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A New Class of Integrals Connected with Polynomials and Extended Generalized Mittag-Leffler Function

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Abstract. The aim of the present investigation is to deal with integrals, which are connected with the extended generalized Mittag-Leffler function, Jacobi polynomial, and Bessel-Maitland function. Further, we are also considering the integral formulae, which involve various special functions. Interesting special cases of the main results are also considered. The results obtained here are general in nature and can presume numerous new integral formulae connecting the several types of polynomials.

1. Introduction and Preliminaries

The Mittag-Leffler function has a wide range of applications, including, solving fractional-order integrals and differential equations. The advancement of the Mittag-Leffler function has grown in the last two decades as a result of its application to engineering and applied sciences problems. Some applications of such a function satiate in the study of random Walk, kinetic equation, electric network, fluid flow, levy flights, complex system and in the theory of probability and statistical distribution. The well-known Mittag-Leffler functions $E_u(z)$ and $E_{u,v}(z)$ were introduced by Gosta Mittag-Leffler [8] defined as follows:

$$E_u(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(ul+1)}, \quad z \in \mathbb{C}, \Re(u) > 0,$$

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and further Wiman [4] defined

$$E_{u,v}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(ul+v)}, \quad z,u,v \in \mathbb{C}, \Re(u) > 0, \Re(v) > 0.$$

Prabhakar [19] introduced the generalization of the Mittag-Leffler function as below:

$$E^{w}_{u,v}(z) = \sum_{l=0}^{\infty} \frac{(w)_l z^l}{\Gamma(ul+v)l!}, \quad z,u,v,w \in \mathbb{C}, \Re(u) > 0, \Re(v) > 0, \Re(w) > 0,$$

where

$$(w)_l = \begin{cases} 1, & \text{if} \quad w \in \mathbb{C}, l = 0 \\ w(w+1)(w+2)\cdots(w+l-1), & \text{if} \quad l \in \mathbb{N}, w \in \mathbb{C}, \end{cases}$$

is known as pochhammer symbol.

Many researchers also developed the Mittag-Leffler function in various forms of generalization; see (Chouhan [1], Shukla [2],Padma[3], Suthar [5], Nisar [11], Khan [13], Maneria [15], Joshi [17]).

Özarslan and Yilmaz [14] developed an extended Mittag-Leffler function defined as follows:

$$E^{w;c}_{u,v}(z;p) = \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+l, c-w\right)(c)_l}{\mathfrak{B}\left(w, c-w\right)\Gamma(ul+v)} \frac{z^l}{l!},$$

where

$$u,v,w,c,p \in \mathbb{C}, \textit{such that } \Re(c) > 0, \Re(p) > 0, \Re(w) > 0.$$

And $\mathfrak{B}\left(r,s\right) = \mathfrak{B}(r,s;p)$ is known as extended Euler's beta function [12], defined as:

$$\begin{aligned} \mathfrak{B}(r,s;p) &= \mathfrak{B}_p(r,s) \\ &= \int_0^1 x^{r-1}(1-x)^{s-1} e^{-\frac{p}{x(1-x)}} dx, \end{aligned}$$

for $\Re(p) > 0, \Re(r) > 0, \Re(s) > 0$, respectively.

In this sequel, Mittal et al. [7] introduced a new generalized extended form of the Mittag-Leffler function defined as follows:

$$E^{w,c;q}_{u,v}(z;p) = \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} . \frac{z^l}{l!}, \tag{1.1}$$

where $u,v,w,c,p \in \mathbb{C}, \textit{such that } [\Re(c), \Re(p), \Re(u), \Re(v), \Re(w)] > 0$ and $q < \Re(u) + 1$. And

$$(c)_{lq} = \frac{\Gamma(c+lq)}{\Gamma(c)},$$

denotes the generalized pochhammer symbol.

