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Study of Transitivity, Periodic Density and Sensitivity for Chaotic Non-Autonomous Fuzzified Dynamical System

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Abstract. The present investigations focus on the mathematical analysis and investigation of non-autonomous discrete dynamical systems. A non-autonomous discrete dynamical system has been framework using the series technique map method to elaborate the relationships between the non-autonomous discrete dynamical system in the original (crisp) system and its g-fuzzified system. More specially, for the considered non-autonomous discrete dynamical system, the relationship between transitivity, weakly mixing, periodic density, and sensitive dependence on initial conditions have been examined.

1. Introduction

Chaos originates from the Greek phrase "Khaos", meaning "gaping void". According to mathematicians, Chaos is hard to define but simple to "recognize when you see it". In other terms, the term " chaos" refers to a state of complete ambiguity or predictability in the behavior of a complex natural system. The word "chaos" mathematics was first investigated by Li et al. [22]. One of the underlying concepts in our daily life is chaos theory. The study of intricate, nonlinear dynamic systems is what it is. It is a field of mathematics that examines systems exhibiting chaotic behaviors despite appearing to be ordered (deterministic). Additionally, it deals with systems that seem chaotic but contain order. The concept of chaotic dynamical system studied in [9]. In [4], it observed

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that f is weakly mixing iff $\bar{f}$ is transitive and identical to the weakly mixing features of $\bar{f}$. Aulbach and Rasmussen [1] approximated the attractors of a non-autonomous dynamical system. Some other authors [5, 10, 34] have observed that chaotic behavior of the dynamical system $\{f_n\}_{n=0}^{\infty}$ leading to a map f. Guirao et al. in [11] have investigated chaos as (Li-Yorke chaos, distributional chaos, w-chaos, specification property, topological chaos, total Devaney chaos, exact Devaney chaos). In the article [5, 6] authors studied the property of continuous maps sequences, which is specified on such a compact metric space uniformly converges to a continuous map and the relation between the $w-$ limit sets of sequences of continuous maps and its continuous maps, the behaviour at the limit of these sequences is this form $(f_n \circ \cdots \circ f_1)(x), x \in [0,1]$ together with simplification (respectively chaoticity) of continuous maps suggest the simplicity (respectively chaoticity) of a sequence of continuous maps. So many authors, such as [1, 3, 12, 16, 24, 31, 33, 35] have studied chaotic behaviour in nonautonomous discrete systems, and some authors [2, 28, 29, 32] studied fuzzy metric space.

In [23], the concept of mixing, transitivity and chaos is investigated for the class of set - valued mappings. In response to the standard fuzzification (Zadeh's extension) of such a dynamical system, R*ó*man-Flores and Chalco-Cano [26] acquired various chaotic features (e.g., transitive, periodic density and sensitive dependency).

Firstly, applying the $t-$ norm or a $t-$ conorm to create fuzzy discrete dynamical systems along with numerous fundamental features of Γ- fuzzifications and studying the relation amid Li-Yorke chaos and with positive topological entropy are discussed in [8]. It's established that certain set-valued dynamical systems $\left(\mathbb{K}(\Psi), \widetilde{f}\right)$ are coupled with $g-$ fuzzifications in dynamical systems $\left(\mathbb{F}(\Psi), \widehat{f_g}\right)$, which the original (crisp) dynamical system itself induces (Ψ, f). In [19], the fundamental percepts properties of $g-$ fuzzifications and numerous dynamical (chaotic and mixing) aspects of g-fuzzification have been introduced and studied.

Interestingly, several of Devaney's chaotic characteristics connected to fuzzy systems have been introduced, and periodic density, transitivity, and sensitive dependence in crisp systems have also been examined in [26]. In difference to [26], Kupka [19] currently, no specific fuzzy dynamical system that has been studied cannot be transitive, i.e., no dynamical system that has been fuzzified and defined on the whole space of fuzzy sets cannot be Devaney chaotic.

Researchers have looked at the connection in [17] with Devaney chaos in the crisp and fuzzified systems (Zadeh's extension). According to his research, the density of $P(f)$ indicates the density of $P(\widehat{f})$ in $\mathbb{F}(\Psi)$

(and also in $\mathbb{F}^1(\Psi)$). Recently, in [18] the idea of $g-$ fuzzification, which generalises the concept of Zadeh's expansion ($g = id_l$) and $\Gamma-$ fuzzification ($g = \Delta\Gamma$) established that if and only if the study's fuzzy discrete dynamical system, which was caused by g-fuzzification, is continuous, is a crisp dynamical system continuous. Furthermore, he studied that g-fuzzifications preserve (semi-) conjugacies. In the recent done in [19], Kupka studied the essential characteristics of discrete dynamical systems revealed by $g-$ fuzzification, e.g., topological or w-chaos, Li-Yorke chaos, transitivity, total transitivity, sensitive dependence, exactness, weakly mixing, mild mixing, strong mixing. Recently, Wu et al. [33] investigated the transitivity and sensitivity of g- fuzzification. He proved some results such as sensitivity, multi-sensitivity and negative answers on sensitivity.

