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Coefficient Estimate and Fekete-Szegö Problems for Certain New Subclasses of Bi-univalent Functions Defined by Generalized Bivariate Fibonacci Polynomial

Rumeysa Öztürk[1] and İbrahim Aktaş[2,*]

Abstract. This article deals with two new subclasses of analytic and bi-univalent functions in the open unit disk, which is defined by applying subordination principle between analytic functions and the generalized Bivariate Fibonacci polynomials. Bounds for coefficients $|a_2|$ and $|a_3|$ of functions in these subclasses are estimated in terms of generalized Bivariate Fibonacci polynomials. In addition, the Fekete-Szegö problem is handled for the members of these subclasses and several consequences and examples of the main results are presented. The results of article generalize some of the previously published papers in the literature.

1. Introduction

Geometric Function Theory is a fruitful topic of complex analysis in mathematics. This sub-branch of complex analysis has achieved to attract attention of mathematicians in recent years. The class of normalized analytic functions in the open unit disk $\mathfrak{U} = \{z \in \mathbb{C} : |z| < 1\}$ is indicated by $\mathcal{A}$ and the members of $\mathcal{A}$ are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathfrak{U}). \tag{1.1}$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions. There are many subclasses of $\mathcal{S}$ which consist of functions having different geometric properties like starlikeness, convexity, close-to convexity and so

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on. These subclasses of functions may be defined by using an important tool known as subordination principle. For $f, h \in \mathcal{A}$, we say that f is subordinate to h if there is a function w such that

$$w(0) = 0, |w(z)| < 1, \qquad f(z) = h\left(w(z)\right), \quad (z \in \mathfrak{U}),$$

where w is a Schwarz function. This notion is symbolised by:

$$f \prec h \quad \text{or} \quad f(z) \prec h(z), \quad (z \in \mathfrak{U}).$$

The function h is univalent in $\mathfrak{U}$, then this relation is equivalent to

$$f(0) = h(0), \qquad f\left(\mathfrak{U}\right) \subset h\left(\mathfrak{U}\right).$$

A detailed knowledge for subordination notion one may refer to [40].

In fact, in the earlier of 20th century the first studies were started to publish on Geometric Function Theory. The first important study in this topic is known as Bieberbach's study [12]. In this study, Bieberbach verified that the second coefficient of function $f \in \mathcal{S}$ satisfy the coefficient inequality $|a_2| \leq 2$. Also, in the same study, Bieberbach conjectured that nth coefficient of every function $f \in \mathcal{S}$ satisfy $|a_n| \leq n$ for $n \geq 2$. This coefficient estimation is known as Bieberbach conjecture in the literature. Moreover, the proof of this prediction occupied mathematicians for many years. Finally, in 1985 French-American mathematician Luis De Branges verified Bieberbach conjecture for every natural number $n \geq 2$ in [19]. From 1916 to 1985, many familiar researchers published papers on Bieberbach conjecture and proved this estimation for certain special values of n. For example, Lowner [39], Garabedian and Schiffer [29], Pederson [47], Ozawa [45], Pederson and Schiffer [48] verified mentioned conjecture for certain special values of n. Of course, many other scientific papers which we cannot mentioned here on this conjecture were published from 1916 to nowadays.

A fascinated problem in Geometric Function Theory is known as Fekete-Szegö problem. This problem is related to coefficients of the functions $f \in \mathcal{S}$ and in [23] Fekete and Szegö proved the following sharp result for $f \in \mathcal{S}$:

$$\left|a_3 - \kappa a_2^2\right| \leq \begin{cases} 4\kappa - 3, & \kappa \geq 1, \\ 1 + 2e^{\left(\frac{-2\kappa}{1-\kappa}\right)}, & 0 \leq \kappa < 1, \\ 3 - 4\kappa, & \kappa < 0. \end{cases}$$

The fundamental inequality $\left|a_3 - \kappa a_2^2\right| \leq 1$ is acquired when $\kappa \to 1$. The coefficients' combination of the form $F_\kappa(f) = a_3 - \kappa a_2^2$, on the functions $f \in \mathcal{A}$ has a considerable impact on Geometric Function Theory. Finding bounds for $|F_\kappa(f)|$ is a maximization problem.

The problems related to the coefficients of functions $f \in \mathcal{S}$ do not limited above problems mentioned. This substantiality of problems in the class $\mathcal{S}$ attract attentions of resarchers. As a result of this attraction, numerous new subclasses of $\mathcal{S}$ were defined and a number of papers were published in this subject line.

