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Application of Gegenbauer Polynomials with Two Variables to Bi-univalency of Generalized Discrete Probability Distribution Via Zero-Truncated Poisson Distribution Series

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Abstract. The present study is unique in exploring bi-univalent functions, which has recently garnered attention from many researchers in Geometric Function Theory (GFT). The uniqueness lies in utilizing a generalized discrete probability distribution and a zero-truncated Poisson distribution combined with generalized Gegenbauer polynomials featuring two variables. We aim to obtain coefficient bounds, the classical Fekete-Szegö inequality, and Hankel and Toeplitz determinants to generalize the probability of a gambler's ruin. Additionally, using the defined bi-univalent function classes contributes to the uniqueness of the obtained results.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The family $\mathcal{A}$ consists of all analytic functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ normalized by $f(0) = f'(0) - 1 = 0$ has Maclaurin series of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

Stand by S the subfamily of $\mathcal{A}$ consisting of univalent functions and stand by Ω the family of analytic functions $\mathbb{D} \longrightarrow \mathbb{C}$ with $w(0) = 0$ and $|w(z)| < 1$. If f, g are analytic in $\mathbb{D}$, we call that f is subordinate to g in $\mathbb{D}$, written by $f(z) \prec g(z)$ if there is a function w such that $f(z) = g(w(z))$.

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Based on the Koebe-One Quarter Theorem [9], every univalent function $f \in \mathcal{A}$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z \ \ (z \in \mathbb{D})$ and $f\left(f^{-1}(w)\right) = w \ \ \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right)$, where

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w \ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \end{aligned}$$

A function $f \in \mathcal{A}$ is considered to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Stand by Σ the family of bi-univalent functions. The concept of bi-univalent functions dates back to the mid-1960s (see, e.g., [6, 14, 19] and the references cited therein). Since then, bounds for the first few coefficients $|a_2|$, $|a_3|$ of various subfamilies of Σ have been introduced by several sequels to [1] including (among others) [5, 15, 16]. However, determination of the bounds for each of

$$|a_n|, \quad (n \in \mathbb{N} \backslash \{1, 2\};\ \mathbb{N} = \{1, 2, 3, \ldots\})$$

remains an open problem for functions in Σ.

Here, we shall stand by S^* and K, the two familiar subfamilies of S consisting of starlike and convex functions, respectively. The two subfamilies have a nice geometrical interpretation naturally generalized respectively as follows

$$S^*(\gamma) = \left\{ f \in S : \Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma,\ z \in \mathbb{D},\ 0 \leq \gamma < 1 \right\}$$

and

$$K(\gamma) = \left\{ f \in S : \Re\left(1 + \frac{zf'(z)}{f(z)}\right) > \gamma,\ z \in \mathbb{D},\ 0 \leq \gamma < 1 \right\}.$$

A function $f \in \Sigma$ is in the class $S_\Sigma^*(\gamma)$ of bi-starlike funtion of order γ and of $K_\Sigma(\gamma)$ of bi-convex function of order γ.

For $\mu > 0$, μ (constant real) a generating function of Gegenbauer polynomials is expressed by

$$\begin{aligned} H_\mu(x, z) &= \frac{1}{[1 - 2xz + (1 + y)z^2]^\mu} \\ &= 1 + \sum_{n=1}^{\infty} C_n^\mu(x, y) z^n. \end{aligned}$$

For fixed x, the function $H_\mu(x, z)$ is analytic in $\mathbb{D}$ and it is expanded by

$$H_\mu(x, z) = 1 + \sum_{n=1}^{\infty} C_n^\mu(x, y) z^n,$$

where $C_n^\mu(x)$ is the Gegenbauer polynomials of degree n with single variable. The Gegenbauer polynomials $C_n^\mu(x)$ is a special case of Jacobi

polynomial and essentially equivalent to ultra-spherical polynomials [2, 4, 11].