In the last two decades, authors [11, 25] studied the crisp (set-valued) discrete dynamical systems connected to the fuzzy discrete dynamical systems produced via g-fuzzification. Although it deviates significantly from Zadeh's extension concept, the usage of g-fuzzifications is instead natural. This study aims to explore certain chaotic characteristics of the nonautonomous discrete dynamical system and its g-fuzzified system. We also study Devaney's chaotic properties of the crisp nonautonomous discrete systems and its g- fuzzified system, i.e., for some reason, we observe the paper [21], who studied the nonautonomous discrete systems achieve from the usual fuzzification. Motivated by the above results, we will describe the relationships of transitivity, weakly mixing, periodic density and sensitive dependence on initial conditions in usual fuzzified systems and its g-fuzzified systems in the present paper.

The initiative of this paper is to study some basic properties of g-fuzzifications in a nonautonomous discrete dynamical system. Section 2, presents some basic concepts of chaotic properties and some main results about g-fuzzification. In Section 3, we investigate the connections between the transitivity of the original system's g-fuzzified system and its nonautonomous discrete system. In Section 4, we investigate the relationships between the periodic density of the original system and its g-fuzzified systems in a nonautonomous discrete system. Section 5, investigates the relationship between the g-fuzzified system's sensitivity and that of the original system's nonautonomous discrete system.

2. Basic Concepts of Nonautonomous Discrete System

Before discussing non-autonomous discrete systems, it is important to review certain key ideas relating to transitivity, weakly mixing, periodic density and sensitive dependence on beginning circumstances. Several

chaotic features of nonautonomous systems has been examined by [5, 10, 27, 34].

Supposed that $\{g_n\}_{n=0}^{\infty}$ is a collection of continuous self-maps in a compact metric space (Ψ, d). The orbit of a point $x \in \Psi$ is the set $\{g_1^n(x) : n \in \mathbb{N}\}$ = orbit $(x, \{g_n\}_{n=0}^{\infty})$ is given as

$$x_n = g_n(x_{n-1}), \quad n = 1, 2, \cdots .$$

We call $(x, \{g_n\}_{n=0}^{\infty})$ is a nonautonomous discrete system (NADS). If $g_n = g$ the system becomes autonomous discrete dynamical system.

Now a map define $G_n : X \to \Psi$ by

$G_n(x) = g_n o g_{n-1} ... o g_2 o g_1(x)$.

Definition 2.1. For any point in the metric space $x \in \Psi$ is **periodic** if $F_n(x) = \Psi$, for some $n \geq 1$.

First, the transitivity and sensitivity of nonautonomous discrete systems as presented by Shi and Chen in [30].

Definition 2.2 ([30])**.** For any sequence of function $\{g_n\}_{n=0}^{\infty}$ is **transitive** if for any chosen nonempty open subsets $M, N \subseteq \Psi$, there exists $k \geq 1$ such that $G_k(M) \cap N \neq \emptyset$.

Definition 2.3 ([3])**.** A sequence $\{g_n\}_{n=0}^{\infty}$ is **weakly mixing** if for chosen nonempty open sets M_1, N_2, N_1 and M_2 there exists $k \geq 1$ such that $G_k(M_1) \cap N_1 \neq \emptyset$ and $G_k(M_2) \cap N_2 \neq \emptyset$.

Definition 2.4 ([30])**.** If a sequence $\{g_n\}_{n=0}^{\infty}$ is said to be **periodic dense**, then periodic point set $\{g_n\}_{n=0}^{\infty}$ is dense in Ψ.

Definition 2.5 ([30])**.** For any sequence of function $\{g_n\}_{n=0}^{\infty}$ is said to be sensitive dependence on initial condition if there exist constant $\delta > 0$ such that for every point x and every neighborhood M about x, there is a $y \in M$ and a $k \geq 1$ such that $d\left(G_k(x), G_k(y)\right) \geq \delta$.

Considering $H(\Psi)$ denote the class of all nonempty compact subset of Ψ. Suppose $A \in H(\Psi)$ now we define the "$\epsilon-$ neighborhood of Q" as the set $O(Q, \epsilon) = \{x \in \Psi : d(x, Q) < \epsilon\}$, where $d(x, Q) = \inf_{a \in Q} \|x - a\|$.

The Hausdorff separation $\rho(Q, B)$ of $Q, B \in H(\Psi)$ is defined by

$$\rho(Q, B) = \inf \left\{\epsilon > 0 : Q \subseteq O(B, \epsilon)\right\},$$

where $H(\Psi)$ is considered as class of nonempty compact subsets of Ψ. In [15], it has been noticed that $(H(\Psi), D)$ is compact, complete and separable when the space Ψ is compact, complete and separable.

The Hausdorff metric space on $H(\Psi)$ is defined by

$$h(Q, B) = \max \left\{\rho(Q, B), \rho(B, Q)\right\}.$$

Now define the class $\mathscr{G}(\Psi)$of all upper semi continuous fuzzy sets $u : \Psi \to [0,1]$ such that $[u]_\alpha \in H(\Psi)$, where $\alpha-$ cut and the support of u are defined by

$$[u]_\alpha = \{x \in \Psi : u(x) \geq \alpha\}, \quad \alpha \in [0,1],$$
$$supp(u) = cl\{x \in \Psi : u(x) > 0\},$$

respectively.