One of the most remarkable subclassess of $\mathcal{S}$ is bi-univalent functions class Σ consisting of functions f such that both $f \in \mathcal{S}$ and $f^{-1} \in \mathcal{S}$. By Koebe one quarter theorem (see [22]) every $f \in \mathcal{S}$ has an inverse function f^{-1} such that

$$f^{-1}\left(f(z)\right) = z, \quad (z \in \mathfrak{U}),$$

and

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

(1.2)
$$\begin{aligned} f^{-1}(w) &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \\ &=: g(w). \end{aligned}$$

Although, every inverse function $f^{-1} \in \mathcal{A}$ do not need to be univalent, the class Σ is not an empty set since the functions $\frac{z}{1-z}$, $-\log(1-z)$ and $\log\sqrt{\frac{1+z}{1-z}}$ are in the class Σ. But, the famous Koebe function $\frac{z}{(1-z)^2}$ and the function $\frac{z}{1-z^2}$ are not member of Σ. Sistematically, the first studies on the class Σ were made by Lewin [38], Netenyahu [42], Brannan and Clunie [14], Brannan and Taha [15]. Although, the first remarkable results on coefficients of bi-univalent functions were given in [14, 15, 38, 42], of course they are not limited to those studies. For instance, Ali et al. presented several results on the coefficient bounds of bi-univalent Ma-Minda starlike and convex functions in [4]. It is worthy to emphasize here that finding a coefficient estimation on $|a_n|$ for the functions in the subclasses of analytic and bi-univalent function is still an open problem for $n \in \mathbb{N}$ and $n \geq 4$.

Mostly, certain number sequences, orthogonal polynomials and their generalizations were used by authors when a new function class was defined in the last decades. Especially, the authors made use of Faber polynomial in [57], Horadam polynomial in [58, 59], Chebyshev polynomial in [5, 16], (p, q)-Chebyshev polynomial in [9], Gegenbauer polynomial in [33], (p, q)-Lucas polynomial in [6, 8, 10, 44], generalized Fibonacci-like polynomial in [1], Bell numbers in [21], Fibonacci numbers in [30, 51] and k-Fibonaci numbers in [27, 28, 31]. On the other hand, various other subclasses of bi-univalent functions were defined in [17, 18, 20, 25, 26, 35, 55] by using different point of view. Also, in the defined new subclasses

some differential, integral and difference operators were used (see for example [3, 7, 24, 32, 34, 43, 46, 50]). The most attractive studies on bi-univalent functions were published Srivastava and his co-authors in the recent years. Especially, comprehensive information and historical developments about the class Σ may be found in [52–54, 56] authorized by Srivastava and his co-authors, and references therein.

Motivated by the developments in the field we define two new subclasses of Σ by using a generalization of the Fibonacci polynomials. Moreover, we argue coefficient inequality and Fekete-Szegö problems for these classes. It is known that generalized bivariate Fibonacci polynomial reduces to the certain polynomials and number sequences upon specializing parameters. As a result of this, main results of paper reduce to some previously published results in [13] and [60] for special values of parameters.

The workflow of paper is as follows: in the rest of this part basic information on the generalized bivariate Fibonacci polynomials is presented. In Section 2 and Section 3, firstly we define new subclasses and then, we discuss bounds for $|a_2|$, $|a_3|$ and $\left|a_3 - \eta a_2^2\right|$ $(\eta \in \mathbb{R})$, which the coefficients a_2 and a_3 are the coefficients of the functions belonging to the subclasses introduced here. Furthermore, we depict a few remarks about the coefficient bounds and Fekete-Szegö functional.

The familiar Fibonacci numbers occur different places in nature and science, and it achieved attract attention of scientist. So, numerous researches were made on this important number sequence. Also, several generalizations of it were introduced by the mathematicians. A generalization of the mentioned numbers is the generalized bivariate Fibonacci polynomials. For detailed knowledges of the generalized bivariate Fibonacci polynomials and Fibonacci numbers one can look at [36] and [37], respectively.

The famous Fibonacci numbers have the following recursive relation:

$$F_n = F_{n-1} + F_{n-2}, \quad (F_0 = 0,\ F_1 = 1,\ n \geq 2).$$

Fibonacci polynomial which generalizes Fibonacci numbers is defined by (see [11])

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad (F_0(x) = 0,\ F_1(x) = 1,\ n \geq 2).$$

A new generalization of $F_n(x)$ is introduced in [36] and it is expressed as below:

Let $n \geq 2$. If the polynomials $p(x, y)$ and $q(x, y)$ have real coefficients, then the generalized bivariate Fibonacci polynomials are defined as (see [36]):

$$H_n(x, y) = p(x, y)H_{n-1}(x, y) + q(x, y)H_{n-2}(x, y), \tag{1.3}$$

such that $H_0(x,y)=0$, $H_1(x,y)=1$, and $p^2(x,y)+4q(x,y)>0$. The generating function of $H_n(x,y)$ is

$$H(x,y,z)=\sum_{n=0}^{\infty}H_n(x,y)z^n = \frac{z}{1-p(x,y)z-q(x,y)z^2}. \tag{1.4}$$

Remark 1.1. Upon the specializing $p(x,y)$ and $q(x,y)$ it is possible to deduce several polynomial sequences. These cases are presented below:

1. Taking $p(x,y)=x$ and $q(x,y)=y$, we have the bivariate Fibonacci polynomial $F_n(x,y)$.
2. Taking $p(x,y)=x$ and $q(x,y)=1$, we deduce the Fibonacci polynomial $F_n(x)$.
3. Taking $p(x,y)=2x$ and $q(x,y)=1$, we arrive at the Pell polynomial $P_n(x)$.
4. Taking $p(x,y)=1$ and $q(x,y)=x$, we have the Jacobsthal polynomial $J_n(x)$.
5. Taking $p(x,y)=2x$ and $q(x,y)=-1$, we get the Chebyshev polynomial $U_{n-1}(x)$ of the second kind.
6. Taking $p(x,y)=3x$ and $q(x,y)=-2$, we obtain the Fermat polynomial $\mathcal{F}_n(x)$.