The Gegenbauer polynomials of two variables are expressed by the following generating function

$$H_\mu(x,y,z) = \frac{1}{[1-2xz+(1+y)z^2]^\mu} = \sum_{n=0}^{\infty} C_n^\mu(x,y)z^n,$$

for $x, y \in [-1,1]$ with series expansion

$$C_n^\mu(x,y) = \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^k(k)_{m-k}(2x)^{m-2k}(y+1)^k}{k!(m-2k)!}, \tag{1.2}$$

where $\mu > \frac{-1}{2}$ [11]. For various specialization of variables involved in (1.2), we have the following first few recurrence and generated recurrence polynomials:

$$\begin{aligned}
[C_1^\mu(x,y)] &= 2\mu x,\\
[C_2^\mu(x,y)] &= \frac{(\mu)_2}{2!}(2x)^2 - \mu(1+y),\\
[C_3^\mu(x,y)] &= \frac{(\mu)_3}{2!}(2x)^3 - \frac{(\mu)_2}{1!}(2x)(1+y),\\
[C_4^\mu(x,y)] &= \frac{(\mu)_4}{2!}(2x)^4 - \frac{(\mu)_3}{2!}(2x)^2(1+y) + \frac{(\mu)_2}{2!}(1+y)^2
\end{aligned}$$

and

$$\begin{aligned}
n(1+y)C_n^\mu(x,y) = (2n-2+\mu)xC_{n-1}^\mu(x,y)\\
- (n+2\mu-2)(1+y)^2C_{n-2}^\mu C_{n-2}^\mu(x,y).
\end{aligned}$$

Gegenbauer polynomials with two variables are specific solutions of partial differential equations of the form

$$(1+y-x^2)\frac{\partial^2}{\partial x^2}C_n^\mu(x,y) - (2\mu+1)x\frac{\partial}{\partial x}C_n^\mu(x,y) + n(n+2\mu)C_n^\mu(x,y) = 0.$$

For $y = 0$, Gegenbauer polynomial reduces to specific solution of ordinary differential equation of the form

$$(1+x^2)\frac{d^2}{dy^2} - (2\mu+1)x\frac{d}{dy} + n(n+2\mu)y = 0.$$

Recently, the concept of discrete probability distribution has attracted the researchers' attention GFT, as evident in the following literature [23–25].

Now, we apply the generalized discrete probability distribution and zero-truncated Poisson (ZTP) distribution to generalize the probability of a gambler ruins. Zero-truncated Poisson distribution, which is a certain probability distribution whose support is the set of positive integers, is also known as conditional probability of a Poisson distributed random variable, such that the value of the random variable is not zero, which implies that a ZTP random variable can't be zero [7, 8]. For example, one expects every shopper on the queue in the pay-point line to have at least one commodity to pay for; that is a shopper does not stand in line with no item. Furthermore, since ZTP is a truncated distribution with the truncation stipulated at $k > 0$, it is then possible to derive the probability mass function $g(k, \lambda)$ from a standard Poisson distribution $f(k, m)$ as

$$\begin{aligned} g(k, m) &= P(X = k \; x > 0) \\ &= \frac{f(k, m)}{1 - f(0, m)}, \end{aligned} \tag{1.3}$$

the mean is given as

$$\frac{m}{1 - e^{-m}} = \frac{me^m}{e^m - 1},$$

while the variance is

$$\begin{aligned} Var[x] &= \frac{m + m^2}{1 - e^{-m}} - \frac{m^2}{(1 - e^{-m})^2} \\ &= E[x](1 + m - E[x]). \end{aligned}$$

It is observed from (1.3) that the probability density function of the zero-truncated Poisson distribution is represented by

$$P_0(X = k) = \frac{m^k}{(e^m - 1)k!}, \quad k = 1, 2, 3, 4, \ldots, \; k > 0. \tag{1.4}$$

It is further observed from (1.4) that the function denoted by $P_0(X = k)$ does not belong to the family $\mathcal{A}$. In [4], Amoura et al. provide the normalization of zero-truncated Poisson distribution by

$$P_0 f(m, z) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(e^m - 1)(k - 1)!} z^k, \quad z \in \mathbb{D},$$

where $m > 0$. This series has an infinite radius of convergence, as determined by the ratio test.