Moreover, let

$$\mathscr{G}^\lambda(\Psi) = \{u \in \mathscr{G}(\Psi) : u(x) \geq \lambda \text{ for some} x \in \Psi\}$$

and let $\mathscr{G}^1(\Psi)$ denote the space of all normal fuzzy sets on Ψ and $\emptyset_\Psi$ denote the empty fuzzy set ($\emptyset_\Psi(x) = 0$) for all $x \in \Psi$.

A levelwise metric D on $\mathscr{G}(\Psi)$ is defined by

$$D(u,v) = \sup_{\alpha \in [0,1]} h([u]_\alpha, [v]_\alpha),$$

for all $u, v \in \mathscr{G}(\Psi)$. It is well known that if (Ψ, d) is complete, then $(\mathscr{G}(\Psi), D)$ is also complete but is not compact and not separable [7, 13, 19].

In [17] Kupka introduced that the set of a piecewise constant fuzzy set is dense in $\mathscr{G}(\Psi)$. Further, in [18] Kupka studied the notion of g-fuzzification to generalize the usual fuzzification, also called Zadeh's extension [15] or Γ- fuzzification [20].

Let $g : \Psi \to \Psi$ be continuous. For a given nonautonomous discrete system $(\Psi, \{\widehat{g}_n\}_{n=0}^\infty)$, the zadeh's extension principle (or usual fuzzification) is a sequence of map $\widehat{g}_n : \mathscr{G}(\Psi) \to \mathscr{G}(\Psi)$ defined by

$$\widehat{g}_n(u)(x) = \sup\left\{u(y) : y \in g_n^{-1}(x)\right\}$$

for any $u \in \mathscr{G}(\Psi)$ and $x \in \Psi$.

Now, we recall g-fuzzifications. Denote $D_m(I)$ the set of all nondecreasing right-continuous functions $g : I \to I$ for which $g(x) = x$ if $x = 0$ and $x = 1$ and $C_m(I)$ denote the set of all continuous maps from $D_m(I)$. For a nonautonomous discrete system $(\Psi, \{g_n\}_{n=0}^\infty)$, where $\{g_n\} \subseteq C(\Psi)$ and for any $g \in D_m(I)$ a map $\widehat{g}_{g,n} : \mathscr{G}(\Psi) \to \mathscr{G}(\Psi)$ defined by

$\widehat{g}_{g,n}(Q)(x) = \sup\left\{g(Q(y)) : y \in g_n^{-1}(x)\right\}$, for any $Q \in \mathscr{G}(\Psi), x \in \Psi$ is called a g-fuzzification of the crisp dynamical system $(\Psi, \{g_n\}_{n=0}^\infty)$. $\widehat{g}_{g,n}$ denote the g-fuzzification induced from a given map g_n for $n \geq 0$. For more details of g-fuzzification see [18].

Now, we recall some propositions which are very often used in this paper.

Proposition 2.6 ([19, 26])**.** *Suppose M and N be two subsets of Ψ. Define $e(M) = \{m \in \mathscr{G}(\Psi) : [m]_0 \subseteq M\}$, and then*

(1) *$e(M) \neq \emptyset$ iff $M \neq \emptyset$,*

(2) *If M is an open set, then $e(M)$ is an open subset of $\mathscr{G}(\Psi)$,*
(3) $e(M \cap N) = e(M) \cap e(N)$.

Proposition 2.7 ([18])**.** *An $\alpha-$ cut $[u]^f_\alpha$ of a fuzzy set $m \in \mathscr{G}(\Psi)$ with respect to $g \in D_m(I)$ is $[u]^f_\alpha = \{x \in supp(u) : g(m(x)) \geq \alpha\}$,*
for $\alpha \in (0, 1]$.

Proposition 2.8 ([18])**.** *Suppose $f : \Psi \to \Psi$ be continuous function and $\widehat{f}_{g,n}$ be g-fuzzification. Then, $f([u]^g_\alpha) = \left[\widehat{f}_g(u)\right]_\alpha$*
satisfy for any $u \in \mathscr{F}_0(\Psi), g \in D_m(I)$ and $\alpha \in (0, 1]$.

Proposition 2.9 ([18])**.** *Now let $g \in D_m(I)$, $u \in \mathscr{F}_0(\Psi)$ and $\alpha \in (0, 1]$. If $[u]^g_\alpha \neq \infty$, then there exists $c \in (0, 1]$ such that $[u]^g_\alpha = [u]_c$.*

In [18, 19], Kupka derived the g-fuzzification in discrete dynamical system. The main aim of this study is to use g-fuzzification in nonautonomous discrete dynamical system.

3. Transitivity in Nonautonomous Discrete System

In the prsent part. we will study the relation between transitivity of nonautonomous discrete system $(\Psi, \{f_n\}_{n=0}^\infty)$ and its g-fuzzified system $\left(\mathscr{F}(\Psi), \left\{\widehat{f}_{g,n}\right\}_{n=0}^\infty\right)$.