Hereafter, $p(x,y)$ and $q(x,y)$ will be denoted by p and q for the brevity, respectively.

2. The Function Class $\mathcal{R}_{\Sigma,}^{\lambda,\delta}(p,q)$

We firstly introduce the function class $\mathcal{R}_{\Sigma,}^{\lambda,\delta}(p,q)\subset\Sigma$. Afterwards, we discuss coefficient bounds and Fekete Szegö problem for the functions in $\mathcal{R}_{\Sigma,}^{\lambda,\delta}(p,q)$. Also, we indicate several remarks, examples and corollaries of them.

Definition 2.1. Let $\lambda\geq 1$, $\delta\geq 0$, $p\neq 0$, $p^2+4q>0$ and $r(z)=\frac{1}{1-pz-qz^2}$. If $f\in\Sigma$ satisfies the next subordinations

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\delta z f''(z)\prec r(z), \tag{2.1}$$

$$(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)\prec r(w), \tag{2.2}$$

then we say that $f\in\mathcal{R}_{\Sigma,}^{\lambda,\delta}(p,q)$, where $z,w\in\mathfrak{U}$ and g is of the form (1.2).

By specializing the variables λ, δ, p and q the class $\mathcal{R}_{\Sigma,}^{\lambda,\delta}(p,q)$ reduces to the following cases.

Remark 2.2. **i.** If we take $\delta = 0$ in Definition 2.1, we get the class $\mathcal{R}_{\Sigma,}^{\lambda,0}(p,q) = \mathcal{R}_{\Sigma,}^{\lambda}(p,q)$ which satifies the following relations:

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \frac{1}{1-pz-qz^2},$$
$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \prec \frac{1}{1-pw-qw^2},$$

where $g = f^{-1}$.

ii. If we take $\lambda = \delta = 0$ in Definition 2.1, we get the class $\mathcal{R}_{\Sigma,}^{0,0}(p,q) = \mathcal{R}_{\Sigma,}(p,q)$ which satifies the following relations:

$$\frac{f(z)}{z} \prec \frac{1}{1-pz-qz^2}, \qquad \frac{g(w)}{w} \prec \frac{1}{1-pw-qw^2},$$

where $g = f^{-1}$.

iii. If we take $\lambda = 1$, $\delta = 0$ in Definition 2.1, we get the class $\mathcal{R}_{\Sigma,}^{1,0}(p,q) = \mathcal{R}_{\Sigma,}^{1}(p,q)$ which satifies the following relations:

$$f'(z) \prec \frac{1}{1-pz-qz^2}, \qquad g'(w) \prec \frac{1}{1-pw-qw^2},$$

where $g = f^{-1}$.

vi. Upon taking $\delta = 0, p = 2t$ and $q = -1$ in Definition 2.1 for $t \in (\frac{1}{2}, 1)$, then we get the class $\mathcal{R}_{\Sigma,}^{\lambda,0}(2t,-1) = \mathfrak{B}_{\Sigma}(\lambda,t)$ introduced by Bulut et al. in [13, Definition 1], and it satisfies the next subordination:

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \frac{1}{1-2tz+z^2},$$
$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \prec \frac{1}{1-2tw+w^2},$$

where $g = f^{-1}$.

v. Upon taking $\lambda = 1, \delta = 0, p = 2t$ and $q = -1$ in Definition 2.1 for $t \in (\frac{1}{2}, 1)$, then we get the class $\mathcal{R}_{\Sigma,}^{1,0}(2t,-1) = \mathfrak{B}_{\Sigma}(t)$ mentioned by Bulut et al. in [13], and it satisfies the next subordination:

$$f'(z) \prec \frac{1}{1-2tz+z^2}, \qquad g'(w) \prec \frac{1}{1-2tw+w^2},$$

where $g = f^{-1}$.

It is worthy to mention here that the class $\mathcal{R}_{\Sigma,}^{\lambda,\delta}(p,q)$ is not an empty set. If we choose $\lambda = 1$, $\delta = 0$, $p = \frac{19}{10}$ and $q = -\frac{9}{10}$, then we get the following example.