Stand by κ the sum of the convergent series of form $\kappa = \sum_{n=0}^{\infty} a_n$, $a_n \geq 0$ for all $n \in \mathbb{N}$. The generalized discrete probability distribution having

probability mass function is defined as

$$p(n) = \frac{a_n}{\kappa}, \quad n = 0, 1, 2, \ldots$$

where $p(n)$ is the probability mass function, $p(n) \geq 0$ and $\sum_n p_n = 1$. In addition, let

$$\xi(x) = \sum_{n=0}^{\infty} a_n z^n, \tag{1.5}$$

then from $\kappa = \sum_{n=0}^{\infty} a_n$ series (1.5) converges for $|z| < 1$ and $z = 1$. Suppose X is a discrete random variable that takes values $x_1, x_2, \ldots$ associated with probabilities $p_1, p_2, \ldots$ then the expected X symbolized by $E(x)$ is defined by

$$E(x) = \sum_{n=1}^{\infty} p_n x^n.$$

The moment is given as $\mu'_r = E(x^r)$, variance is given as $\mu'_2 - (\mu'_1)^2$ and moment about the origin is defined as $Mean = \mu'_1 = \frac{\xi'}{\kappa}$ and the variance is expressed as

$$\text{variance} = \mu'_2 - (\mu'_1)^2 = \frac{1}{\kappa}\left[\xi''(1) + \xi'(1) - \frac{(\xi'(1))^2}{\kappa}\right],$$

where μ'_1 is the mean of the distribution.

Furthermore, the moment generating function of a random variable X is indicated by

$$M_x(t) = E(e^{xt})$$

while the moment generating function of the generalized discrete probability is expressed by

$$M_x(t) = \frac{\xi(e^t)}{\kappa}.$$

The polynomial whose coefficients are probabilities of the generalized distribution are expressed by

$$K_\varphi(z) = z + \sum_{n=2}^{\infty} \frac{a_{n-1}}{\kappa} z^n,$$

where $\kappa = \sum_{n=0}^{\infty} a_n$ in [25].

The Hadamard product (or convolution) of $f(z)$ given by (1.1) and

$$\varphi(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n,$$

is expressed by

$$\begin{aligned}(f*\varphi)(z) &= z+\sum_{n=2}^{\infty} a_n\varphi_n z^n \\ &= (\varphi * f)(z).\end{aligned}$$

Let $K : \mathcal{A} \longrightarrow \mathcal{A}$. Next, by using the notation

$$\begin{aligned}K_\tau f(z) &= P_0 f(m,z) * K_\varphi(z) \\ &= z+\sum_{n=2}^{\infty} \frac{m^{n-1}}{(e^m-1)(n-1)!}\frac{a_{n-1}}{\kappa} z^n,\end{aligned} \tag{1.6}$$

we have by using the Salagean derivative operator [27] in (1.6)

$$D^\delta K_\tau f(z) = z+\frac{2^\delta m}{(e^m-1)}\frac{a_1}{\kappa}z^2+\frac{3^\delta m^2}{2(e^m-1)}\frac{a_2}{\kappa}z^3+\frac{4^\delta m^3}{6(e^m-1)}\frac{a_3}{\kappa}z^4+\cdots,$$

where $\delta \in N_0, m > 0$ and Salagean derivative operator is given by

$$D^0 f(z) = f(z), \qquad D^1 f(z) = Df(z) = zf'(z),$$

$$D^\delta f(z) = D\left(D^{n-1}f(z)\right) = z\left(D^{n-1}f(z)\right)' = z+\sum_{n=2}^{\infty} n^\delta a_n z^n.$$

Next, the concept of the Hankel determinant has been the center of attraction to the researchers in GFT. Various authors in literature investigated the Hankel determinant; see examples and details in [3, 12, 13, 21, 22, 26]. Noonan and Thomas in [20], studied the $n-th$ class Hankel determinant whose elements are coefficients of function $f \in S$ defined in (1.1) and denoted by $H_{m,n}(f)$. They defined it as

$$H_{m,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+m-1} & a_{n+m-2} & \cdots & a_{n+2m-2} \end{vmatrix}. \tag{1.7}$$

Several authors have investigated various classes and subclasses of S, particularly when $a_1 = 1,\ m = 2,\ n = 1$ and $a_1 = 1,\ m = 2,\ n = 2$. It is observed from (1.7) that

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = (a_3 - a_2^2)$$

and

$$H_{2,1}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = (a_2a_4 - a_3^2),$$

where $H_{2,1}(f)$ and $H_{2,2}(f)$ are respectively the Fekete-Szegö classical funtional and second Hankel determinant.