The Jacobi polynomial $P_\mu^{\eta,\theta}(t)$ [6, 10] is defined as follows:

$$P_\mu^{\eta,\theta}(t) = \frac{(\eta+1)_\mu}{\mu!}\,{}_2F_1\left[\begin{matrix} -\mu, \mu+\eta+\theta+1; \\ \eta+1; \end{matrix}\ \frac{1-t}{2}\right]. \tag{1.2}$$

If we substitute $\eta = \theta = 0$ in (1.2), then it will convert into a Legendre polynomial [6]. For $t = 1$ in equation (1.2), the $P_\mu^{\eta,\theta}(t)$ is also a polynomial of degree n and it becomes

$$P_\mu^{\eta,\theta}(1) = \frac{(\eta+1)_\mu}{\mu!}.$$

In our current study, we use the integral expressions derived by Saxena [21] and also used in [18], given as follows:

$$\begin{aligned}\int_{-1}^{1} t^\lambda (1-t)^\eta (1+t)^\delta P_\mu^{\eta,\theta}(t)\,dt \\ = \frac{(-1)^\mu\, 2^{\eta+\delta+1}\Gamma(\delta+1)\Gamma(\mu+\eta+1)\Gamma(\delta+\theta+1)}{\mu!\,\Gamma(\delta+\theta+\mu+1)\Gamma(\delta+\eta+\mu+2)} \\ \times {}_3F_2\left[\begin{matrix} -\lambda, \delta+\theta+1, \delta+1; \\ \delta+\theta+\mu+1, \delta+\eta+\mu+2; \end{matrix}\ 1\right],\end{aligned} \tag{1.3}$$

and

$$\int_0^\infty t^\lambda J_\nu^\zeta(t)\,dt = \frac{\Gamma(\lambda+1)}{\Gamma(1+\nu-\zeta-\nu\zeta)}, \quad \Re(\lambda) > -1, 0 < \zeta < 1. \tag{1.4}$$

We also use the following integral formula defined by Rainville and Srivastava et. al [6, 10], derived as follows:

$$\int_{-1}^{1} (1-t)^{\alpha+n}(1+t)^{\beta+n}\,dt = 2^{2n+\alpha+\beta+1}\,\mathfrak{B}(\alpha+n+1, \beta+n+1). \tag{1.5}$$

The Bessel-Maitland function or the Wright generalized Bessel function [16, 20] in the form of series representation is defined as

$$J_\nu^\zeta(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\zeta k+\nu+1)k!} = \phi(\zeta, \nu+1; -z), \quad (\zeta > 0; z \in C). \tag{1.6}$$

This function has many applications in the areas of biology, chemistry, physics, and engineering. For more details about applications of the Bessel function, see [9]. In this study, we develop different integral formulae involving the extended generalized Mittag-Leffler function (1.1), the Wright generalized Bessel (Bessel-Maitland) function (1.6), and Jacobi's polynomial (1.2).

2. Main Results

In this segment, we acquire some new integral formulae connected with extended generalized Mittag-Leffler function, Jacobi polynomial, and Bessel-Maitland function. Our findings are stated in underneath the theorems 2.1, 2.2, 2.4, 2.6, 2.7 and 2.8, respectively.

Theorem 2.1. *Let $c, u, v, w, p \in \mathbb{C}$, such that*

$$[\Re(c), \Re(p), \Re(u), \Re(v), \Re(w)] > 0,$$

$q < \Re(u) + 1$,, $\Re(\lambda) > -1, \Re(\delta) > -1, \eta > -1$ and $\theta > -1$, then there holds the following formula:

$$\begin{aligned}(2.1)\qquad &\int_{-1}^{1} t^{\lambda}(1-t)^{\eta}(1+t)^{\delta} P_{\mu}^{\eta,\theta}(t) E_{u,v}^{w,c;q}\left(z(1+t)^{h}; p\right) dt \\ &= \frac{(-1)^{\mu}\, 2^{\eta+\delta+1}\Gamma(\mu+\eta+1)}{\mu!} \\ &\quad \times \sum_{l=0}^{\infty} \frac{\Gamma(\delta+hl+1)\Gamma(\delta+hl+\theta+1)}{\Gamma(\delta+hl+\theta+\mu+1)\Gamma(\delta+hl+\eta+\mu+2)} \\ &\quad \times \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} \frac{(z.2^{h})^{l}}{l!} \\ &\quad \times {}_3F_2\left[\begin{matrix} -\lambda, \delta+hl+\theta+1, \delta+hl+1; \\ \delta+hl+\theta+\mu+1, \delta+hl+\eta+\mu+2; \end{matrix} 1\right].\end{aligned}$$

