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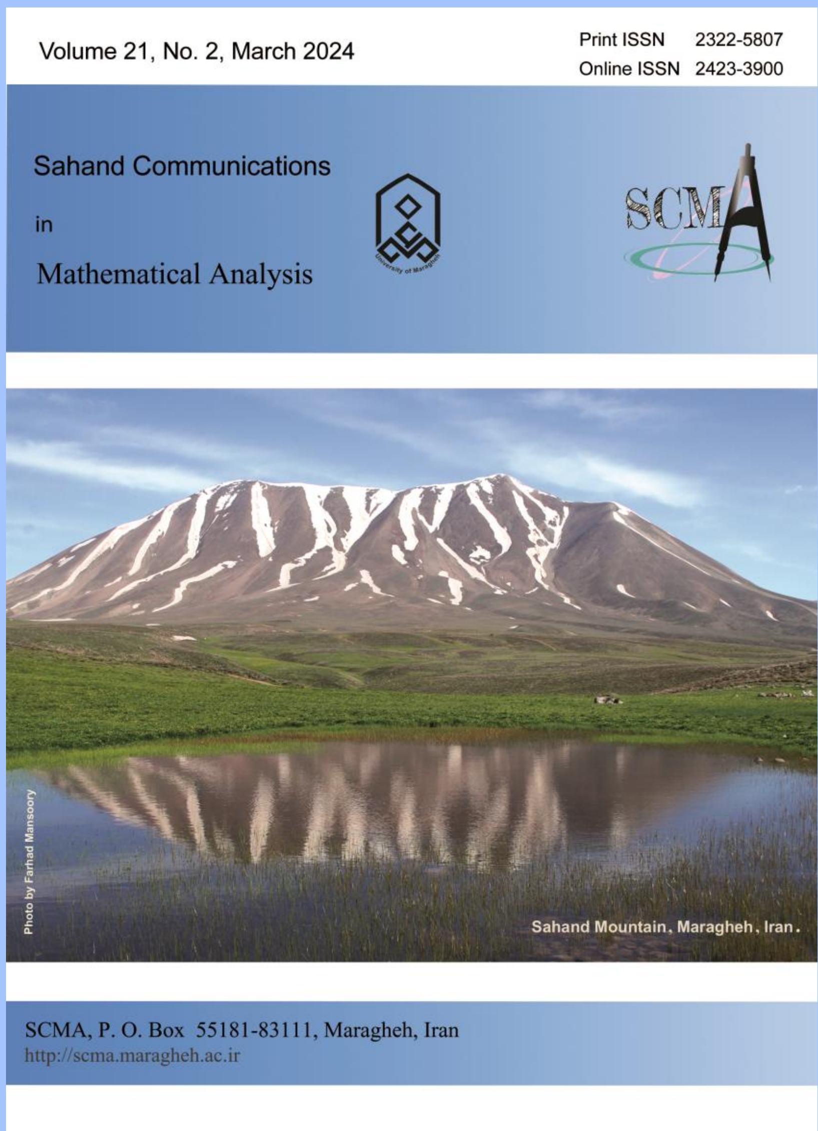

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On Continuous Frames in Hilbert C^*-Modules

Hadi Ghasemi[1] and Tayebe Lal Shateri[2] *

Abstract. In the present paper, we study continuous frames in Hilbert C^*-modules and present some results of these frames. Next, we give the concept of dual continuous frames in Hilbert C^*-modules and investigate some properties of them. Also, by introducing the notion of the similarity of the continuous frames, characterizing it, and stating some of its properties, we refer to the investigation of the effect of similarity on the dual continuous frames in Hilbert C^*-modules.

1. Introduction And Preliminaries

Frames for Hilbert spaces were introduced in 1952 by Duffin and Schaeffer [13] to study some problems in nonharmonic Fourier series and widely studied from 1986 since the great work by Daubechies, Grossmann and Meyer [11]. Now, frames have been widely applied in signal processing, sampling, filter bank theory, system modeling, Quantum information, cryptography, and more. [6, 14, 16, 29]. Various generalizations of frames such as frames of subspaces, wavelet frames, g-frames, and weighted and controlled frames have been developed, (see [5, 7, 30]). We refer to [8, 9] for an introduction to frame theory and its applications. The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by G. Kaiser [20] and independently by Ali, Antoine and Gazeau [1]. These frames are known as continuous frames.

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Frank and Larson [17] extended the notion of a frame for an operator on a Hilbert C^*-module. For a discussion of frames in Hilbert C^*-modules, we refer to Refs. [3, 4, 19, 26–28]. A Hilbert C^*-module is a generalization of a Hilbert space that allows the inner product to take values in a C^*-algebra rather than the field of complex numbers. The extended results of this more general framework are not a routine generalization, because there are essential differences between Hilbert C^*-module and Hilbert Space. For example, we know that every bounded operator on a Hilbert space has a unique adjoint, while this fact does not hold for bounded operators on a Hilbert C^*-module. Also, any closed subspace in a Hilbert space has an orthogonal complement, but it is not true, in general, for a Hilbert C^*-module.

The paper is organized as follows. First, we recall the basic definitions and some notations about Hilbert C^*-modules, and we also give some properties of them which we will use in the subsequent sections. In Section 2, we recall the notion of continuous frames in Hilbert C^*-modules. We present some results of frames in the view of continuous frames and give many interesting properties of the operators related to these frames. In section 3, we define duals of continuous frames in Hilbert C^*-modules and we introduce a special category of continuous frames in Hilbert C^*-modules called Riesz-type frames, and describe a characterization of it. Finally, in section 4, by introducing the notion of the similarity of the continuous frames, and provide a characterization of it. Also, we give some of its properties and application in recognizing Riesz-type frames express, and we refer to the investigation of the effect of similarity on the dual continuous frames in Hilbert C^*-modules.

Now, we give only a brief introduction to the theory of Hilbert C^*-modules to make our explanations self-contained. For comprehensive accounts, we refer readers to [21, 23, 24]. Throughout this paper, $\mathcal{A}$ denotes a unital C^*-algebra.

Definition 1.1. A *pre-Hilbert module* over a unital C^*-algebra $\mathcal{A}$ is a complex vector space U that is also a left $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : U \times U \to \mathcal{A}$, which is $\mathbb{C}$-linear and $\mathcal{A}$-linear in its first variable, satisfying the following conditions:

(i) $\langle f, f \rangle \geq 0$,
(ii) $\langle f, f \rangle = 0$ iff $f = 0$,
(iii) $\langle f, g \rangle^* = \langle g, f \rangle$,
(iv) $\langle af, g \rangle = a \langle f, g \rangle$,

for all $f, g \in U$ and $a \in \mathcal{A}$.

A pre-Hilbert $\mathcal{A}$-module U is called *Hilbert $\mathcal{A}$-module* if U is complete with respect to the topology determined by the norm $\|f\| = \| \langle f, f \rangle \|^{\frac{1}{2}}$.