Other matrices are closely related to the Hankel determinat, such as the Toeplitz determinant. A Toeplitz determinant can be considered an upsidedown or inversion of the Hankel determinant in that Hankel determinant has constant entries along the reverse diagonal, in contrast, Toeplitz matrices have constant entries along the diagonal. Symmetric Toeplitz determinant $T_q(n)$ is given as follows

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+m-1} & a_{n+m-2} & \cdots & a_{n+2m-2} \end{vmatrix}.$$

In particular

$$T_2(2) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = (a_1a_3 - a_2^2)$$

and

$$T_2(3) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = (a_2a_4 - a_3^2).$$

The author's main aim is to derive various properties such as coefficient bounds, classical Fekete-Szegö inequality, Hankel and Toeplitz determinants of bi-univalent functions by using the newly introduced generalized Gegenbauer polynomials of two variables $C_m^{\mu}(x, y)$ [18], generalized discrete probability distribution and zero-truncated Poisson distribution series. Furthermore, the results obtained in this investigation establish a new stairway in the relationship between GFT and statistics in general and between GFT and discrete probability in particular. Again, to the best of our knowledge, we stress that there is no such result in the existing literature in this direction.

2. Preliminaries

Firstly, we will introduce the new class $ZT^{\delta}(\alpha, \beta, \mu, x, y)$ of bi-univalent functions.

Definition 2.1. Let $f \in ZT^{\delta}(\alpha, \beta, \mu, x, y),\ \alpha \geq 0,\ \beta \geq 0,\ \delta \in N_0,\ \mu > 0,\ x \in (\frac{1}{2}, 1),\ y \geq 0$, then

$$\frac{(1-\alpha+2\beta)D^{\delta}K_{\tau}(z,m) + (\alpha-2\beta)D^{\delta+1}K_{\tau}(z,m) + \alpha D^{\delta+2}K_{\tau}(z,m) - \alpha z}{z} \prec C_n^{\mu}(z,x,y)$$

and

$$\frac{(1-\alpha+2\beta)D^{\delta}g_{\tau}(w,m) + (\alpha-2\beta)D^{\delta+1}g_{\tau}(w,m) + \beta D^{\delta+2}g_{\tau}(w,m) - \beta w}{w} \prec C_n^{\mu}(w,x,y).$$

Now, we need the following lemmas.

Lemma 2.2 ([17]). *If a function $p \in P$ is given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \quad (z \in \mathbb{D}),$$

then

$$|p_k| \leq 2, \quad (k \in N),$$

where P is a family of every p analytic in $\mathbb{D}$ for which

$$p(0) = 1 \quad \textit{and} \quad R(p(z)) > 0, \quad (z \in \mathbb{D}).$$

Lemma 2.3 ([10]). *Let $p \in P$ then*

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right)$$
$$4p_2 = p_1^3 + 2x\left(4 - p_1^2\right)p_1 - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z,$$

for some x, z, with $|x| < 1$ and $|z| < 1$.

Proposition 2.4. *Let $u(z) = 1 + \sum\limits_{n\geq 1}^{\infty} u_n z^n$, $v(z) = 1 + \sum\limits_{n\geq 1}^{\infty} v_n z^n \in P$ and $u_1 = -v_1$, then*

$$2u_2 = u_1^2 + x\left(4 - u_1^2\right), \qquad 2v_2 = v_1^2 + x(4 - v_1^2)$$

implies that

$$u_2 - v_2 = \frac{\left(4 - u_1^2\right)(x - y)}{2}, \qquad u_2 + v_2 = u_1^2 + \frac{\left(4 - u_1^2\right)(x + y)}{2}$$