Proof. It is logical to leverage our understanding of the function ${}_2F_1$ [6] to improve its usability with the aid of Jacobi polynomials. It is possible to write the L.H.S. of (2.1) as

$$\int_{-1}^{1} t^{\lambda}(1-t)^{\eta}(1+t)^{\delta} P_{\mu}^{\eta,\theta}(t) E_{u,v}^{w,c;q}\left(z(1+t)^{h}; p\right) dt = \Im_1(say)$$

$$\begin{aligned}\Im_1 = &\int_{-1}^{1} t^{\lambda}(1-t)^{\eta}(1+t)^{\delta} P_{\mu}^{\eta,\theta}(t) \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} \\ &\times \frac{\left(z(1+t)^{h}\right)^{l}}{l!}\, dt.\end{aligned}$$

Now, by altering the order of integration and summation, which is true under the circumstances, it becomes

$$\begin{aligned}(2.2)\qquad \Im_1 = &\sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} . \frac{(z)^{l}}{l!} \\ &\times \int_{-1}^{1} t^{\lambda}(1-t)^{\eta}(1+t)^{\delta+hl} P_{\mu}^{\eta,\theta}(t)\, dt,\end{aligned}$$

by using equation (1.3), equation (2.2) will become as

$$\begin{aligned}\Im_1 &= \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)}.\frac{(z)^l}{l!} \\ &\times \frac{(-1)^{\mu}\, 2^{\eta+\delta+hl+1}\Gamma(\delta+hl+1)}{\mu!\,\Gamma(\delta+hl+\theta+\mu+1)} \times \frac{\Gamma(\mu+\eta+1)\Gamma(\delta+hl+\theta+1)}{\Gamma(\delta+hl+\eta+\mu+2)} \\ &\times {}_3F_2\left[\begin{matrix} -\lambda, \delta+hl+\theta+1, \delta+hl+1; \\ \delta+hl+\theta+\mu+1, \delta+hl+\eta+\mu+2; \end{matrix} 1\right].\end{aligned}$$

After rearranging the terms, we will get the desired result. □

Special case: For $c, u, v, w, p \in \mathbb{C}$, $[\Re(c), \Re(p), \Re(u), \Re(v), \Re(w)] > 0, q < \Re(u)+1$ and $\Re(\lambda) > -1, \Re(\delta) > -1, \eta > -1, \theta > -1$, then we have

$$\begin{aligned}(2.3)\qquad &\int_{-1}^{1} e^t (1-t)^{\eta}(1+t)^{\delta} P_{\mu}^{\eta,\theta}(t) E_{u,v}^{w,c;q}\left(z(1+t)^h; p\right) dt \\ &= \frac{(-1)^{\mu}\, 2^{\eta+\delta+1}\Gamma(\mu+\eta+1)}{\mu!} \\ &\times \sum_{\lambda=0}^{\infty}\sum_{l=0}^{\infty} \frac{\Gamma(\delta+hl+1)\Gamma(\delta+hl+\theta+1)}{\Gamma(\delta+hl+\theta+\mu+1)\Gamma(\delta+hl+\eta+\mu+2)} \\ &\times \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} \frac{(z.2^h)^l}{(l!)(\lambda!)} \\ &\times {}_3F_2\left[\begin{matrix} -\lambda, \delta+hl+\theta+1, \delta+hl+1; \\ \delta+hl+\theta+\mu+1, \delta+hl+\eta+\mu+2; \end{matrix} 1\right].\end{aligned}$$

Proof. The L.H.S. of (2.3), considering $e^t = \sum_{\lambda=0}^{\infty} \frac{(t)^{\lambda}}{\lambda!}$ and using theorem 2.1, we get above solution. □