By [19, Example 2.46], if $\mathcal{A}$ is a C^*-algebra, then it is a Hilbert $\mathcal{A}$-module with respect to the inner product

$$\langle a, b\rangle = ab^*, \quad (a, b \in \mathcal{A}).$$

Example 1.2. Let $\ell^2(\mathcal{A})$ be the set of all sequences $\{a_n\}_{n\in\mathbb{N}}$ of elements of a C^*-algebra $\mathcal{A}$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ is convergent in $\mathcal{A}$. Then $\ell^2(\mathcal{A})$ is a Hilbert $\mathcal{A}$-module with respect to the pointwise operations and inner product defined by

$$\langle \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}\rangle = \sum_{n=1}^{\infty} a_n b_n^*.$$

In the following lemma the *Cauchy-Schwartz inequality* extends to Hilbert C^*-modules.

Lemma 1.3 ([24, Lemma 15.1.3] (Cauchy-Schwartz inequality))**.** *Let U be a Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$. Then*

$$\| \langle f, g\rangle \|^2 \leq \| \langle f, f\rangle \| \, \| \langle g, g\rangle \|$$

for all $f, g \in U$.

Definition 1.4. Let U and V be two Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$. A map $T : U \to V$ is said to be *adjointable* if there exists a map $T^* : V \to U$ satisfying

$$\langle Tf, g\rangle = \langle f, T^*g\rangle$$

for all $f \in U, g \in V$. Such a map T^* is called the *adjoint* of T. By $End^*_{\mathcal{A}}(U)$ we denote the set of all adjointable maps on U.

It is surprising that every adjointable operator is automatically linear and bounded. We need the following results in next sections.

Lemma 1.5 ([31, Lemma 1.1])**.** *Let U and V be two Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$ and $T \in End^*_{\mathcal{A}}(U, V)$ have closed range. Then T^* has closed range and*

$$U = Ker(T) \oplus R(T^*), \qquad V = Ker(T^*) \oplus R(T).$$

Lemma 1.6. *[4] Let U and V be two Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$ and $T \in End^*_{\mathcal{A}}(U, V)$. Then the following are equivalent:*

(i) *T is surjective,*

(ii) *T^* is bounded below with respect to the norm; i.e.*

$$\exists m > 0 \quad s.t \quad \|T^*f\| \geq m\|f\|.$$

(iii) *T^* is bounded below with respect to the inner product; i.e.*

$$\exists m > 0 \quad s.t \quad \langle T^*f, T^*f\rangle \geq m \langle f, f\rangle,$$

for all $f \in V$.

Theorem 1.7 ([23, Theorem 2.1.4]). *Let U and V be two Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$ and $T \in End^*_{\mathcal{A}}(U,V)$. Then The following are equivalent:*

(i) *T is bounded and $\mathcal{A}$-linear.*
(ii) *There exists $k > 0$ such that*

$$\langle Tf, Tf \rangle \leq k \langle f, f \rangle ,$$

for all $f \in U$.

The next theorem is *Douglas theorem* [12] extention for Hilbert modules.

Theorem 1.8 ([15, Theorem1.1]). *Let $T_1 \in End^*_{\mathcal{A}}(U,V)$ and $T_2 \in End^*_{\mathcal{A}}(K\ ,V)$ with $\overline{R(T_2^*)}$ orthogonally complemented. Then the following are equivalent:*

(i) *$T_1T_1^* \leq \lambda T_2T_2^*$ for some $\lambda > 0$,*
(ii) *there exists $\mu > 0$ such that $\|T_1^*z\| \leq \mu \|T_2^*z\|$, for all $z \in V$,*
(iii) *there exists $D \in End^*_{\mathcal{A}}(U,K)$ such that $T_1 = T_2D$, i.e. the equation $T_2X = T_1$ has a solution,*
(iv) *$R(T_1) \subseteq R(T_2)$.*

From now, we assume that $\mathcal{A}$ is a unital C^*-algebra, U is a Hilbert C^*-module over $\mathcal{A}$ and (Ω, μ) is a measure space.

2. Continuous Frames in Hilbert C^*-Modules

In this section, we recall continuous frames in Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$, and then we give some results for these frames.

Let $\mathcal{Y}$ be a Banach space, $(\mathcal{X}, \mu)$ a measure space, and $f : \mathcal{X} \to \mathcal{Y}$ a measurable function. The integral of the Banach-valued function f has been defined by Bochner and others. Most properties of this integral are analogous to those of the integral of real-valued functions (see [10, 32]). Since every C^*-algebra and Hilbert C^*-module is a Banach space, hence we can use this integral in these spaces.

Definition 2.1. Let (Ω, μ) be a measure space and $\mathcal{A}$ is a unital C^*-algebra. We define,

$$L^2(\Omega, \mathcal{A}) = \left\{ \varphi : \Omega \to \mathcal{A}; \quad \int_{\Omega} \|\varphi(\omega)\varphi(\omega)^*\| \, d\mu(\omega) < \infty \right\}.$$

For any $\varphi, \psi \in L^2(\Omega, \mathcal{A})$, the inner product is defined by

$$\begin{aligned} \langle \varphi, \psi \rangle &= \int_{\Omega} \langle \varphi(\omega), \psi(\omega) \rangle \, d\mu(\omega) \\ &= \int_{\Omega} \varphi(\omega)\psi(\omega)^* d\mu(\omega), \end{aligned}$$

and the norm is defined by $\|\varphi\| = \| \langle \varphi, \varphi \rangle \|^{\frac{1}{2}}$. It was shown in [21] that $L^2(\Omega, \mathcal{A})$ is a Hilbert $\mathcal{A}$-module.

Now we recall the continuous frames in Hilbert $\mathcal{A}$-modules [18, 22]. We prove some interesting properties of the frame operator.

Definition 2.2. A mapping $F : \Omega \to U$ is called a continuous frame for U if

(i) F is weakly-measurable, i.e, the mapping $\omega \longmapsto \langle f, F(\omega) \rangle$ is measurable on Ω,
(ii) there exist constants $A, B > 0$ such that

$$A \langle f, f \rangle \leq \int_{\Omega} \langle f, F(\omega) \rangle \langle F(\omega), f \rangle \, d\mu(\omega) \leq B \langle f, f \rangle \,, \tag{2.1}$$

for any $f \in U$.

The constants A, B are called *lower* and *upper* frame bounds, respectively. The mapping F is called *Bessel* if only the right inequality in (2.1) holds and is called *tight* if $A = B$.

Definition 2.3. A continuous frame $F : \Omega \to U$ is called *exact* if for every measurable subset $\Omega_1 \subseteq \Omega$ with $0 < \mu(\Omega_1) < \infty$, the mapping $F : \Omega \backslash \Omega_1 \to U$ is not a continuous frame for U.

Now, we give an example of a continuous frame in a Hilbert C^*-module.

Example 2.4. Assume that $\mathcal{A} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : x, y \in \mathbb{C} \right\}$, then $\mathcal{A}$ is a unital C^*-algebra. Also $\mathcal{A}$ is a Hilbert C^*-module over itself, with the following inner product,

$$\begin{aligned} \langle ., . \rangle : \mathcal{A} \times \mathcal{A} \ &\rightarrow \quad \mathcal{A} \\ (M, N) &\longmapsto M(\overline{N})^t. \end{aligned}$$

Suppose that (Ω, μ) is a measure space where $\Omega = [0, 1]$ and μ is the Lebesgue measure. Consider the mapping $F : \Omega \to \mathcal{A}$ defined as $F(\omega) = \begin{pmatrix} 2\omega & 0 \\ 0 & \omega - 1 \end{pmatrix}$, for any $\omega \in \Omega$.