and

$$4u_3 = u_1^3 + 2\left(4 - u_1^2\right)u_1 x - \left(4 - u_1^2\right)u_1 x^2 + 2\left(4 - u_1^2\right)\left(1 - |x|^2 z\right)$$
$$4v_3 = m_1^3 + 2(4 - v_1^2)v_1 y - (4 - v_1^2)v_1 y^2 + 2(4 - v_1^2)\left(1 - |y|^2 w\right)$$

then

$$u_3 - m_3 = \frac{u_1^3}{2} + \frac{u_1\left(4 - u_1^2\right)(x + y)}{2} - \frac{u_1\left(4 - u_1^2\right)(x^2 + y^2)}{2} + \frac{\left(4 - u_1^2\right)\left[\left(1 - |x|^2 z\right) - \left(1 - |y|^2 w\right)\right]}{2}$$

for some x, y, z with $|y|, |x|, |z|, |w| \leq 1$, $|u_1| \in [0, 2]$.

3. Main Results

Theorem 3.1. *Let $f \in ZT^{\delta}(\alpha, \beta, \mu, x, y)$, then*

$$\left|\frac{a_1}{\kappa}\right| \leq \frac{x\sqrt{2\mu}}{\sqrt{\left|2K_3x^2\mu + K_2^2[(x+y) - 2x^2(\mu+1)+1]\right|}},$$

$$\left|\frac{a_2}{\kappa}\right| \leq \frac{2x\mu}{K_3},$$

$$\left|\frac{a_3}{\kappa}\right| \leq \frac{5K_4C_1^{2\mu}(x.y) + 2K_2K_3\left[C_2^{\mu}(x,y) + 8C_3^{\mu}(x,y)\right]}{8K_2K_3K_4},$$

and $\eta \in \mathbb{R}$

$$\left|\frac{a_2}{\kappa} - \eta\frac{a_1^2}{\kappa^2}\right| \leq \begin{cases} 2\left|1-\eta\right| M(x,y,\mu), & \left[\left|1-\eta\right| M(x,y,\mu) \geq \frac{x\mu}{2K_3}\right], \\ \frac{x\mu}{K_3}, & \left[\left|1-\eta\right| M(x,y,\mu) \leq \frac{x\mu}{2K_3}\right], \end{cases}$$

where

$$K_n = (1-\alpha+2\beta)\left(\frac{j^{\delta}m^{n-1}}{(e^m-1)(n-1)!}\right) + (\alpha-2\beta)\left(\frac{j^{\delta+1}m^{n-1}}{(e^m-1)(n-1)!}\right) + \beta\left(\frac{j^{\delta+2}m^{n-1}}{(e^m-1)(n-1)!}\right),$$

for $n = 2, 3, 4, \ldots$ *and*

$$M(x,y,\mu) = \frac{8x^3\mu^3}{4K_3x^2\mu^2 + K_2^2[2xy + \mu(1+y) - 2\mu(1+y)x^2]}.$$

Proof. Let $f \in ZT^{\delta}(\alpha, \beta, \mu, x, y)$. From Definition 2.1, we have

(3.1)
$$\frac{(1-\alpha+2\beta)D^{\delta}K_{\tau}(z,m) + (\alpha-2\beta)D^{\delta+1}K_{\tau}(z,m) + \alpha D^{\delta+2}K_{\tau}(z,m) - \alpha z}{z} = C_n^{\mu}(z,x,y)$$

and

(3.2)
$$\frac{(1-\alpha+2\beta)D^{\delta}g_{\tau}(w,m) + (\alpha-2\beta)D^{\delta+1}g_{\tau}(w,m) + \beta D^{\delta+2}g_{\tau}(w,m) - \beta w}{w} = C_n^{\mu}(w,x,y).$$

where

$$u(z) = \frac{1+\nu(z)}{1-\nu(z)} = 1 + u_1 z + u_2 z^2 + u_3 z^3 + \cdots \tag{3.3}$$

and

$$v(w) = \frac{1+\varepsilon(w)}{1-\varepsilon(w)} = 1 + v_1 w + v_2 w^2 + v_3 w^3 + \cdots, \tag{3.4}$$

where

$$\nu(z) = \frac{u(z)-1}{u(z)+1}, \qquad \varepsilon(w) = \frac{v(w)-1}{v(w)+1}, \quad (z, w \in \mathbb{D}).$$