Theorem 2.2. *Let $c, u, v, w, p \in \mathbb{C}$, such that $[\Re(c), \Re(p), \Re(u), \Re(v), \Re(w)] > 0, q < \Re(u)+1$, $\Re(\delta) > -1, \eta > -1, \theta > -1$, then there holds the following formula:*

(2.4)

$$\begin{aligned}&\int_{-1}^{1} (1-t)^{\delta}(1+t)^{\theta} P_{\mu}^{\eta,\theta}(t) P_{\tau}^{\rho,\sigma}(t) E_{u,v}^{w,c;q}\left(z(1-t)^h; p\right) dt \\ &= \frac{2^{\delta+\theta+1}\Gamma(\rho+\tau+1)\Gamma(\eta+\mu+1)}{\mu!\,\tau!} \sum_{r=0}^{\infty} \frac{(-\mu)_r(\eta+\theta+\mu+1)_r}{r!\,\Gamma(\eta+r+1)} \\ &\times \sum_{s=0}^{\infty} \frac{(-\tau)_s(\rho+\sigma+\tau+1)_s}{s!\,\Gamma(\rho+s+1)} \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)}.\frac{(z.2^h)^l}{l!} \\ &\times \mathfrak{B}(\delta+hl+r+s+1, \theta+1).\end{aligned}$$

Proof. The L.H.S. of (2.4) can be written as

$$\int_{-1}^{1}(1-t)^{\delta}(1+t)^{\theta}P_{\mu}^{\eta,\theta}(t)P_{\tau}^{\rho,\sigma}(t)E_{u,v}^{w,c;q}\left(z(1-t)^{h};p\right)dt=\Im_2(say)$$

$$\Im_2=\int_{-1}^{1}(1-t)^{\delta}(1+t)^{\theta}P_{\mu}^{\eta,\theta}(t)P_{\tau}^{\rho,\sigma}(t)\times\sum_{l=0}^{\infty}\frac{\mathfrak{B}_p\left(w+lq,c-w\right)}{\mathfrak{B}\left(w,c-w\right)}\frac{(c)_{lq}}{\Gamma(ul+v)}.\frac{\left(z(1-t)^{h}\right)^{l}}{l!}\,dt, \tag{2.5}$$

now using (1.2) in (2.5), we have

$$\begin{aligned}\Im_2=&\sum_{l=0}^{\infty}\frac{\mathfrak{B}_p\left(w+lq,c-w\right)}{\mathfrak{B}\left(w,c-w\right)}\frac{(c)_{lq}}{\Gamma(ul+v)}.\frac{(z)^{l}}{l!}\\&\times\int_{-1}^{1}(1-t)^{\delta+hl}(1+t)^{\theta}\,\frac{(\eta+1)_{\mu}}{\mu!}\\&\times\sum_{r=0}^{\infty}\frac{(-\mu)_r(\eta+\theta+\mu+1)_r}{r!\,(\eta+1)_r}\left(\frac{1-t}{2}\right)^{r}\\&\times\frac{(\rho+1)_{\tau}}{\tau!}\sum_{s=0}^{\infty}\frac{(-\tau)_s(\rho+\sigma+\tau+1)_s}{s!\,(\rho+1)_s}\left(\frac{1-t}{2}\right)^{s}dt.\end{aligned}$$

After simplification and rearranging the terms, we have

$$\begin{aligned}\Im_2=&\sum_{l=0}^{\infty}\frac{\mathfrak{B}_p\left(w+lq,c-w\right)}{\mathfrak{B}\left(w,c-w\right)}\frac{(c)_{lq}}{\Gamma(ul+v)}.\frac{(z)^{l}}{l!}\frac{\Gamma(\eta+\mu+1)\Gamma(\rho+\tau+1)}{\mu!\,\tau!}\\&\times\sum_{r=0}^{\infty}\frac{(-\mu)_r(\eta+\theta+\mu+1)_r}{r!\,2^{r}\,\Gamma(\eta+r+1)}\sum_{s=0}^{\infty}\frac{(-\tau)_s(\rho+\sigma+\tau+1)_s}{s!\,2^{s}\,\Gamma(\rho+s+1)}\\&\times\int_{-1}^{1}(1-t)^{\delta+hl+r+s}(1+t)^{\theta}\,dt.\end{aligned} \tag{2.6}$$