Example 2.3. The function $f(z) = \frac{1}{2}\log\frac{1+z}{1-z} \in \Sigma$ is in the class $\mathcal{R}^{1,0}_{\Sigma,}(\frac{19}{10}, -\frac{9}{10})$. Because, its inverse is $g(w) = \frac{e^{2w}-1}{e^{2w}+1} \in \mathcal{S}$ and putting $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$, $g(w) = \frac{e^{2w}-1}{e^{2w}+1}$, $\lambda = 1$,$\delta = 0$, $p = \frac{19}{10}$ and $q = -\frac{9}{10}$ in (2.1) and (2.2), then we get the subordinations

$$f_1(z) = \frac{1}{1-z^2} \prec f_2(z) = \frac{1}{1-\frac{19}{10}z+\frac{9}{10}z^2}, \tag{2.3}$$

$$h_1(w) = \frac{4e^{2w}}{(1+e^{2w})^2} \prec h_2(w) = \frac{1}{1-\frac{19}{10}w+\frac{9}{10}w^2}, \tag{2.4}$$

which hold true for all $z, w \in \mathfrak{U}$. In fact, the function $f_2(z)$ (or $h_2(w)$) is analytic and univalent in $\mathfrak{U}$, since its simple poles are $z = 1 \notin \mathfrak{U}$ and $z = \frac{10}{9} \notin \mathfrak{U}$. Also, $f_2(z_1) = f_2(z_2)$ if and only if $(z_1 - z_2)[19 - 9(z_1 + z_2)]) = 0$. But, there are no any pairs of points $z_1, z_2 \in \mathfrak{U}$ such that $z_1 + z_2 = \frac{19}{9} > 2$ and so it must be $z_1 = z_2$. Namely, the function $f_2(z)$ (or $h_2(w)$) is analytic and univalent in $\mathfrak{U}$. In addition, $f_1(0) = f_2(0) = 1$, $h_1(0) = h_2(0) = 1$, $f_1(\mathfrak{U}) \subseteq f_2(\mathfrak{U})$ (see FIGURE 1 (c)) and $h_1(\mathfrak{U}) \subseteq h_2(\mathfrak{U})$ (see FIGURE 2 (f)) for $\forall z, w \in \mathfrak{U}$. By applying [49, Lemma 2.1, p.36] the subordinations (2.3) and (2.4) hold true.

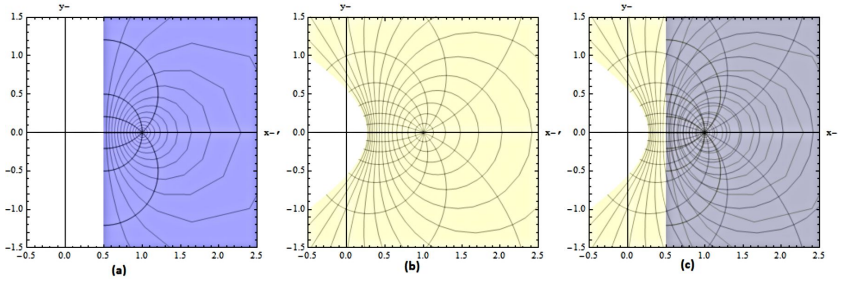


FIGURE 1. (a)$f_1(\mathfrak{U})$ (b)$f_2(\mathfrak{U})$ (c) $f_1(\mathfrak{U}) \subseteq f_2(\mathfrak{U})$

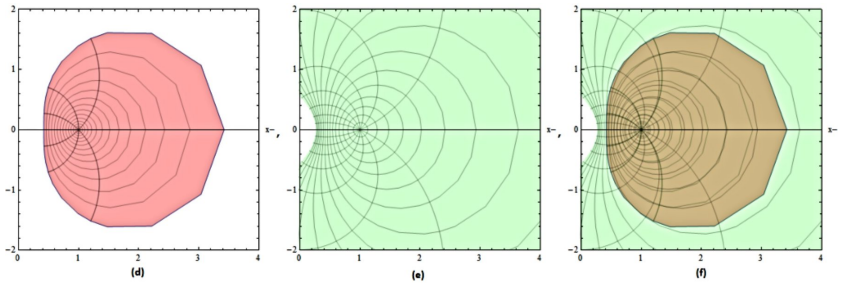


FIGURE 2. (d)$h_1(\mathfrak{U})$ (e)$h_2(\mathfrak{U})$ (f) $h_1(\mathfrak{U}) \subseteq h_2(\mathfrak{U})$

Theorem 2.4. *Let $\lambda \geq 1$, $\delta \geq 0$, $p \neq 0$ and $p^2 + 4q > 0$. If $f \in \mathcal{R}_{\Sigma,}^{\lambda,\delta}(p,q)$, then*

$$|a_2| \leq \frac{|p|\sqrt{|p|}}{\sqrt{|p^2(1+2\lambda+6\delta) - (p^2+q)(1+\lambda+2\delta)^2|}}, \tag{2.5}$$

$$|a_3| \leq \frac{p^2}{(1+\lambda+2\delta)^2} + \frac{|p|}{(1+2\lambda+6\delta)}, \tag{2.6}$$

and

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|p|}{1+2\lambda+6\delta}, & |1-\mu| \leq \frac{|A(p,q,\lambda,\delta)|}{p^2(1+\lambda+6\delta)}, \\ \\ \frac{|p|^3|1-\mu|}{|A(p,q,\lambda,\delta)|}, & |1-\mu| \geq \frac{|A(p,q,\lambda,\delta)|}{p^2(1+\lambda+6\delta)}, \end{cases} \tag{2.7}$$

where $\mu \in \mathbb{R}$ and $A(p,q,\lambda,\delta) = p^2(1+2\lambda+6\delta) - (p^2+q)(1+\lambda+2\delta)^2$.