For each $f = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathcal{A}$, we have

$$\begin{aligned} &\int_{\Omega} \langle f, F(\omega) \rangle \langle F(\omega), f \rangle \, d\mu(\omega) \\ &\quad = \int_{[0,1]} \left\langle \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 2\omega & 0 \\ 0 & \omega - 1 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 2\omega & 0 \\ 0 & \omega - 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\rangle d\mu(\omega) \end{aligned}$$

$$= \int_{[0,1]} \begin{pmatrix} 2\omega a & 0 \\ 0 & (\omega-1)b \end{pmatrix} \begin{pmatrix} 2\omega \overline{a} & 0 \\ 0 & (\omega-1)\overline{b} \end{pmatrix} d\mu(\omega)$$
$$= \int_{[0,1]} \begin{pmatrix} 4\omega^2 & 0 \\ 0 & (\omega-1)^2 \end{pmatrix} \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} d\mu(\omega)$$
$$= \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \int_{[0,1]} \begin{pmatrix} 4\omega^2 & 0 \\ 0 & (\omega-1)^2 \end{pmatrix} d\mu(\omega)$$
$$= \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix},$$

hence

$$\frac{1}{3} \langle f, f \rangle \leq \int_{\Omega} \langle f, F(\omega) \rangle \langle F(\omega), f \rangle \, d\mu(\omega) \leq \frac{4}{3} \langle f, f \rangle .$$

Therefore, F is a continuous frame with bounds $A = \frac{1}{3}$ and $B = \frac{4}{3}$.

Now, we introduce some important operators for continuous frames in Hilbert C^*-modules.

Definition 2.5. Let $F : \Omega \to U$ be a continuous frame. Then

(i) The *synthesis operator* or *pre-frame operator* $T_F : L^2(\Omega, \mathcal{A}) \to U$ weakly defined by

$$\langle T_F \varphi, f \rangle = \int_{\Omega} \varphi(\omega) \langle F(\omega), f \rangle \, d\mu(\omega), \quad (f \in U). \tag{2.2}$$

(ii) The adjoint of T, called The *analysis operator* $T_F^* : U \to L^2(\Omega, \mathcal{A})$ is defined by

$$(T_F^* f)(\omega) = \langle f, F(\omega) \rangle , \quad (\omega \in \Omega, f \in U). \tag{2.3}$$

Definition 2.6. Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U. Then the *continuous frame operator* $S_F : U \to U$ is weakly defined by

$$\langle S_F f, f \rangle = \int_{\Omega} \langle f, F(\omega) \rangle \langle F(\omega), f \rangle \, d\mu(\omega), \quad (f \in U). \tag{2.4}$$

In following theorem, we investigate some properties of pre-frame operator and analysis operator.

Theorem 2.7. *Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U with bounds A, B. Then the pre-frame operator $T_F : L^2(\Omega, A) \to U$ is well defined, surjective, adjointable $\mathcal{A}$-linear map and* *bounded with* $\|T\| \leq \sqrt{B}$. *Moreover, the analysis operator* $T_F^* : U \to L^2(\Omega, A)$ *is injective and has closed range.*

Proof. Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U with bounds A, B. Then

(i) T is adjointable and T^* is its adjoint. Because for $f \in U$ and $\varphi \in L^2(\Omega, A)$ we have

$$\begin{aligned}
\langle \varphi, T^* f \rangle &= \int_\Omega \langle \varphi(\omega), (T^* f)(\omega) \rangle \, d\mu(\omega) \\
&= \int_\Omega \langle \varphi(\omega), \langle f, F(\omega) \rangle \rangle \, d\mu(\omega) \\
&= \int_\Omega \varphi(\omega) \langle F(\omega), f \rangle \, d\mu(\omega) \\
&= \left\langle \int_\Omega \varphi(\omega) F(\omega) d\mu(\omega), f \right\rangle \\
&= \langle T\varphi, f \rangle .
\end{aligned}$$

(ii) The pre-frame operator T is well defined and bounded with $\|T\| \leq \sqrt{B}$ because for $\varphi \in L^2(\Omega, A)$ we have

$$\begin{aligned}
\|T\varphi\|^2 &= \left\| \int_\Omega \varphi(\omega) F(\omega) d\mu(\omega) \right\|^2 \\
&= \sup_{f \in U, \, \|f\|=1} \left\| \left\langle \int_\Omega \varphi(\omega) F(\omega) d\mu(\omega), f \right\rangle \right\| \\
&= \sup_{f \in U, \, \|f\|=1} \left\| \int_\Omega \langle \varphi(\omega), \langle f, F(\omega) \rangle \rangle \, d\mu(\omega) \right\| \\
&= \sup_{f \in U, \, \|f\|=1} \| \langle \varphi, T^* f \rangle \| \\
&\leq \sup_{f \in U, \, \|f\|=1} \| \langle \varphi, \varphi \rangle \| \, \| \langle T^* f, T^* f \rangle \| \\
&= \sup_{f \in U, \, \|f\|=1} \|\varphi\|^2 \, \left\| \int_\Omega \langle f, F(\omega) \rangle \langle F(\omega), f \rangle \, d\mu(\omega) \right\|,
\end{aligned}$$

since F is a continuous frame, so

$$\begin{aligned}
\|T\varphi\|^2 &\leq \sup_{f \in U, \quad \|f\|=1} \|\varphi\|^2 B \|f\|^2 \\
&\leq B \|\varphi\|^2,
\end{aligned}$$

i.e., $\|T\| \leq \sqrt{B}$.

(iii) T is surjective. Indeed, by definition of continuous frames in Hilbert C^*-modules, for each $f \in U$,

$$A\langle f, f\rangle \leq \langle T^*f, T^*f\rangle \leq B\langle f, f\rangle .$$

Then T^* is bounded below with respect to the inner product and by lemma 1.6, T is surjective.

(iv) T^* is injective. Indeed, if $f \in U$ and $T^*f = 0$, then

$$\begin{aligned} \|\langle f, f\rangle\| &= \|A^{-1}A\langle f, f\rangle\| \\ &= A^{-1}\|A\langle f, f\rangle\| \\ &\leq A^{-1}\|\langle T^*f, T^*f\rangle\| \\ &= A^{-1}\|T^*f\|^2. \end{aligned}$$

Thus $\|\langle f, f\rangle\| = 0$ and $f = 0$. Now, we show that T^* has closed range. Let $\{T^*f_n\}_{n=1}^{\infty}$ be a sequence in $R(T^*)$ such that $\lim_{n\to\infty} T^*f_n = g$. By definition of continuous frames in Hilbert C^*-modules, for $n, m \in N$

$$\begin{aligned} \|A\langle f_n - f_m, f_n - f_m\rangle\| &\leq \|\langle T^*(f_n - f_m), T^*(f_n - f_m)\rangle\| \\ &= \|T^*(f_n - f_m)\|^2. \end{aligned}$$

Since $\{T^*f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega, A)$ so,

$$\lim_{n,m\to\infty} \|A\langle f_n - f_m, f_n - f_m\rangle\| = 0.$$

Also

$$\|\langle f_n - f_m, f_n - f_m\rangle\| \leq A^{-1}\|A\langle f_n - f_m, f_n - f_m\rangle\|,$$

thus the sequence $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in U and so there exists $f \in U$ such that $\lim_{n\to\infty} f_n = f$. Definition of continuous frames, implies that

$$\|T^*(f_n - f)\|^2 \leq B\|\langle f_n - f, f_n - f\rangle\|,$$

then $\lim_{n\to\infty} \|T^*f_n - T^*f\| = 0$ and $T^*f = g$. Therefore $R(T^*)$ is close. □

In the next theorem, we give some properties of continuous frame operator.