Proposition 3.1. $\left[\widehat{F}_{g,n}(u)\right]_\alpha = F_n([u]^g_\alpha) = F_n([u]_c)$, *where $g \in D_m(I)$ and $c \in (0, 1]$.*

Proof. Suppose that $\widehat{f}_{g,i}(u_i) = u_{i+1}$.

$$\begin{aligned}
\left[\widehat{F}_{g,n}(u_1)\right]_\alpha &= \left[\widehat{f}_{g,n} \circ \ \widehat{f}_{g,n-1} \circ \cdots \circ \widehat{f}_{g,1}(u_1)\right]_\alpha \\
&= \left[\widehat{f}_{g,n} \circ \widehat{f}_{g,n} \circ \cdots \circ \widehat{f}_{g,2}(u_2)\right]_\alpha \\
&= \left[\widehat{f}_{g,n} \circ \widehat{f}_{g,n-1} \circ \cdots \circ \widehat{f}_{g,3}(u_3)\right]_\alpha \\
&\ \ \vdots \\
&= \left[\widehat{f}_{g,n}(u_n)\right]_\alpha \\
&= f_n([u_n]^g_\alpha) \\
&= f_n \circ f_{n-1}([u_{n-1}]^g_\alpha) \\
&\ \ \vdots \\
&= f_n \circ f_{n-1} \circ \cdots \circ f_1([u_1]^g_\alpha) \\
&= F_n([u]^g_\alpha)
\end{aligned}$$

$$= F_n([u]_c). \qquad \square$$

Proposition 3.2. *Consider Ψ be a compact metric space, $\{f_n\} \subseteq C(I), g \in D_m(I)$ and $\widehat{f}_{g,n}$ is a g-fuzzification of f_n. Define $[u]_0^g = \{x \in [u]_0 : g(u(x)) > 0\}$ for any $\alpha \in [0,1]$. Then $[u]_0^g = [u]_0$.*

Proof. Since g is non decreasing, we have

$$\begin{aligned}[u]_0^g &= \{x \in [u]_0 : g(u(x)) > 0\} \\ &= \{x \in [u]_0 : u(x) > 0\}.\end{aligned}$$

Then $[u]_0^g = [u]_0$. $\square$

Proposition 3.3. $\widehat{F}_{g,n}[e(U)] \subseteq e[F_n(U)]$.

Proof. Let $u \in \widehat{F}_{g,n}[e(U)]$, then there exists $s \in e(U)$ such that $u = \widehat{F}_{g,n}(s)$. Thus, from Proposition (3.2) we have

$$[u]_0 = \left[\widehat{F}_{g,n}(s)\right]_0 = F_n\left([s]_0^g\right) = F_n([s]_0).$$

Now, $[s]_0 \subseteq U$, then $[u]_0 = F_n([s]_0) \subseteq F_n(U)$. Hence $u \in e[F_n(U)]$. $\square$

Theorem 3.4. *Let $(\Psi, \{f_n\}_{n=0}^{\infty})$ be a nonautonomous discrete system. $\{f_n\} \subseteq C(\Psi)$ and $g \in D_m(I)$. If $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ is transitive then $\{f_n\}_{n=0}^{\infty}$ is transitive, where $\widehat{f}_{g,n}$ denote the g-fuzzification induced from a given map f_n for $n \geq 0$.*

Proof. Let $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ be transitive. We have to show that $\{f_n\}_{n=0}^{\infty}$ is transitive, i.e., for any nonempty open set U and V, there is a $k \geq 1$ such that $F_k(U) \cap V \neq \emptyset$.

From the Proposition (2.6)(2), $e(U)$ and $e(V)$ are open subsets of $\mathscr{F}(\Psi)$ and therefore, $e(U) \cap e(V)$ is also open. Now, from transitivity of $\widehat{F}_{g,n}$ and from Proposition (3.3), there is a $k \geq 1$ such that

$$\begin{aligned}\emptyset \neq \widehat{F}_{g,k}[e(U)] \cap e(V) &\subset e[F_k(U)] \cap e(V) \\ &= e[F_k(U) \cap V].\end{aligned}$$

Now, applying Proposition (2.6)(1), we obtain $F_k(U) \cap V \neq \emptyset$. Hence $\{f_n\}_{n=0}^{\infty}$ is transitive. $\square$

We define an example to show that the converse of the Theorem (3.4) does not hold.

Example 3.5. Let $\lambda \in \mathbb{R}$ be an irrational number and $F_\lambda : S^1 \to S^1$ is defined by $F_\lambda(e^{i\theta}) = e^{i(\theta + 2\pi\lambda)}$, for each $z \in S^1$ the orbit $\left(F_\lambda^k(e^{i\theta}) : k \in \mathbb{N}\right)$ is dense in S^1 and f_λ is transitive which is studied in [9]. Now, let $u \in \mathscr{F}(S^1)$ and $\operatorname{diam}([u]_0^g) = 1$ then consider $0 < \epsilon < \frac{1}{2}$ and $U =$

$B(\widehat{1},\frac{\epsilon}{2})$, $V = B(u,\frac{\epsilon}{2})$. We have $w \in U = B(\widehat{1},\frac{\epsilon}{2}) = \text{diam}([w]_0^g) \leq \frac{\epsilon}{2}$, $v \in V = B(u,\frac{\epsilon}{2}) = \text{diam}([v]_0^g) \geq (1-\epsilon)$. Thus $\text{diam}\Big([\widehat{F}_{g,\lambda}^k(v)]_0\Big) = \text{diam}\big(F_\lambda^k[v]_0^g\big) \geq 1-\epsilon$ for $k \in \mathbb{N}$. Thus $U \cap \widehat{F}_{g,\lambda}^k(v) = \emptyset$. Hence $\Big\{\widehat{f}_{g,n}\Big\}_{n=0}^{\infty}$ is not transitive on $\mathscr{F}(S^1)$.