Now, using the expression given by equation (1.5), equation (2.6) gives the desired result. □

Corollary 2.3. *In the above theorem, if we replace* δ *by* $\lambda-1$ *and* $\eta=\theta=\rho=\sigma=0$, *then we have*

$$\begin{aligned}&\int_{-1}^{1}(1-t)^{\lambda-1}P_{\mu}^{2}(t)E_{u,v}^{w,c;q}\left(z(1-t)^{h};p\right)dt\\&\quad=\sum_{r=0}^{\infty}\frac{2^{\lambda}(-\mu)_r(\mu+1)_r}{(r!)^{2}}\sum_{s=0}^{\infty}\frac{2^{\lambda}(-\tau)_s(\tau+1)_s}{(s!)^{2}}\end{aligned}$$

$$\times \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} . \frac{(z.2^h)^l}{l!} \mathfrak{B}(\lambda+hl+r+s, 1).$$

Theorem 2.4. *Let $c, u, v, w, p \in \mathbb{C}$, such that*

$$[\Re(c), \Re(p), \Re(u), \Re(v), \Re(w)] > 0,$$

$q < \Re(u)+1$, $\eta > -1$ and $\theta > -1$, then there holds the following formula:

$$\begin{aligned} (2.7) \qquad & \int_{-1}^{1} (1-t)^{\rho}(1+t)^{\sigma} P_{\mu}^{\eta,\theta}(t) E_{u,v}^{w,c;q}\left(z(1-t)^h(1+t)^k; p\right) dt \\ &= \frac{2^{\rho+\sigma+1}\Gamma(\eta+\mu+1)}{\mu!} \sum_{r=0}^{\infty} \frac{(-\mu)_r(\eta+\theta+\mu+1)_r}{\Gamma(\eta+r+1)\, r!} \\ &\quad \times \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} . \frac{(z.2^{h+k})^l}{l!} \\ &\quad \times \mathfrak{B}(\rho+hl+r+1, \sigma+kl+1). \end{aligned}$$

Proof. The L.H.S. of (2.7) can be written as

$$\int_{-1}^{1} (1-t)^{\rho}(1+t)^{\sigma} P_{\mu}^{\eta,\theta}(t) E_{u,v}^{w,c;q}\left(z(1-t)^h(1+t)^k; p\right) dt = \Im_3(say)$$

$$\begin{aligned} (2.8) \qquad \Im_3 &= \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} . \frac{(z)^l}{l!} \\ &\quad \times \int_{-1}^{1} (1-t)^{\rho+hl}(1+t)^{\sigma+kl} P_{\mu}^{\eta,\theta}(t)\, dt, \end{aligned}$$

now using (1.2) in equation (2.8), we get

$$\begin{aligned} (2.9) \quad \Im_3 &= \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} . \frac{(z)^l}{l!} \frac{(\eta+1)_{\mu}}{\mu!} \\ &\quad \times \sum_{r=0}^{\infty} \frac{(-\mu)_r(\eta+\theta+\mu+1)_r}{2^r\,(\eta+1)_r\, r!} \int_{-1}^{1} (1-t)^{\rho+hl+r}(1+t)^{\sigma+kl}\, dt. \end{aligned}$$

Finally using equation (1.5) in (2.9), gives the required result. □

Corollary 2.5. *In above theorem, if we replace ρ by $\rho-1$, σ by $\sigma-1$ and $\eta = \theta = 0$, then we have*

$$\begin{aligned} &\int_{-1}^{1} (1-t)^{\rho-1}(1+t)^{\sigma-1} P_{\mu}(t)\, E_{u,v}^{w,c;q}\left(z(1-t)^h(1+t)^k; p\right) dtv \\ &\quad = \sum_{r=0}^{\infty} \frac{2^{\rho+\sigma+1}(-\mu)_r(\mu+1)_r}{\Gamma((r!)^2} \end{aligned}$$

$$\times \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} . \frac{(z.2^{h+k})^l}{l!}\, \mathfrak{B}(hl+r+, kl).$$

Similarly, the following results stated in theorems 2.6 and 2.7 can also be obtained, here we omitted the proof.