Theorem 2.8. *Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U with bounds A, B and continuous frame operator S and pre-frame operator T. Then $S = TT^*$ is positive, adjointable, self-adjoint and invertible and $\|S\| \leq B$.*

Proof. Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U with bounds A, B. Then

(i) $S = TT^*$ because for $f \in U$ we have

$$\begin{aligned}\langle TT^* f, f\rangle &= \left\langle T(\{\langle f, F(\omega)\rangle\}_{\omega\in\Omega}), f\right\rangle \\ &= \left\langle \int_\Omega \langle f, F(\omega)\rangle F(\omega) d\mu(\omega), f\right\rangle \\ &= \langle Sf, f\rangle .\end{aligned}$$

(ii) For $f, g \in U$, we have

$$\begin{aligned}\langle Sf, g\rangle &= \left\langle \int_\Omega \langle f, F(\omega)\rangle F(\omega) d\mu(\omega), g\right\rangle \\ &= \int_\Omega \langle f, F(\omega)\rangle \langle F(\omega), g\rangle d\mu(\omega) \\ &= \int_\Omega (\langle g, F(\omega)\rangle \langle F(\omega), f\rangle)^* d\mu(\omega) \\ &= \left(\int_\Omega \langle g, F(\omega)\rangle \langle F(\omega), f\rangle d\mu(\omega)\right)^* \\ &= (\langle Sg, f\rangle)^* \\ &= \langle f, Sg\rangle .\end{aligned}$$

Therefore S is adjointable and self-adjoint.

(iii) By definition of continuous frames, for each $f \in U$ we get

$$\begin{aligned}A\langle f, f\rangle &\leq \langle Sf, f\rangle \\ &\leq B\langle f, f\rangle ,\end{aligned}$$

by [23, Proposition 2.1.3], this implies that S is positive. Since $AI \leq S \leq BI$, so

$$\begin{aligned}0 \leq I - B^{-1}S &\leq I - B^{-1}AI \\ &= \frac{B-A}{B} I \\ &< I,\end{aligned}$$

hence $\|I - B^{-1}S\| < 1$. Therefore S is invertible.

(iv) Since

$$\begin{aligned}\|S\| &= \sup_{f\in U,\ \|f\|\leq 1} \|\langle Sf, f\rangle\| \\ &= \sup_{f\in U,\ \|f\|\leq 1,} \left\| \int_\Omega \langle f, F(\omega)\rangle \langle F(\omega), f\rangle d\mu(\omega)\right\| \\ &\leq \sup_{f\in U,\ \|f\|\leq 1} \|B\langle f, f\rangle\| \\ &\leq B.\end{aligned}$$

Hence, S is bounded.

□

Remark 2.9. Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U with the frame operator S and the pre-frame operator T. Invertibility of S implies that T is surjective.
In fact, since S is invertible, so for all $f \in U$ there exists $g \in U$ such that

$$\begin{aligned} f &= Sg \\ &= \int_\Omega \langle g, F(\omega)\rangle F(\omega) d\mu(\omega). \end{aligned}$$

On the other hand, $S = TT^*$ then

$$\begin{aligned} f &= TT^* g \\ &= \int_\Omega \langle g, F(\omega)\rangle F(\omega) d\mu(\omega), \end{aligned}$$

this implies that

$$T^* g = \{\langle g, F(\omega)\rangle\}_{\omega\in\Omega} \in L^2(\Omega, A),$$

i.e., T is surjective.

Remark 2.10. Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U with the frame operator S and the pre-frame operator T. Since $T^* : U \to L^2(\Omega, A)$ is injective and has closed range so by [3, Lemma 0.1], $S = TT^*$ is invertible and

$$\|S^{-1}\|^{-1} I \leq S \leq \|T^*\|^2 I.$$

Moreover, lower and upper bounds of F are respectively $\|S^{-1}\|^{-1}$ and $\|T\|^2$.

Theorem 2.11. *Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U with bounds $A, B > 0$, continuous frame operator S_F and pre-frame operator T_F.Also let $K \in End^*_{\mathcal{A}}(U)$ be surjective.Then KF is a continuous frame for U with continuous frame operator KS_FK^*. Moreover, $T_{KF} = KT_F$ and $T^*_{KF} = T^*_F K^*$.*

Proof. For each $f \in U$ we have

$$\begin{aligned} A\langle K^*f, K^*f\rangle &\leq \int_\Omega \langle K^*f, F(\omega)\rangle \langle F(\omega), K^*f\rangle d\mu(\omega) \\ &\leq B\langle K^*f, K^*f\rangle, \end{aligned}$$

thus

$$\begin{aligned} A\langle K^*f, K^*f\rangle &\leq \int_\Omega \langle f, KF(\omega)\rangle \langle KF(\omega), f\rangle d\mu(\omega) \\ &\leq B\langle K^*f, K^*f\rangle. \end{aligned}$$

Since K is surjective, then by [3, Lemma 0.1],

$$\|(KK^*)^{-1}\|^{-1} = \|(KK^*)^{-1}\|^{-1} \leq KK^*$$

and

$$\begin{aligned}\|(KK^*)^{-1}\|^{-1} \langle f, f\rangle &\leq \langle KK^* f, f\rangle \\ &\leq \langle K^* f, K^* f\rangle \\ &\leq \|K^2\| \langle f, f\rangle .\end{aligned}$$

Hence,

$$\begin{aligned}A\|(KK^*)^{-1}\|^{-1} \langle f, f\rangle &\leq \int_\Omega \langle f, KF(\omega)\rangle \langle KF(\omega), f\rangle \, d\mu(\omega) \\ &\leq B\|K^2\| \langle f, f\rangle ,\end{aligned}$$

i.e., KF is a continuous frame for Hilbert C^*-module U.
Also, for each $f \in U$ we have

$$\begin{aligned}KS_F K^* f &= K \int_\Omega \langle K^* f, F(\omega)\rangle F(\omega) d\mu(\omega) \\ &= \int_\Omega \langle f, KF(\omega)\rangle KF(\omega) d\mu(\omega),\end{aligned}$$

then $KS_F K^*$ is the continuous frame operator for KF. Moreover,

$$\begin{aligned}\langle T_{KF}\varphi, f\rangle &= \left\langle \int_\Omega \varphi(\omega) KF(\omega) d\mu(\omega), f \right\rangle \\ &= \left\langle K \int_\Omega \varphi(\omega) F(\omega) d\mu(\omega), f \right\rangle \\ &= \langle KT_F\varphi, f\rangle ,\end{aligned}$$

for each $f \in U$ and $\varphi \in L^2(\Omega, \mathcal{A})$. Then $T_{KF} = KT_F$ and $T^*_{KF} = T^*_F K^*$. □

Since $AI \leq S \leq BI$, so

$$B^{-1}I \leq S^{-1} \leq A^{-1}I.$$

Hence, we get the following corollary.