Proof. Suppose that $f(z) \in \mathcal{R}_{\Sigma,}^{\lambda,\delta}(p,q)$ and $g = f^{-1}$ given by (1.2). By Definition 2.1, from the subordinations (2.1) and (2.2) we can write

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) = r(u(z)), \tag{2.8}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) = r(v(w)), \tag{2.9}$$

where $u, v : \mathfrak{U} \to \mathfrak{U}$, $u(z) = \sum_{j=1}^{\infty} u_j z^j$ and $v(w) = \sum_{j=1}^{\infty} v_j w^j$ are Schwarz functions such that $u(0) = 0 = v(0)$, $|u(z)| < 1$ and $|v(w)| < 1$ for all $z, w \in \mathfrak{U}$. Also, definition of the Schwarz function imply that

$$|u_j| < 1, \tag{2.10}$$

and

$$|v_j| < 1, \tag{2.11}$$

for $j \in \mathbb{N}$. By using some basic technics in mathematics we can write that

$$\begin{aligned} &(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) \\ &\quad = 1 + (1+\lambda+2\delta)a_2 z + (1+2\lambda+6\delta)a_3 z^2 + \ldots, \end{aligned} \tag{2.12}$$

$$\begin{aligned} &(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) \\ &\quad = 1 - (1+\lambda+2\delta)a_2 w + (1+2\lambda+6\delta)(2a_2^2 - a_3)w^2 + \ldots, \end{aligned} \tag{2.13}$$

(2.14)
$$r(u(z)) = 1 + H_2(x,y)u_1 z + \left[H_2(x,y)u_2 + H_3(x,y)u_1^2\right] z^2 + \cdots$$

(2.15)
$$r(v(w)) = 1 + H_2(x,y)v_1 w + \left[H_2(x,y)v_2 + H_3(x,y)v_1^2\right] w^2 + \cdots .$$

Due to the equality (2.8), comparing the terms with the same degree in (2.13) and (2.15) we deduce

$$(1+\lambda+2\delta)a_2 = H_2(x,y)u_1 \tag{2.16}$$

$$(1+2\lambda+6\delta)a_3 = H_2(x,y)u_2 + H_3(x,y)u_1^2. \tag{2.17}$$

Similarly, by virtue of the equality (2.9), comparing the equalities in (2.12) and (2.14) we arrive at

$$-(1+\lambda+2\delta)a_2 = H_2(x,y)v_1 \tag{2.18}$$

$$(1+2\lambda+6\delta)(2a_2^2 - a_3) = H_2(x,y)v_2 + H_3(x,y)v_1^2. \tag{2.19}$$

Now, from the equations (2.16) and (2.18) it can be easily written that

$$u_1 = -v_1, \tag{2.20}$$

$$2(1+\lambda+2\delta)^2 a_2^2 = H_2^2(x,y)(u_1^2 + v_1^2). \tag{2.21}$$

Also, summation of the equations (2.17) and (2.19) yields that

$$2(1+\lambda+2\delta)^2 a_2^2 = H_2(x,y)(u_2+v_2) + H_3(x,y)(u_1^2+v_1^2). \tag{2.22}$$

If we replace $u_1^2 + v_1^2$ obtained from the equation (2.21) in (2.22) we get

$$a_2^2 = \frac{H_2^3(x,y)(u_2+v_2)}{2H_2^2(x,y)(1+2\lambda+6\delta) - 2H_3(x,y)(1+\lambda+2\delta)^2}. \tag{2.23}$$

Here, it is known from recurrence relation (1.3) that $H_2(x,y) = p$ and $H_3(x,y) = p^2+q$. Subtituting these values in (2.23) and using triangular inequality with the inequalities (2.10) and (2.11) we deduce (2.5).

On the other hand, subtracting equation (2.19) from equation (2.17) we have

$$a_3 = a_2^2 + \frac{H_2(x,y)(u_2 - v_2)}{2(1+2\lambda+6\delta)}. \tag{2.24}$$

If we replace a_2^2 obtained from the equation (2.21) in (2.24) we get

$$a_3 = \frac{H_2(x,y)(u_2-v_2)}{2(1+2\lambda+6\delta)} + \frac{H_2^2(x,y)(u_1^2+v_1^2)}{2(1+\lambda+2\delta)^2}. \tag{2.25}$$

Here, using the fact that $H_2(x,y) = p$, $H_3(x,y) = p^2+q$ and using triangular inequality with (2.10) and (2.11) in (2.25) we deduce (2.6). In addition, by using the equations (2.23) and (2.24) one can write that

$$a_3 - \mu a_2^2 = H_2(x,y)\left[(\kappa(\mu) + B(\lambda,\delta))\, u_2 + (\kappa(\mu) - B(\lambda,\delta))\, v_2\right], \tag{2.26}$$

where $\mu \in \mathbb{R}$, $B(\lambda,\delta) = \frac{1}{2(1+2\lambda+6\delta)}$ and

$$\kappa(\mu) = \frac{(1-\mu)H_2^2(x,y)}{2H_2^2(x,y)(1+2\lambda+6\delta) - 2H_3(x,y)(1+\lambda+2\delta)^2}.$$