In [17, 19], it has been shown that no fuzzification can be transitive on the fuzzy set $\mathscr{F}(\Psi)$, but there exists a transitive fuzzification on the space of normal fuzzy set $\mathscr{F}^1(\Psi)$. In this section, we introduce a method to show that $\{f_n\}_{n=0}^{\infty}$ is transitive and it implies $\Big\{\widehat{f}_{g,n}\Big\}_{n=0}^{\infty}$ is transitive. Let $\mathcal{U}$ be a subset of $\mathscr{F}^1(\Psi)$. We have

$r(\mathcal{U}) = \{Q \in \mathscr{K}(\Psi) : \exists\ u \in \mathcal{U}\ s.t.\ A \subseteq [u]_0\}$.

Proposition 3.6. *Let $\mathcal{U}$ be a subset of $\mathscr{F}^1(\Psi)$, $g \in D_m(I)$, then $F_n[r(\mathcal{U})] \subseteq r[\widehat{F}_{g,n}(\mathcal{U})]$.*

Proof. Let $z \in F_n[r(\mathcal{U})]$, then there exists $w \in A \subseteq [u]_0$ such that $z = F_n(w)$. Now, from Proposition (3.2) we have

$$z = F_n(w) \in F_n([u]_0) = F_n([u]_0^g) = [\widehat{F}_{g,n}(u)]_0.$$

Hence $z \in r[\widehat{F}_{g,n}(\mathcal{U})]$. Thus $F_n[r(\mathcal{U})] \subseteq r[\widehat{F}_{g,n}(\mathcal{U})]$. □

Before starting the next theorem, we recall the following proposition.

Proposition 3.7 ([21])**.** *Let $\mathcal{U}$ and $\mathcal{V}$ be subsets of $\mathscr{F}^1(\Psi)$.*

1. *$r(\mathcal{U}) \neq \emptyset$ if and only if $\mathcal{U} \neq \emptyset_\Psi$, where $\emptyset_\Psi$ is the empty fuzzy set ($\emptyset_\Psi = 0$) for each $x \in \Psi$.*
2. *Assume that $u \neq v$ implies $[u]_0 \cap [v]_0 = \emptyset$, then $r(\mathcal{U} \cap \mathcal{V}) = r(\mathcal{U}) \cap r(\mathcal{V})$.*
3. *If $\mathcal{U}$ is a nonempty open subset of $\mathscr{F}^1(\Psi)$, then $r(\mathcal{U})$ is a nonempty open subset of Ψ.*

Theorem 3.8. *Let $\Big\{\widehat{f}_{g,n}\Big\}_{n=0}^{\infty}$ be a sequence of maps on normal fuzzy sets $\mathscr{F}^1(\Psi)$. If $\{f_n\}_{n=0}^{\infty}$ is transitive then $\Big\{\widehat{f}_{g,n}\Big\}_{n=0}^{\infty}$ is transitive, where $\widehat{f}_{g,n}$ denote the g-fuzzification induced by a given map f_n for $n \geq 0$.*

Proof. Consider any nonempty open subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathscr{F}^1(\Psi)$. Let $\{f_n\}_{n=0}^{\infty}$ be transitive. We have to show that $\Big\{\widehat{f}_{g,n}\Big\}_{n=0}^{\infty}$ is transitive, i.e., there is a $k \geq 1$ such that $\widehat{F}_{g,k}(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$.

As $\mathcal{U}$ and $\mathcal{V}$ are open. From Proposition (3.7)(3), $r(\mathcal{U})$ and $r(\mathcal{V})$ are also open sets. Again, from transitivity of $\{f_n\}_{n=0}^{\infty}$, there is a $k \geq 1$ such that $F_k(r(\mathcal{U})) \cap r(\mathcal{V}) \neq \emptyset$.

Now, from Proposition (3.7) (2) and (3.6), we have

$$\begin{aligned}\emptyset \neq F_k[r(\mathcal{U})] \cap r(\mathcal{V}) &\subseteq r\left[\widehat{F}_{g,k}(\mathcal{U})\right] \cap r(\mathcal{V}) \\ &= r\left[\widehat{F}_{g,k}(\mathcal{U}) \cap \mathcal{V}\right].\end{aligned}$$

Thus, from Proposition (3.7) (1), we have $\widehat{F}_{g,k}(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. Hence $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ is transitive. □

Theorem 3.9. *Let $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ be a sequence of maps on normal fuzzy sets $\mathscr{F}^1(\Psi)$. If $\{f_n\}_{n=0}^{\infty}$ is weakly mixing then $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ is weakly mixing, where $\widehat{f}_{g,n}$ denote the g-fuzzification induced from a given map f_n for $n \geq 0$.*

Proof. Consider any nonempty open subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathscr{F}^1(\Psi)$. Let $\{f_n\}_{n=0}^{\infty}$ be weakly mixing. We have to show that $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ is weakly mixing, i.e., there is a $k \geq 1$ such that

$$\widehat{F}_{g,k}(\mathcal{U}) \cap \mathcal{V} \neq \emptyset, \qquad \widehat{F}_{g,k}(\mathcal{V}) \cap (\mathcal{V}) \neq \emptyset.$$