Corollary 2.12. *Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U with bounds A, B and frame operator S. Then $S^{-1}F$ is a continuous frame for Hilbert C^*-module U with bounds $B^{-1}, A^{-1} > 0$ and frame operator S^{-1}.*

Lemma 2.13. *Let U be a Hilbert C^*-module. Then the following are equivalent:*

(i) *$F : \Omega \to U$ is a continuous Bessel mapping for U.*

(ii) *The mapping* $\Omega \to \langle f, F(\Omega)\rangle$ *is measurable and there exists constant* $D > 0$ *such that*

$$\left\| \int_{\Omega} \langle f, F(\omega)\rangle \langle F(\omega), f\rangle \, d\mu(\omega) \right\| \leq D \left\| \langle f, f\rangle \right\|.$$

Proof. $(i) \Longrightarrow (ii)$ Obvious.
$(ii) \Longrightarrow (i)$ Define an operator $V : U \to L^2(\Omega, A)$ by $Vf = \{\langle f, F(\Omega)\rangle\}_{\omega\in\Omega}$. It is clear that V is well-defined and $\mathcal{A}$-linear. Also,

$$\begin{aligned} \|Vf\|^2 &= \|\langle Vf, Vf\rangle\| \\ &= \left\| \int_{\Omega} \langle f, F(\omega)\rangle \langle F(\omega), f\rangle \, d\mu(\omega) \right\| \\ &\leq D\| \langle f, f\rangle \| \\ &= D\|f\|^2. \end{aligned}$$

Then V is bounded and by theorem 1.7 $\langle Vf, Vf\rangle \leq D \langle f, f\rangle$. □

Now, we give an equivalent condition for the mapping $F : \Omega \to U$ to become a continuous frame.

Theorem 2.14. *Let* U *be a Hilbert* C^**-module. Then the following are equivalent:*

(i) $F : \Omega \to U$ *is a continuous frame for* U*.*

(ii) *The mapping* $\Omega \to \langle f, F(\Omega)\rangle$ *is measurable and there exist constants* $A, B > 0$ *such that*

$$\begin{aligned} A\| \langle f, f\rangle \| &\leq \left\| \int_{\Omega} \langle f, F(\omega)\rangle \langle F(\omega), f\rangle \, d\mu(\omega) \right\| \\ &\leq B\| \langle f, f\rangle \|. \end{aligned}$$

Proof. $(i) \Longrightarrow (ii)$ Obvious.
$(ii) \Longrightarrow (i)$ Suppose that there exist constants $A, B > 0$ such that

$$\begin{aligned} A\| \langle f, f\rangle \| &\leq \left\| \int_{\Omega} \langle f, F(\omega)\rangle \langle F(\omega), f\rangle \, d\mu(\omega) \right\| \\ &\leq B\| \langle f, f\rangle \|. \end{aligned}$$

then

$$\begin{aligned} \left\| \int_{\Omega} \langle f, F(\omega)\rangle \langle F(\omega), f\rangle \, d\mu(\omega) \right\| &= \|\langle Sf, f\rangle\| \\ &= \left\| \left\langle S^{\frac{1}{2}} f, S^{\frac{1}{2}} f \right\rangle \right\| \\ &= \|S^{\frac{1}{2}} f\|^2, \end{aligned}$$

then, $A\|f\|^2 \leq \|S^{\frac{1}{2}}f\|^2$ implies that

$$\sqrt{A}\|f\| \leq \|S^{\frac{1}{2}}f\|.$$

Hence by Lemma 1.6,

$$\sqrt{A}\langle f,f\rangle \leq \left\langle S^{\frac{1}{2}}f, S^{\frac{1}{2}}f\right\rangle = \langle Sf,f\rangle$$

Since S is self-adjoint so by theorem 1.7, there exists $D>0$ such that

$$\langle Sf,f\rangle = \left\langle S^{\frac{1}{2}}f, S^{\frac{1}{2}}f\right\rangle \leq D\langle f,f\rangle .$$

Hence

$$\begin{aligned}\sqrt{A}\langle f,f\rangle &\leq \langle Sf,f\rangle \\ &= \int_\Omega \langle f,F(\omega)\rangle \langle F(\omega),f\rangle \, d\mu(\omega) \\ &\leq D\langle f,f\rangle .\end{aligned}$$

□

Lemma 2.15. *Let $K:\Omega\to U$ be a continuous Bessel mapping for U with bound B and $V\in End^*_{\mathcal{A}}(U)$ be adjointable. Then $VK:\Omega\to U$ is continuous Bessel mapping for U.*

Proof. By theorem 1.7, there exists a constant $D>0$ such that

$$\begin{aligned}\int_\Omega \langle f,VK(\omega)\rangle \langle VK(\omega),f\rangle \, d\mu(\omega) &= \int_\Omega \langle V^*f,K(\omega)\rangle \langle K(\omega),V^*f\rangle \, d\mu(\omega) \\ &\leq B\langle V^*f,V^*f\rangle \\ &\leq BD\langle f,f\rangle ,\end{aligned}$$

for each $f\in U$. □

Theorem 2.16. *Let U be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. Then a mapping $F:\Omega\to U$ is a continuous frame for U if and only if the synthesis operator $T:L^2(\Omega,\mathcal{A})\to U$ is well-defined and onto.*

Proof. (Necessity) It is shown in Theorem 2.7 .
(Sufficiency) Let T be well-defined. Then T is adjointable and

$$\begin{aligned}T^* :& U \to L^2(\Omega,\mathcal{A}) \\ & f \longmapsto \{\langle f,F(\omega)\rangle\}_{\omega\in\Omega} .\end{aligned}$$

Also

$$\begin{aligned}&\left\|\int_\Omega \langle f,F(\omega)\rangle \langle F(\omega),f\rangle \, d\mu(\omega)\right\|^2 \\ &\quad = \left\|\left\langle f, \int_\Omega \langle f,F(\omega)\rangle F(\omega) d\mu(\omega)\right\rangle\right\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \|f\|^2 \left\|T(\{\langle f, F(\omega)\rangle\}_{\omega\in\Omega})\right\|^2 \\
&\leq \|f\|^2\|T\|^2 \left\|\{\langle f, F(\omega)\rangle\}_{\omega\in\Omega}\right\|^2 \\
&= \|f\|^2\|T\|^2 \left\|\int_\Omega \langle f, F(\omega)\rangle \langle F(\omega), f\rangle \, d\mu(\omega)\right\|.
\end{aligned}
$$

Then

$$
\begin{aligned}
\left\|\int_\Omega \langle f, F(\omega)\rangle \langle F(\omega), f\rangle \, d\mu(\omega)\right\| &\leq \|f\|^2\|T\|^2 \\
&= \| \langle f, f\rangle \| \|T\|^2.
\end{aligned}
$$

