Table 1 and Figure 1 which were generated using Matlab where the color blue represents the Right Hand Side (RHS) and red represents the Left Hand Side (LHS) of the obtained result.

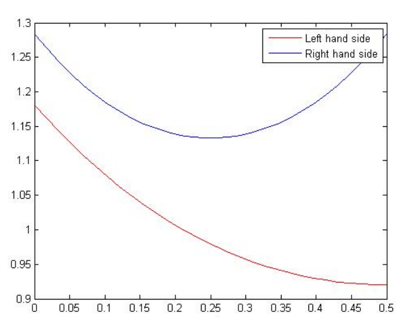

Figure 1. $x \in \left[0, \frac{1}{2}\right]$

TABLE 1. Numerical validation of Theorem 3.2

x	Values of the Left term	Values of the right term
0.00	1.1814	1.2840
0.05	1.1266	1.2275
0.10	1.0797	1.1853
0.15	1.0399	1.1560
0.20	1.0067	1.1388
0.25	0.9794	1.1331
0.30	0.9576	1.1388
0.35	0.9410	1.1560
0.40	0.9293	1.1853
0.45	0.9223	2.2275
0.50	0.9200	2.2840

Based on the information provided in Table 1 and Figure 1, it is evident that the inequality stated in Theorem 3.2 holds for $x \in [1, 2]$. This demonstrates the validity of the inequality across various values of x.

4.2. **Applications to Special Means.**

We will now discuss different types of means which provide various ways of averaging two numbers and have different properties and applications in mathematical and statistical contexts.

- Arithmetic mean: $A(c,d) = \frac{c+d}{2}$;
- Harmonic mean: $H(c,d) = \frac{2cd}{c+d}$, $c,d > 0$;
- Logarithmic means: $L(c,d) = \frac{d-c}{\ln d - \ln c}$, $c,d > 0$ and $c \neq d$;
- p-Logarithmic mean: $L_p(c,d) = \left(\frac{d^{p+1}-c^{p+1}}{(p+1)(d-c)}\right)^{\frac{1}{p}}$, $c,d > 0, c \neq d$ and $p \in \mathbb{R} \setminus \{-1, 0\}$.

Proposition 4.1. *Let $c,d \in \mathbb{R}$ with $0 < c < d$, then we have*

$$e^{A(c^p,d^p)-L_p^p(c,d)} \leq e^{p\frac{d-c}{8}\left(c^{p-1}+d^{p-1}\right)}.$$

Proof. The statement can be derived by applying Theorem 3.2 to the function $\mathcal{F}(t) = e^{t^p}$ $(p \geq 2)$, where $\mathcal{F}^*(t) = e^{pt^{p-1}}$ and $\left(\int_c^d \mathcal{F}(t)^{dt}\right)^{\frac{1}{c-d}} = \exp\left(-L_p^p(c,d)\right)$. □

Proposition 4.2. *Let $c,d \in \mathbb{R}$ with $0 < c < d$, then we have*

$$e^{A^{-1}(c,d)-L^{-1}(c,d)} \leq e^{-\frac{d-c}{6}\left(\frac{1}{4c^2}+\frac{1}{4d^2}+A^{-2}(c,d)\right)}.$$

Proof. The statement can be derived by applying Theorem 3.2 to the function $\mathcal{F}(t) = e^{1/t}$, where $\mathcal{F}^*(t) = e^{-1/t^2}$ and $\left(\int_c^d \mathcal{F}(t)^{dt}\right)^{\frac{1}{c-d}} = \exp\left(-L^{-1}(c,d)\right)$. □

5. Conclusion

In conclusion, this study has successfully achieved its objectives by introducing a new identity for multiplicatively differentiable functions. The derived identity has enabled us to establish a customized Companion of Ostrowski inequality specifically tailored for multiplicatively convex functions. These novel results contribute to the existing body of knowledge in the field and provide a foundation for further research. Moreover, the practical applications demonstrated in this study highlight the relevance and significance of multiplicative calculus in various domains.

Acknowledgment. The work of the second author was supported by DGRSDT, MESRS of Algeria (PRFU Project A14N01EP230220230001).

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