Now, using the fact that $H_2(x,y) = p$, $H_3(x,y) = p^2+q$ and using triangular inequality with (2.10) and (2.11) in (2.26) we deduce (2.7). □

Example 2.5. The function $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$ fits the requirements of Theorem 2.4. Taking $\lambda = 1$, $\delta = 0$, $p = \frac{19}{10}$ and $q = -\frac{9}{10}$ we can see that $|a_2| = 0 \leq \frac{19\sqrt{19}}{10}$ and $|a_3| = \frac{1}{3} \leq \frac{1843}{1200}$.

For the some special values of λ, δ, p and q, we obtain the followings.

Corollary 2.6. *Taking $\delta = 0, p = 2t$ and $q = -1$ in Theorem 2.4 for $t \in (\frac{1}{2}, 1)$, then we get the results given in [13, Theorem 1].*

Corollary 2.7. *Taking $\lambda = 1, \delta = 0, p = 2t$ and $q = -1$ in Theorem 2.4 we get the corresponding Corollaries given by [13].*

By putting $\mu = 1$ in Theorem 2.4, we are led to the following result on Fekete-Szegö type inequality.

Corollary 2.8. *Assume that $\lambda \geq 1$, $\delta \geq 0$, $p \neq 0$ and $p^2+4q > 0$. If $f(z) \in \mathcal{R}_{\Sigma,}^{\lambda,\delta}(p,q)$, then*

$$\left|a_3 - a_2^2\right| = \frac{|p|}{1+2\lambda+6\delta}. \tag{2.27}$$

3. The Function Class $\mathcal{R}_{\Sigma}^{\alpha}(p,q)$

Here, we introduce the function class $\mathcal{R}_{\Sigma}^{\alpha}(p,q) \subset \Sigma$ and then, we discuss coefficient bounds and Fekete Szegö problem for the functions in $\mathcal{R}_{\Sigma}^{\alpha}(p,q)$. Also, we present several remarks, examples and corollaries of them.

Definition 3.1. Let $\alpha \geq 0$, $p \neq 0$, $p^2+4q > 0$ and $r(z) = \frac{1}{1-pz-qz^2}$. If $f \in \Sigma$ satisfies the following subordinations

$$\frac{\left(zf'(z) + \alpha z^2 f''(z)\right)'}{f'(z)} \prec r(z), \tag{3.1}$$

and

$$\frac{\left(wg'(w) + \alpha w^2 g''(w)\right)'}{g'(w)} \prec r(w), \tag{3.2}$$

then we say that $f \in \mathcal{R}_{\Sigma}^{\alpha}(p,q)$, where $z, w \in \mathfrak{U}$ and g is of the form (1.2).

Remark 3.2. Putting $\alpha = 0$ in Definition 3.1, the class $\mathcal{R}_{\Sigma}^{0}(p,q)$ reduces to the subclass of bi-convex functions defined in [60] as Definition 4 and it satisfies the following relations:

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1}{1 - pz - qz^2},$$
$$1 + \frac{wg''(w)}{g'(w)} \prec \frac{1}{1 - pw - qw^2},$$

where $g = f^{-1}$.

It is important to mention that the class $\mathcal{R}_{\Sigma}^{\alpha}(p,q)$ is not an empty set. Taking $\alpha = 0$, $p = 1$ and $q = 0$ we have the next example.

Example 3.3. The function $t(z) = -3\log(1 - \frac{z}{3}) = z + \frac{z^2}{6} + \frac{z^3}{27} + \cdots \in \mathcal{S}$ is in the class $\mathcal{R}_{\Sigma}^{0}(1,0)$. Because, its inverse is $s(w) = \frac{3\left(e^{\frac{w}{3}} - 1\right)}{e^{\frac{w}{3}}} = w - \frac{w^2}{6} + \frac{w^3}{54} + \cdots \in \mathcal{S}$. In order to see that the functions t and s are univalent in $\mathfrak{U}$, see FIGURE 3 (a) and (b), respectively. So, we get that $t(z) = -3\log(1 - \frac{z}{3}) \in \Sigma$. Also, puttting $t(z) = -3\log(1 - \frac{z}{3})$, $s(w) = \frac{3e^{\frac{w}{3}} - 1}{e^{\frac{w}{3}}}$, $\alpha = q = 0$ and $p = 1$ in 3.1 and 3.2, then we have

$$f_3(z) = \frac{3}{3 - z} \prec f_4(z) = \frac{1}{1 - z}, \tag{3.3}$$

and

$$h_3(w) = 1 - \frac{w}{3} \prec h_4(w) = \frac{1}{1 - w}, \tag{3.4}$$

which hold true for all $z, w \in \mathfrak{U}$. The relation (3.3) holds true since there exist a function $\chi(z) = \frac{z}{3}$ such that $\chi(0) = 0$ and $|\chi(z)| = \frac{|z|}{3} < 1$ for all $z \in \mathfrak{U}$. Also, the relation (3.4) can be proven from [49, Lemma 2.1, p.36]. In fact, the function $h_4(w) = \frac{1}{1-w}$ is analytic and univalent in $\mathfrak{U}$, $h_3(0) = h_4(0) = 1$ and $h_3(\mathfrak{U}) \subset h_4(\mathfrak{U})$ (see FIGURE 3 (c)).