Moreover, since T is onto, so T^* is bounded below by lemma 1.6, and $T^*\|_{R(T^*)}$ is invertible. Then for each $f \in U$ we have $(T^*\|_{R(T^*)})^{-1}T^*f = f$. Then

$$
\begin{aligned}
\|f\|^2 &= \|(T^*\|_{R(T^*)})^{-1}T^*f\|^2 \\
&\leq \|(T^*\|_{R(T^*)})^{-1}\|^2\|T^*f\|^2 \\
&= \|(T^*\|_{R(T^*)})^{-1}\|^2 \left\|\int_\Omega \langle f, F(\omega)\rangle \langle F(\omega), f\rangle \, d\mu(\omega)\right\|,
\end{aligned}
$$

hence

$$
\|(T^*\|_{R(T^*)})^{-1}\|^{-2}\| \langle f, f\rangle \| \leq \left\|\int_\Omega \langle f, F(\omega)\rangle \langle F(\omega), f\rangle \, d\mu(\omega)\right\|. \qquad \square
$$

Corollary 2.17. *Let U be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. Then a mapping $F : \Omega \to U$ is a continuous Bessel mapping with bound B for U if and only if the synthesis operator $T : L^2(\Omega, \mathcal{A}) \to U$ is well-defined and bounded with $\|T\| \leq \sqrt{B}$.*

3. Duals continuous frames in Hilbert C^*-modules

In this section, we introduce the consept of duals continuous frames in Hilbert C^*-modules and give some properties of continuous frames and their duals. Duals are very important for the reconstruction of Hilbert C^*-module elements.

Definition 3.1. Let $F : \Omega \to U$ be a continuous Bessel mapping. A continuous Bessel mapping $G : \Omega \to U$ is called a *dual* for F if

$$
\langle f, g\rangle = \int_\Omega \langle f, G(\omega)\rangle \langle F(\omega), g\rangle \, d\mu(\omega), \quad (f, g \in U), \tag{3.1}
$$

and equivalently $T_F T_G^* = I_U$, where T_F and T_G denote the synthesis operators of F and G, respectively. In this case (F, G) is called a *dual pair*. Since $T_F T_G^* = I_U$ if and only if $T_G T_F^* = I_U$, so (3.1) is equivalent to

$$
\langle f, g\rangle = \int_\Omega \langle f, F(\omega)\rangle \langle G(\omega), g\rangle \, d\mu(\omega), \quad (f, g \in U).
$$

Remark 3.2. Let $F:\Omega\to U$ be a continuous frame for Hilbert C^*-module U. Then by reconstructin formula we have

$$\begin{aligned} f &= \int_\Omega \langle f, F(\omega)\rangle\, S^{-1}F(\omega)d\mu(\omega) \\ &= \int_\Omega \left\langle f, S^{-1}F(\omega)\right\rangle F(\omega)d\mu(\omega), \end{aligned}$$

for all $f\in U$.

Then $S^{-1}F$ is a dual for F, which is called *canonical dual.*

Definition 3.3. Let $F:\Omega\to U$ be a continuous frame for Hilbert C^*-module U. If F has only one dual, we call F a *Riesz-type frame.*

The following theorem states a useful characterizatoin of Riesz-type frames.

Theorem 3.4. *Let $F:\Omega\to U$ be a continuous frame for Hilbert C^*-module U over a unital C^*-algebra $\mathcal{A}$. Then F is a Riesz-type frame if and only if the analysis operator $T_F^*:U\to L^2(\Omega,A)$ is onto.*

Proof. (Necessity) Since T_F is adjointable and has closed range, so $L^2(\Omega,A) = Ker(T_F)\oplus R(T_F^*)$ by Lemma 1.5. Also by [24, Lemma 15.3.4],

$$R(T_F^*)\neq L^2(\Omega,A) \quad \text{if and only if} \quad R(T_F^*)^\perp \neq \{0\}\,.$$

Now let $G=S^{-1}F$ be the canonical dual of F and $R(T_F^*)\neq L^2(\Omega,A)$. Then $R(T_F^*)^\perp\neq\{0\}$.
Suppose that $h\in R(T_F^*)^\perp$ such that $\|\int_\Omega \|h(\omega)^*\|^2 d\mu(\omega)\| = 1$ i.e., $\|h\|=1$. Define

$$\begin{aligned} K :&\Omega \to L^2(\Omega,A) \\ &\omega \longmapsto h(\omega)^*h. \end{aligned}$$

Then for $\varphi\in L^2(\Omega,A)$ we have,

$$\begin{aligned} \left\|\int_\Omega \langle\varphi, K(\omega)\rangle \langle K(\omega),\varphi\rangle\, d\mu(\omega)\right\| &= \left\|\int_\Omega \langle\varphi, h(\omega)^*h\rangle \langle h(\omega)^*h,\varphi\rangle\, d\mu(\omega)\right\| \\ &= \left\|\int_\Omega \langle\varphi, h\rangle\, h(\omega)h(\omega)^* \langle h,\varphi\rangle\, d\mu(\omega)\right\| \\ &\le \|\langle\varphi,h\rangle\| \left\|\int_\Omega \|h(\omega)^*\|^2 d\mu(\omega)\|\langle h,\phi\rangle\|\right\| \\ &= \|\langle\varphi,h\rangle\|\|\langle h,\varphi\rangle\| \\ &\le \|h\|^2\|\varphi\|^2 \\ &= \|h\|^2\|\langle\varphi,\varphi\rangle\|, \end{aligned}$$

so K is a continuous Bessel mapping.
Let $V : L^2(\Omega, A) \to U$ be a adjointable operator such that $Vh \neq 0$. Then $VK : \Omega \to U$ is a continuous Bessel mapping and so $G + VK$ is. If we show that

$$\int_\Omega \langle f, VK(\omega)\rangle \langle F(\omega), g\rangle \, d\mu(\omega) = 0, \quad (f, g \in U),$$

then $G + VK$ is a dual of F. Because for $f, g \in U$,

$$\begin{aligned}
&\int_\Omega \langle f, G(\omega) + VK(\omega)\rangle \langle F(\omega), g\rangle \, d\mu(\omega) \\
&\quad = \int_\Omega \langle f, G(\omega)\rangle \langle F(\omega), g\rangle \, d\mu(\omega) + \int_\Omega \langle f, VK(\omega)\rangle \langle F(\omega), g\rangle \, d\mu(\omega) \\
&\quad = \langle f, g\rangle + 0 \\
&\quad = \langle f, g\rangle ,
\end{aligned}$$

i.e., F is not Riesz-type.For this we have

$$\begin{aligned}
\int_\Omega \langle f, VK(\omega)\rangle \langle F(\omega), g\rangle \, d\mu(\omega) &= \int_\Omega \langle f, Vh(\omega)^* h\rangle \langle F(\omega), g\rangle \, d\mu(\omega) \\
&= \int_\Omega \langle V^* f, h\rangle \, h(\omega) \langle F(\omega), g\rangle \, d\mu(\omega) \\
&= \langle V^* f, h\rangle \int_\Omega h(\omega) \langle F(\omega), g\rangle \, d\mu(\omega) \\
&= \langle V^* f, h\rangle \left\langle \{h(\omega)\}_{\omega\in\Omega}, \{\langle g, F(\omega)\rangle\}_{\omega\in\Omega} \right\rangle \\
&= 0.
\end{aligned}$$

Note that $\{h(\omega)\}_{\omega\in\Omega} \in R(T_F^*)^\perp$ and $\{\langle g, F(\omega)\rangle\}_{\omega\in\Omega} \in R(T_F^*)$.