As $\mathcal{U}$ and $\mathcal{V}$ are open. From Proposition (3.7)(3), $r(\mathcal{U})$ and $r(\mathcal{V})$ are also open sets. From the weakly mixing of $\{f_n\}_{n=0}^{\infty}$, there is a $k \geq 1$ such that

$$F_k\left(r(\mathcal{U})\right) \cap r(\mathcal{V}) \neq \emptyset, \qquad F_k\left(r(\mathcal{V})\right) \cap r(\mathcal{V}) \neq \emptyset.$$

From the Proposition (3.6), we have

$$r\left(\widehat{F}_{g,k}(\mathcal{U})\right) \cap r(\mathcal{V}) \neq \emptyset, \qquad r\left(\widehat{F}_{g,k}(\mathcal{V})\right) \cap r(\mathcal{V}) \neq \emptyset.$$

Now, applying Proposition (3.7) (2), we have

$$r\left(\widehat{F}_{g,k}(\mathcal{U}) \cap \mathcal{V}\right) \neq \emptyset, \qquad r\left(\widehat{F}_{g,k}(\mathcal{V}) \cap \mathcal{V}\right) \neq \emptyset.$$

Thus, from Proposition (3.7)(1),

$$\widehat{F}_{g,k}(\mathcal{U}) \cap \mathcal{V} \neq \emptyset, \qquad \widehat{F}_{g,k}(\mathcal{V}) \cap \mathcal{V} \neq \emptyset.$$

Hence $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ is weakly mixing. □

4. Periodic Density in Nonautonomous Discrete System

This section gives an overview of the relation between periodic density of nonautonomous discrete system $(\Psi, \{f_n\}_{n=0}^{\infty})$ and its g-fuzzified system $\left(\mathscr{F}(\Psi), \left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}\right)$ has been discussed.

Theorem 4.1. *Let $(\Psi, \{f_n\}_{n=0}^{\infty})$ be a nonautonomous discrete system, $\{f_n\} \subseteq C(\Psi)$ and $g \in D_m(I)$. If $\{f_n\}_{n=0}^{\infty}$ has periodic density then $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ has periodic density, where $\widehat{f}_{g,n}$ denote the g-fuzzification induced from a given map f_n for $n \geq 0$.*

Proof. The proof of this theorem is a slight similar to Theorem (5) in [26]. □

The converse of this theorem is true under this condition.

In [13], Kaleva studied the fixed point property of some spaces of fuzzy sets which we are going to use.

Let $\mathscr{H}$ be a subspace of $\mathscr{F}(\Psi)$, $\widehat{f}_{g,\mathscr{H}}(u) = \widehat{f}_g(u)$, $\forall\ u \in \mathscr{H}$ and $g \in D_m(I)$. We say that a topological space Ψ has the fixed point property (f.p.p.) if every continuous map $f : \Psi \to \Psi$ has a fixed point. We define $\mathscr{K}_p(\Psi)$, the family of all nonempty compact subsets of Ψ which has the f.p.p, where $\mathscr{F}_p(\Psi) = \{u \in \mathscr{F}(\Psi) : [u]_\alpha \in \mathscr{K}_p(\Psi)\}$.

Proposition 4.2. *Let Ψ be a nonempty compact set and $U \subseteq \Psi$, $e_{\mathscr{H}}(U) = \{u \in \mathscr{H} : [u]_0 \subseteq U\}$. If U is an open subset of Ψ then $e_{\mathscr{H}}(U)$ is an open subset of $\mathscr{F}(\Psi)$.*

In the next theorem, we show that if $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ has periodic density then $\{f_n\}_{n=0}^{\infty}$ has periodic density.

Theorem 4.3. *Let $\mathscr{H}$ be a subspace of $\mathscr{F}(\Psi)$ and $\mathscr{H} \subseteq \mathscr{F}_p(\Psi)$. If $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ has periodic density then $\{f_n\}_{n=0}^{\infty}$ has periodic density.*

Proof. Let $x_0 \in \Psi$, $\epsilon > 0$ and $\chi_{\{x\}} \in \mathscr{H}$, from the periodic density of $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$, $\exists\ \nu \in \mathscr{H}$ and $n \in N$ such that

$$D\left(\chi_x, \nu\right) < \epsilon, \tag{4.1}$$

$$\widehat{F}_{g,n}(\nu) = F_n(\nu) = \nu. \tag{4.2}$$

Now, from combination of both (4.1) and (4.2) we have $d(x_0, F_n(\nu)) < \epsilon$, for all $\nu \in \mathscr{H}$. From Schauder's fixed-point theorem, there exists $\nu_p \in \mathscr{F}_p(\Psi)$ such that $F_n(\nu_p) = \nu_p$. Therefore, ν_p a periodic point of $\{f_n\}_{n=0}^{\infty}$ and from above we obtain $d(x_0, \nu_p) < \epsilon$. Hence $\{f_n\}_{n=0}^{\infty}$ has periodic density on Ψ. □

Proposition 4.4. *Let Ψ be a compact metric space, $\{f_n\} \subseteq C(\Psi)$ and $g \in D_m(I)$. If $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ is transitive then $\{f_n\}_{n=0}^{\infty}$ is weakly mixing, where $\widehat{f}_{g,n}$ denote the g-fuzzification induced from a given map f_n for $n \geq 0$.*