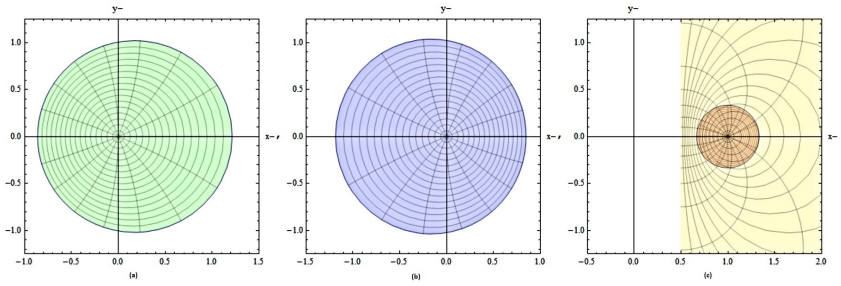


FIGURE 3. (a) $t(\mathfrak{U})$ (b) $s(\mathfrak{U})$ (c) $h_3(\mathfrak{U}) \subseteq h_4(\mathfrak{U})$

Theorem 3.4. *Let $\alpha \geq 0$, $p \neq 0$ and $p^2 + 4q > 0$. If $f(z) \in \mathcal{R}^{\alpha}_{\Sigma,}(p,q)$, then*

$$|a_2| \leq \frac{|p|\sqrt{|p|}}{\sqrt{|2p^2(1+5\alpha) - 4(p^2+q)(1+2\alpha)^2|}},$$
$$|a_3| \leq \frac{|p|}{6(1+3\alpha)} + \frac{p^2}{4(1+2\alpha)^2},$$

and

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|p|}{6(1+3\alpha)}, & |1-\nu| \leq \frac{|C(p,q,\alpha)|}{3p^2(1+3\alpha)}, \\ \\ \frac{|p|^3|1-\nu|}{2|C(p,q,\alpha)|}, & |1-\nu| \geq \frac{|C(p,q,\alpha)|}{3p^2(1+3\alpha)}, \end{cases} \tag{3.5}$$

where $\nu \in \mathbb{R}$ and $C(p,q,\alpha) = p^2(1+5\alpha) - 2(p^2+q)(1+2\alpha)^2$.

Proof. Let the function $f(z) \in \mathcal{R}^{\alpha}_{\Sigma,}(p,q)$. By Definition 3.1 and subordination principle, there exist the functions $\psi, \phi : \mathfrak{U} \to \mathfrak{U}$ such that

$$\psi(z) = \sum_{i=1}^{\infty} \psi_i z^i = \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots, \tag{3.6}$$

and

$$\phi(w) = \sum_{i=1}^{\infty} \phi_i w^i = \phi_1 w + \phi_2 w^2 + \phi_3 w^3 + \cdots, \tag{3.7}$$

where $\psi(0) = 0 = \phi(0)$, $|\psi(z)| < 1$ and $|\phi(w)| < 1$ for all $z, w \in \mathfrak{U}$. It is known from [41, p. 172] that

$$|\psi_i| \leq 1, \tag{3.8}$$

$$|\phi_i| \leq 1, \tag{3.9}$$

for $\forall i \in \mathbb{N}$. The relations (3.1) and (3.2) imply that

$$\frac{\left(zf'(z) + \alpha z^2 f''(z)\right)'}{f'(z)} = r(\psi(z)), \tag{3.10}$$

and

$$\frac{\left(wg'(w) + \alpha w^2 g''(w)\right)'}{g'(w)} = r(\phi(w)). \tag{3.11}$$

By the series representations of the functions $f(z)$ and $\psi(z)$ given by (1.1) and (3.6), respectively, we can write that

$$\begin{aligned} &1 + 2(1+2\alpha)a_2 z + 2\left[(1+3\alpha)3a_3 - 2(1+2\alpha)a_2^2\right] z^2 + \cdots \\ &\quad = H_1(x,y) + H_2(x,y)\psi_1 z + \left[H_2(x,y)\psi_2 + H_3(x,y)\psi_1^2\right] z^2 + \cdots, \end{aligned} \tag{3.12}$$

where $H_1(x,y)=1$. By the similar point of view, from the series expansions of the functions $g(w)$ and $\phi(w)$ given by (1.2) and (3.7), respectively, one can easily write that

(3.13)
$$1-2(1+2\alpha)a_2w+\left[4a_2^2(2+7\alpha)-6a_3(1+3\alpha)\right]w^2+\cdots = H_1(x,y)+H_2(x,y)\phi_1w+\left[H_2(x,y)\phi_2+H_3(x,y)\phi_1^2\right]w^2+\cdots.$$