(Sufficiency) Let G_1, G_2 be two duals of F and $G_1 \neq G_2$. Then

$$\begin{aligned}
&\int_\Omega \langle f, G_1(\omega) - G_2(\omega)\rangle \langle F(\omega), g\rangle \, d\mu(\omega) \\
&\quad = \int_\Omega \langle f, G_1(\omega)\rangle \langle F(\omega), g\rangle \, d\mu(\omega) - \int_\Omega \langle f, G_2(\omega)\rangle \langle F(\omega), g\rangle \, d\mu(\omega) \\
&\quad = \langle f, g\rangle - \langle f, g\rangle \\
&\quad = 0.
\end{aligned}$$

Hence

$$\left\langle \{\langle f, G_1(\omega) - G_2(\omega)\rangle\}_{\omega\in\Omega}, \{\langle g, F(\omega)\rangle\}_{\omega\in\Omega} \right\rangle = 0.$$

Since $\{\langle f, G_1(\omega) - G_2(\omega)\rangle\}_{\omega\in\Omega} \in R(T_{G_1-G_2}^*)$ and $\{\langle g, F(\omega)\rangle\}_{\omega\in\Omega} \in R(T_F^*)$, so

$$R(T_{G_1-G_2}^*) \perp R(T_F^*).$$

Also T_F^* is onto and $L^2(\Omega, A) = Ker(T_F) \oplus R(T_F^*)$, then $R(T_F^*)^{\perp} = \{0\}$ and so

$$\langle f, G_1(\omega) - G_2(\omega)\rangle = 0, \quad (f \in U),$$

hence

$$G_1(\omega) - G_2(\omega) = 0, \quad (\omega \in \Omega).$$

Therefore $G_1 = G_2$, that is F is Riesz-type. □

We end this section by the following remark.

Remark 3.5. Let $F : \Omega \to U$ be a Riesz-type frame for Hilbert C^*-module U. Then $F(\omega) \neq 0$ for every $\omega \in \Omega$.

For this, let G be the canonical dual of F and $\omega_0 \in \Omega$ such that $F(\omega_0) = 0$. Define $G_1 : \Omega \to U$ where $G_1(\omega_0) \neq 0$ and $G_1(\omega) = G(\omega)$ for all $\omega \neq \omega_0$. Then G_1 is a continuous Bessel mapping and

$$\begin{aligned} f &= \int_{\Omega} \langle f, G(\omega)\rangle F(\omega) d\mu(\omega) \\ &= \int_{\{\omega_0\}} \langle f, G(\omega)\rangle F(\omega) d\mu(\omega) + \int_{\Omega \setminus \{\omega_0\}} \langle f, G(\omega)\rangle F(\omega) d\mu(\omega) \\ &= \int_{\Omega} \langle f, G_1(\omega)\rangle F(\omega) d\mu(\omega). \end{aligned}$$

Hence G_1 is a dual of F and F is not Riesz-type.

4. Similar Continuous Frames in Hilbert C^*-Modules

The notion "similar" for g-frames in Hilbert spaces and Hilbert C^*-modules has been defined in [2, 25]. We use this concept for continuous frames in Hilbert C^*-modules and obtain some equivalent conditions for them. Then we get some results about it. This concept is a suitable tool to prove some features of Riesz-type frames.

Definition 4.1. Two continuous frames $F, G : \Omega \to U$ on Hilbert C^*-module U are called *similar* if there exists an invertible operator $L \in End_{\mathcal{A}}^*(U)$ such that $F = LG$. Moreover, if L is unitary operator, then F and G are called *unitary equivalent.*

Obviously, F is similar to G if and only if G is similar to F. First, we state an interesting characterization of similar continuous frames, which has many applications in the proofs of other theorems.

Theorem 4.2. *Let $F, G : \Omega \to U$ be two continuous frames for U with pre-frame operators T_F and T_G, respectively. Then F and G are similar if and only if $R(T_F^*) = R(T_G^*)$.*

Proof. Assume that F and G are similar. Then there exists an invertible operator $L \in End^*_{\mathcal{A}}(U)$ such that $F = LG$ and so $T^*_F = T^*_G L^*$ and by Theorem 1.8, $R(T^*_F) \subseteq R(T^*_G)$. Also, $T^*_G = T^*_F (L^{-1})^*$ implies that $R(T^*_G) \subseteq R(T^*_F)$. Hence $R(T^*_F) = R(T^*_G)$.

Conversely, suppose that $R(T^*_F) = R(T^*_G)$. Then there exist $L_1, L_2 \in End^*_{\mathcal{A}}(U)$ such that

$$T^*_F = T^*_G L_1, \qquad T^*_G = T^*_F L_2.$$

Therefore $T^*_F = T^*_F L_2 L_1$ and $T^*_G = T^*_G L_1 L_2$. This shows that $L_1 L_2 = L_2 L_1 = I_U$ and so L_1 , L_2 are invertible and $L_1 = L_2^{-1}$. Hence, $T_F = LT_G$ for some invertible operator $L \in End^*_{\mathcal{A}}(U)$. For each $f \in U$ and $\varphi \in L^2(\Omega, \mathcal{A})$, we have

$$\begin{aligned} \int_\Omega \varphi(\omega) \langle F(\omega), f \rangle \, d\mu(\omega) &= L \int_\Omega \varphi(\omega) \langle G(\omega), f \rangle \, d\mu(\omega) \\ &= \int_\Omega \varphi(\omega) \langle LG(\omega), f \rangle \, d\mu(\omega). \end{aligned}$$

Then

$$\begin{aligned} \left\langle \varphi, \{\langle f, F(\omega) - LG(\omega) \rangle\}_{\omega\in\Omega} \right\rangle &= \int_\Omega \varphi(\omega) \langle F(\omega) - LG(\omega), f \rangle \, d\mu(\omega) \\ &= 0. \end{aligned}$$

This implies that $F(\omega) = LG(\omega)$, for all $\omega \in \Omega$. Therefore F and G are similar continuous frames. ☐

In the following theorem, we give the application of Theorem 4.2 in proving the fact that how an arbitrary continuous frame becomes a Riesz-type frame with the influence of another Riesz-type frame.

Theorem 4.3. *Let $F, G : \Omega \to U$ be two continuous frames for U with pre-frame operators T_F and T_G, respectively. If F is a Riesz-type frame, then G is a Riesz-type frame if and only if there exists a constant $N > 0$ such that*

$$\|T_F\varphi - T_G\varphi\|^2 \leq N. \min\left\{\|T_F\varphi\|^2, \|T_G\varphi\|^2\right\},$$

for each $\varphi \in L^2(\Omega, \mathcal{A})$.