Proof. Let $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ be transitive. We have to show that $\{f_n\}_{n=0}^{\infty}$ is weakly mixing, i.e., for any nonempty open set U and V, there is a $k \geq 1$ such that

$$F_k(U) \cap V \neq \emptyset, \quad F_k(V) \cap V \neq \emptyset.$$

From the Proposition (2.6)(2), $e(U)$ and $e(V)$ are open subsets of Ψ and therefore, $e(U) \cap e(V)$ is open. Now, from transitivity of $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$, there is a $k \geq 1$ such that

$$\emptyset \neq \widehat{F}_{g,k}(e(U) \cap e(V)) \cap e(V) \subset \widehat{F}_{g,k}(e(U)) \cap \widehat{F}_{g,k}(e(V)) \cap e(V).$$

Thus, we have

$$\widehat{F}_{g,k}(e(U)) \cap e(V) \neq \emptyset, \quad \widehat{F}_{g,k}(e(V)) \cap e(V) \neq \emptyset.$$

Since $\widehat{F}_{g,k}(e(U)) \cap e(V) \neq \emptyset$, then there exists $u \in e(U)$ and $v \in e(V)$ such that $\widehat{F}_{g,k}(u) = v$. From Proposition (3.2), we have

$$F_k([u]_0^g) = F_k([u]_0) = [v]_0 \subseteq V.$$

For any $x \in [u]_0 \subseteq U$, or that, $F_k(U) \cap V \neq \emptyset$. On the other hand (c.f., f.p.p), we can choose $v_1, v_2 \in e(V)$ such that $F_k(v_1) = v_2$. Hence from induction that $F_k(V) \cap V \neq \emptyset$. □

5. Sensitive Dependence on Initial Conditions

This section gives in overview of the relation between sensitivity of the original system's $(\Psi, \{f_n\}_{n=0}^{\infty})$ in nonautonomous discrete dynamical system and its g-fuzzified system $\left(\mathscr{F}(\Psi), \left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}\right)$.

Theorem 5.1. *Let Ψ be a compact metric space, $\{f_n\} \subseteq C(\Psi)$ and $g \in D_m(I)$. If $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ is sensitive dependence then $\{f_n\}_{n=0}^{\infty}$ is sensitive dependence, where $\widehat{f}_{g,n}$ denote the g-fuzzification induced from a given map f_n for $n \geq 0$.*

Proof. Let $\left\{\widehat{f}_{g,n}\right\}_{n=0}^{\infty}$ be sensitive dependence then there exists a constant $\delta > 0$ and $\epsilon > 0$, $\exists$ $\nu_0 \in B(u_0, \epsilon)$ and $k \in N$ such that $D\left(\widehat{F}_{g,k}(u_0), \widehat{F}_{g,k}(\nu_0)\right) > \delta$ where $u_0, \nu_0 \in \mathscr{F}(\Psi)$

$$\begin{aligned} D\left(\widehat{F}_{g,k}(u_0), \widehat{F}_{g,k}(\nu_0)\right) &= \sup_{\alpha \in [0,1]} H\left(\left[\widehat{F}_{g,k}(u_0)\right]_{\alpha}, [\widehat{F}_{g,k}(\nu_0)]_{\alpha}\right) \\ &= \sup_{\alpha \in [0,1]} H\left(F_k[u_0]_{\alpha}^g, F_k[\nu_0]_{\alpha}^g\right) \\ &> \delta. \end{aligned}$$

Now, there exists $x_0 \in [u_0]^g_\alpha$ and $y_0 \in [\nu_0]^g_\alpha$ such that $d\left(F_k(x_0), F_k(y_0)\right) > \delta$. Since, $\nu_0 \in B(u_0, \epsilon)$ we have $d(x_0, y_0) < \epsilon$. Hence $\{f_n\}_{n=0}^{\infty}$ is sensitive dependence. □

6. Conclusions

The recent investigations are focused on the analysis of transitivity, periodic density, weakly mixing and sensitive dependence property on initial conditions the non-autonomous discrete dynamical system by g-fuzzification studied by [18, 19]. The relationship between chaotic properties of original (crisp) systems and its g-fuzzified systems has been carried out. Initially, we have proven that the non-autonomous discrete system is transitive in g-fuzzified system, it is transitive in a crisp (original) system, but the converse need not be accurate. Also, in the normal fuzzy set, the converse of transitivity holds and weakly mixes in the crisp (original) system, implying the weakly mixed in the g-fuzzified system. Schauder's fixed point theorem has been used to show the periodic density in a crisp system from the periodic density in a g-fuzzified system. Finally, we have found that sensitive dependency on starting conditions in the g-fuzzified system implies the sensitive dependence the initial position the crisp system.