Theorem 2.6. *Let $c, u, v, w, p \in \mathbb{C}$, such that $[\Re(c), \Re(p), \Re(u), \Re(v), \Re(w)] > 0, q < \Re(u) + 1$ and $\eta > -1, \theta > -1$, then there holds the following formula:*

$$\begin{aligned} (2.10) \qquad & \int_{-1}^{1} (1-t)^{\rho}(1+t)^{\sigma} P_{\mu}^{\eta,\theta}(t) E_{u,v}^{w,c;q}\left(z(1+t)^{-h}; p\right) dt \\ &= \frac{2^{\rho+\sigma+1}\Gamma(\eta+\mu+1)}{\mu!} \sum_{r=0}^{\infty} \frac{(-\mu)_r(\eta+\theta+\mu+1)_r}{\Gamma(\eta+r+1)\, r!} \\ &\quad \times \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} . \frac{(z.2^{-h})^l}{l!} \\ &\quad \times \mathfrak{B}(\rho+r+1, \sigma-hl+1). \end{aligned}$$

Theorem 2.7. *Let $c, u, v, w, p \in \mathbb{C}, [\Re(c), \Re(p), \Re(u), \Re(v), \Re(w)] > 0, q < \Re(u) + 1$ and $\eta > -1, \theta > -1$, then there holds the following formula:*

$$\begin{aligned} (2.11) \qquad & \int_{-1}^{1} (1-t)^{\rho}(1+t)^{\sigma} P_{\mu}^{\eta,\theta}(t) E_{u,v}^{w,c;q}\left(z(1-t)^{h}(1+t)^{-k}; p\right) dt \\ &= \frac{2^{\rho+\sigma+1}\Gamma(\eta+\mu+1)}{\mu!} \sum_{r=0}^{\infty} \frac{(-\mu)_r(\eta+\theta+\mu+1)_r}{\Gamma(\eta+r+1)\, r!} \\ &\quad \times \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} . \frac{(z.2^{h-k})^l}{l!} \\ &\quad \times \mathfrak{B}(\rho+hl+r+1, \sigma-kl+1). \end{aligned}$$

Theorem 2.8. *Let $c, u, v, w, p \in \mathbb{C}, [\Re(c), \Re(p), \Re(u), \Re(v), \Re(w)] > 0, q < \Re(u)+1$ and $\Re(\lambda) > -1, \eta - \eta t > -1, \eta > 0, 0 < \zeta < 1, \Re(\lambda+1) > 0$, then there holds the following formula:*

$$\begin{aligned} (2.12) \qquad & \int_{-\infty}^{\infty} t^{\lambda} J_{\nu}^{\zeta}(t) E_{u,v}^{w,c;q}\left(zt^{\eta}; p\right) dt \\ &= \sum_{l=0}^{\infty} \frac{\mathfrak{B}_p\left(w+lq, c-w\right)}{\mathfrak{B}\left(w, c-w\right)} \frac{(c)_{lq}}{\Gamma(ul+v)} . \frac{(z)^l}{l!} \frac{\Gamma(\lambda+\eta l+1)}{\Gamma(1+\nu-\zeta-\zeta\nu)}. \end{aligned}$$

3. Concluding Remarks

The Bessel-Maitland function with an extended generalized Mittag-Leffler function and the integral formulas using Jacobi polynomials are

defined generally. Jacobi polynomials allow for the use of integral formulae to address a variety of fractional calculus issues. Additionally, Jacobi polynomials and their extended forms with various special functions may be utilized to create several image formulae, which can be used to many engineering and applied science applications. We end this inquiry by noting that the findings are of a general nature and several other intriguing integral formulae involving various polynomial types could be developed.

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