Now, comparing the coefficients in (3.12) and (3.13), respectively, we deduce

$$2(1+2\alpha)a_2=H_2(x,y)\psi_1, \tag{3.14}$$

$$6(1+3\alpha)a_3-4(1+2\alpha)a_2^2=H_2(x,y)\psi_2+H_3(x,y)\psi_1^2, \tag{3.15}$$

$$-2(1+2\alpha)a_2=H_2(x,y)\phi_1, \tag{3.16}$$

$$4(2+7\alpha)a_2^2-6(1+3\alpha)a_3=H_2(x,y)\phi_2+H_3(x,y)\phi_1^2. \tag{3.17}$$

Equalities (3.14) and (3.16) imply that

$$\psi_1=-\phi_1, \tag{3.18}$$

and

$$8(1+2\alpha)^2a_2^2=H_2^2(x,y)(\psi_1^2+\phi_1^2). \tag{3.19}$$

Also, adding equations (3.15) and (3.17), and using equation (3.19) in the obtained result, we get

$$a_2^2=\frac{H_2^3(x,y)(\psi_2+\phi_2)}{4H_2^2(x,y)(1+5\alpha)-8H_3(x,y)(1+2\alpha)^2}. \tag{3.20}$$

Replacing $H_2(x,y)=p$ and $H_3(x,y)=p^2+q$ in (3.20), a basic calculation and the inequalities (3.8)−(3.9) yield that

$$|a_2|\leq\frac{|p|\sqrt{|p|}}{\sqrt{|2p^2(1+5\alpha)-4(p^2+q)(1+2\alpha)^2|}}.$$

On the other hand, if we subtract equation (3.17) from the equation (3.15) and make use of the relation (3.18), then we arrive at

$$a_3=a_2^2+\frac{H_2(x,y)(\psi_2-\phi_2)}{12(1+3\alpha)}. \tag{3.21}$$

Now, considering equation (3.19) in (3.21) we get that

$$a_3=\frac{H_2^2(x,y)(\psi_1^2+\phi_1^2)}{8(1+2\alpha)^2}+\frac{H_2(x,y)(\psi_2-\phi_2)}{12(1+3\alpha)}. \tag{3.22}$$

Here, taking $H_2(x,y)=p$ in (3.22), a straightforward calculation and, the inequalities (3.8) and (3.9) yield that

$$|a_3| \leq \frac{|p|}{6(1+3\alpha)} + \frac{p^2}{4(1+2\alpha)^2}.$$

Finally, making use of the equations (3.20) and (3.22) one can write that

(3.23)

$$\begin{aligned} &a_3 - \nu a_2^2 \\ &= H_2(x,y)\left\{\frac{(1-\nu)H_2^2(x,y)(\psi_2+\phi_2)}{4H_2^2(x,y)(1+5\alpha)-8H_3(x,y)(1+2\alpha)^2} + \frac{\psi_2-\phi_2}{12(1+3\alpha)}\right\} \\ &= H_2(x,y)\left\{\left(l(\nu)+\frac{1}{12(1+3\alpha)}\right)\psi_2 + \left(l(\nu)-\frac{1}{12(1+3\alpha)}\right)\phi_2\right\}, \end{aligned}$$

where $l(\nu)=\frac{(1-\nu)H_2^2(x,y)}{4H_2^2(x,y)(1+5\alpha)-8H_3(x,y)(1+2\alpha)^2}$. If we put $H_2(x,y)=p$ and $H_3(x,y)=p^2+q$ in (3.23), and consider triangle inequality with (3.8) and (3.9), then we conclude (3.5). □

Example 3.5. The function $f(z)=-3\log\left(1-\frac{z}{3}\right)$ fits the requirements of Theorem 3.4. Taking $\alpha=0$, $p=1$ and $q=0$ we can see that $|a_2|=\frac{1}{6}\leq\frac{1}{\sqrt{2}}$ and $|a_3|=\frac{1}{27}\leq\frac{5}{12}$.

Corollary 3.6. *If we take $\alpha=0$ in Theorem 3.4 we obtain the corresponding results in [60] given as Corollary 3 and Corollary 7.*

By putting $\nu=1$ in Theorem 3.4, we are led to the following result on Fekete-Szegö type inequality.

Corollary 3.7. *Let $\alpha\geq 0$, $p\neq 0$ and $p^2+4q>0$. If $f(z)\in\mathcal{R}_{\Sigma,}^{\alpha}(p,q)$, then*

$$|a_3-a_2^2| \leq \frac{|p|}{6(1+3\alpha)}. \tag{3.24}$$

4. Conclusion

In the present paper, we introduced some new function subclasses by applying subordination between analytic functions and generalized bivariate Fibonacci polynomials. By the subordination principle and fundamental rules in analysis we discussed estimations on the first two coefficients of functions in the subclasses defined here. In addition, we handled the Fekete-Szegö problem for functions in these new subclasses. This paper generalizes some previously obtained results in [13] and [60]. We end our investigation by drawing the attention of the interested readers that the Hankel determinant problems can be investigated for these subclasses in the future.

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