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References

1. B. Aulbach and M. Rasmussen, *Approximation of attractors of nonautonomous dynamical systems*, Discrete Contin. Dyn. Syst. Ser. B, 5 (2005), pp. 215-238.
2. M. Abtahi, *Cauchy sequences in fuzzy metric spaces and fixed point theorems*, Sahand Commun. Math. Anal., 20 (2023), pp. 137-152.
3. F. Balibrea and P. Oprocha, *Weak mixing and chaos in nonautonomous discrete systems*, Appl. Math. Lett., 25 (2012), pp. 1135-1141.
4. J. Banks, *Chaos for induced hyperspace maps*, Chaos Solit. Fractals, 25 (2005), pp. 681-685.
5. J.S. Cánovas, *Li-Yorke chaos in a class of nonautonomous discrete systems*, J. Differ. Equ. Appl., 17 (2011), pp. 479-486.
6. J.S. Cánovas, *On w-limit sets of nonautonomous discrete systems*, J. Differ. Equ. Appl., 12 (2006), pp. 95-100.
7. P. Diamond and P.E. Kloeden, *Characterization of compact subsets of fuzzy sets*, Fuzzy Sets Syst., 29 (1989), pp. 341-348.

8. P. Diamond and A. Pokrovskii, *Chaos, entropy and a generalized extension principle*, Fuzzy Sets Syst., 61 (1994), pp. 277-283.
9. R.L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, 1989.
10. J. Dvořáková, *Chaos in nonautonomous discrete dynamical systems*, Commun. Nonlinear Sci. Numer. Simul., 17 (2012), pp. 4649-4652.
11. J.L.G. Guirao, D. Kwietniak, M. Lampart, P. Oprocha and A. Peris, *Chaos on hyperspaces*, Nonlinear Anal. Theory Methods Appl., 71 (2009), pp. 1-8.
12. Q. Huang, Y. Shi and L. Zhang, *Sensitivity of non-autonomous discrete dynamical systems*, Appl. Math. Lett., 39 (2015), pp. 31-34.
13. O. Kaleva, *On the convergence of fuzzy sets*, Fuzzy Sets and Syst., 17 (1985), pp. 53-65.
14. A. Khan and P. Kumar, *Chaotic properties on time varying map and its set valued extension*, Adv. Pure Appl. Math., 3 (2013), pp. 359-364.
15. P.E. Kloeden, *Compact supported endographs and fuzzy sets*, Fuzzy Sets Syst., 4 (1980), pp. 193-201.
16. S. Kolyada and L. Snoha, *Topological entropy of nonautonomous dynamical systems*, Random and Computational Dynamics, 4 (1996), pp. 205-233.
17. J. Kupka, *On Devaney chaotic induced fuzzy and set-valued dynamical systems*, Fuzzy Sets Syst., 177 (2011), pp. 34-44.
18. J. Kupka, *On fuzzifications of discrete dynamical systems*, Inf. Sci., 181 (2011), pp. 2858-2872.
19. J. Kupka, *Some chaotic and mixing properties of fuzzified dynamical systems*, Inf. Sci., 279 (2014), pp. 642-653.
20. K. Kuratowski, *Topology*, vol. II, Academic Press, London, New York, 1968.
21. Y. Lan, *Chaos in nonautonomous discrete fuzzy dynamical systems*, J. Nonlinear Sci. Appl., 9 (2016), pp. 404-412.
22. T. Li and J. Yorke, *Period three implies chaos*, Am. Math. Mon., 82 (1975), pp. 985-992.
23. G. Liao, L. Wang and Y. Zhang, *Transitivity, mixing and chaos for a class of set valued mappings*, Sci. China Math., 49 (2006), pp. 1-8.
24. M. Murillo-Arcila and A. Peris, *Mixing properties for nonautonomous linear dynamics and invariant sets*, Appl. Math. Lett., 26 (2013), pp. 215-218.

25. A. Peris, *Set-valued discrete chaos*, Chaos Solit. Fractals, 26 (2005), pp. 19-23.
26. H. Román-Flores and Y. Chalco-Cano, *Some chaotic properties of Zadeh's extensions*, Chaos Solit. Fractals, 35 (2008), pp. 452-459.
27. H. Román-Flores, Y. Chalco-Cano, G. N. Silva and J. Kupka, *On turbulent, erratic and other dynamical properties of Zadeh's extensions*, Chaos Solit. Fractals, 44 (2011), pp. 990-994.
28. M. Saheli, S.A.M. Mohsenialhossini and H. Saedi, *The operators theorems on fuzzy topological space*, Sahand Commun. Math. Anal., 19 (2022), pp. 57- 76.
29. M.S. Sezen, *Fixed point theorems for fuzzy* (γ, β)*-contractions in non-Archimedean fuzzy metric spaces*, Sahand Commun. Math. Anal., 18 (2021), pp. 31-44.
30. Y. Shi and G. Chen, *Chaos of time -varying discrete dynamical systems*, J. Differ. Equ. Appl., 15 (2009), pp. 429-449.
31. Y. Shi, *Chaos in nonautonomous discrete dynamical systems approached by their induced systems*, Int. J. Bifurcation Chaos, 22 (2012), pp. 1250284.
32. A. talabeigi and M. Talabeigi, *An approach to change the topology of a topological space*, Sahand Commun. Math. Anal., 19 (2022), pp. 55-63.
33. X. Wu and G. Chen, *Sensitivity and transitivity of fuzzified dynamical systems*, Inf. Sci., 396 (2017), pp. 14-23.
34. X. Wu and P. Zhu, *Chaos in a class of nonautonomous discrete systems*, Appl. Math. Lett., 26 (2013), pp. 431-436.
35. C. Yang, Y. Wang and Z. Li, *Some results about sensitivity on nonautonomous dynamical system in hyperspace*, Pure Appl. Math., 30 (2014), pp. 201-206.

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