Proof. Suppose that G is a Riesz-type frame. Then F and G are similar and there exists an invertible operator $L \in End^*_{\mathcal{A}}(U)$ such that $F = LG$, by Theorem 4.2. If $f = T_G\varphi$, for some $\varphi \in L^2(\Omega, \mathcal{A})$, then

$$\begin{aligned} Lf &= \int_\Omega \varphi(\omega) LG(\omega) d\mu(\omega) \\ &= \int_\Omega \varphi(\omega) F(\omega) d\mu(\omega) = T_F\varphi. \end{aligned}$$

Therefore

$$\begin{aligned}\|T_F\varphi - T_G\varphi\|^2 &= \|Lf - f\|^2 \\ &= \| \langle Lf - f, Lf - f\rangle \| \\ &\leq \|L\|^2\|f\|^2 + 2\|L\|\|f\|^2 + \|f\|^2 \\ &= (\|L\| + 1)^2\|T_G\varphi\|^2.\end{aligned}$$

Since $f = L^{-1}Lf$, so

$$\|f\|^2 \leq \|L^{-1}\|^2\|Lf\|^2, \qquad \| \langle Lf, f\rangle \| \leq \|L^{-1}\|\|Lf\|^2.$$

Then

$$\begin{aligned}\|T_F\varphi - T_G\varphi\|^2 &\leq \| \langle Lf, Lf\rangle \| + \| \langle Lf, f\rangle \| + \| \langle f, Lf\rangle \| + \| \langle f, f\rangle \| \\ &\leq \|Lf\|^2 + 2\|L^{-1}\|\|Lf\|^2 + \|L^{-1}\|^2\|Lf\|^2 \\ &= (\|L^{-1}\| + 1)^2\|T_F\varphi\|^2.\end{aligned}$$

Setting $N = \alpha(\|L\| + 1)^2 = \beta(\|L^{-1}\| + 1)^2$, for some $\alpha, \beta \in \mathbb{R}$, we have

$$\|T_F\varphi - T_G\varphi\|^2 \leq N \min\left\{\|T_F\varphi\|^2, \|T_G\varphi\|^2\right\}.$$

Conversely, by surjectivity of T_F and T_G, the mapping

$$\begin{aligned}L : U &\longrightarrow U \\ T_G\varphi &\longmapsto T_F\varphi,\end{aligned}$$

is well-defined for each $\varphi \in L^2(\Omega, \mathcal{A})$. It is easy to see that L is adjointable and invertible. Applying Theorem 4.2, it is enough to show that $F = LG$. We have

$$\begin{aligned}\int_\Omega \varphi(\omega) \langle F(\omega), g\rangle \, d\mu(\omega) &= \left\langle \int_\Omega \varphi(\omega)F(\omega)d\mu(\omega), g \right\rangle \\ &= \left\langle L \int_\Omega \varphi(\omega)G(\omega)d\mu(\omega), g \right\rangle \\ &= \int_\Omega \varphi(\omega) \langle LG(\omega), g\rangle \, d\mu(\omega),\end{aligned}$$

then

$$\left\langle \varphi, \{\langle F(\omega) - LG(\omega), g\rangle\}_{\omega\in\Omega} \right\rangle = 0,$$

for each $g \in U$ and $\varphi \in L^2(\Omega, \mathcal{A})$. Hence $F(\omega) = LG(\omega)$, for all $\omega \in \Omega$, and consequently F and G are similar. □

As a result, we state the equivalent conditions under which a continuous frame is Riesz-type under the influence of a Riesz-type frame.

Corollary 4.4. *Let $F, G : \Omega \to U$ be two continuous frames for U with pre-frame operators T_F and T_G, respectively. If F is a Riesz-type frame, then the followings are equivalent.*

(i) *G is a Riesz-type frame,*
(ii) *F and G are similar,*
(iii) $R(T_F^*) = R(T_G^*)$,
(iv) *there exists a constant $N > 0$ such that*

$$\|T_F\varphi - T_G\varphi\|^2 \leq N. \min\left\{\|T_F\varphi\|^2, \|T_G\varphi\|^2\right\},$$

for each $\varphi \in L^2(\Omega, \mathcal{A})$.

Let F be a continuous frame for U. Duo to Theorem 2.11, we can obtain a family of continuous frames similar to F. In the next theorem, we examine the relationship of the duals of F with the duals of its similar continuous frames.

Theorem 4.5. *Let $F : \Omega \to U$ be a continuous frame for U with the pre-frame operator T_F. Assume that $L \in End_{\mathcal{A}}^*(U)$ is an invertible operator. Then each dual of LF is similar to a dual of F and vice versa.*

Proof. Suppose that $G : \Omega \to U$ is a dual of LF with the pre-frame operator T_G. If T_{LF} is the pre-frame operator of LF, then $T_{LF} = LT_F$ and $T_{LF}^* = T_F^* L^*$. Also,

$$\begin{aligned} f &= T_{LF}T_G^* f \\ &= L(T_F T_G^*)f, \end{aligned}$$

for each $f \in U$. Hence $LT_F T_G^* = I_U$. Since L is invertible, so $T_F T_G^* L = I_U$ and $T_F T_{L^*G}^* = I_U$. It follows that L^*G is a dual of F that is similar to G.
Conversely, Assume that $G_1 : \Omega \to U$ is a dual of F with the pre-frame operator T_{G_1}. Then $T_F T_{G_1}^* = I_U$ and $LT_F T_{G_1}^* L^{-1} = I_U$, so $T_{LF} T_{(L^*)^{-1}G_1}^* = I_U$. Hence $(L^{-1})^* G_1$ is a dual of LF which is similar to the dual continuous frame G_1 of F. □

Based on proof of Theorem 4.5, duals of similar continuous frames are characterized as follows.

Corollary 4.6. *Let $F : \Omega \to U$ be a continuous frame for U and $L \in End_{\mathcal{A}}^*(U)$ is an invertible operator. Then the following statements hold.*

(i) *If $F_1 : \Omega \to U$ is a dual of F, then $(L^{-1})^* F_1$ is a dual of LF,*
(ii) *If $F_2 : \Omega \to U$ is a dual of LF, then $L^* F_2$ is a dual of F.*

By spectral mapping theorem, every real power of the frame operator is invertible and positive. Then the following remark holds.

Remark 4.7. Let $F, G : \Omega \to U$ be two continuous frames for U with continuous frame operators S_F and S_G, respectively. Then we can construct a similar continuous frame to one of them. Put $K := S_G^\alpha S_F^\beta$, for

$\alpha, \beta \in \mathbb{R}$. Then KF is similar to both F and G. Also,

$$\begin{aligned} S_{KF} &= KS_F K^* \\ &= S_G^{\alpha} S_F^{\beta} S_F S_F^{\beta} S_G^{\alpha} \\ &= S_G^{\alpha} S_F^{2\beta+1} S_G^{\alpha}. \end{aligned}$$

Setting $\alpha = -\beta = \dfrac{1}{2}$ in previous remark for arbitrary continuous frames F, G, and by Theorem 4.5, the next corollary shows that we can form a continuous frame similar to F such that its continuous frame operator is same as S_G.

Corollary 4.8. *Let $F, G : \Omega \to U$ be two continuous frames for U with continuous frame operators S_F and S_G, respectively. Then $R := S_G^{\frac{1}{2}} S_F^{\frac{-1}{2}} F$ is a continuous frame similar to F with $S_R = S_G$. Also, if F_1 is a dual of F, then $S_F^{\frac{1}{2}} S_G^{\frac{-1}{2}} F_1$ is a dual of R.*

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