By (3.3) and (3.4), we have

$$\nu(z) = \frac{u_1}{2}z + \left(\frac{u_2}{2} - \frac{u_1^2}{4}\right)z^2 + \left(\frac{u_3}{2} - \frac{u_2 u_1}{2} + \frac{u_1^3}{8}\right)z^3 + \cdots,$$
$$\varepsilon(w) = \frac{v_1}{2}w + \left(\frac{v_2}{2} - \frac{v_1^2}{4}\right)w^2 + \left(\frac{m_3}{2} - \frac{v_2 v_1}{2} + \frac{v_1^3}{8}\right)w^3 + \cdots,$$

which are the expansion of the implication of (3.3) and (3.4).

From (3.1) and (3.2), using (1.2), we obtain

$$\begin{aligned} H(u(z), x, y) = 1 &+ \frac{C_1^\mu(x,y)u_1}{2}z + \left[\frac{C_1^\mu(x,y)}{2}\left(u_2 - \frac{u_1^2}{2}\right) + \frac{C_2^\mu(x,y)u_1}{4}\right]z^2 \\ &+ \left[\frac{C_1^\mu(x,y)}{2}\left(u_3 - u_1 u_2 + \frac{u_1^3}{4}\right) + \frac{C_2^\mu(x,y)u_1}{2}\left(u_2 - \frac{u_1^2}{2}\right)\right. \\ &\left. + \frac{C_3^\mu(x,y)u_1^3}{8}\right]z^3 + \cdots. \end{aligned} \tag{3.5}$$

$$\begin{aligned} H(v(w), x, y) = 1 &+ \frac{C_1^\mu(x,y)v_1}{2}w + \left[\frac{C_1^\mu(x,y)}{2}\left(v_2 - \frac{v_1^2}{2}\right) + \frac{C_2^\mu(x,y)v_1}{4}\right]w^2 \\ &+ \left[\frac{C_1^\mu(x,y)}{2}\left(v_3 - v_1 v_2 + \frac{v_1^3}{4}\right) + \frac{C_2^\mu(x,y)v_1}{2}\left(v_2 - \frac{v_1^2}{2}\right)\right. \\ &\left. + \frac{C_3^\mu(x,y)v_1^3}{8}\right]w^3 + \cdots. \end{aligned} \tag{3.6}$$

From (3.1), (3.2), (3.5) and (3.6), we established that

$$K_2\frac{a_1}{\kappa} = \frac{C_1^\mu(x,y)u_1}{2} \tag{3.7}$$

$$K_3\frac{a_2}{\kappa} = \frac{C_1^{\mu}(x,y)}{2}\left(u_2 - \frac{u_1^2}{2}\right) + \frac{C_2^{\mu}(x,y)u_1}{4} \tag{3.8}$$

$$K_4\frac{a_3}{\kappa} = \frac{C_1^{\mu}(x,y)}{2}\left(u_3 - u_1u_2 + \frac{u_1^3}{4}\right) + \frac{C_2^{\mu}(x,y)u_1}{2}\left(u_2 - \frac{u_1^2}{2}\right) + \frac{1}{8}C_3^{\mu}(x,y)u_1^3 \tag{3.9}$$

$$-K_2\frac{a_1}{\kappa} = \frac{C_1^{\mu}(x,y)v_1}{2} \tag{3.10}$$

$$K_3\left(2\frac{a_1^2}{\kappa^2} - \frac{a_2}{\kappa}\right) = \frac{C_1^{\mu}(x,y)}{2}\left(v_2 - \frac{v_1^2}{2}\right) + \frac{C_2^{\mu}(x,y)v_1}{4} \tag{3.11}$$

and

$$-K_4\left(5\frac{a_1^3}{\kappa^3} - 5\frac{a_1}{\kappa}\frac{a_2}{\kappa} + \frac{a_3}{\kappa}\right) = \frac{C_1^{\mu}(x,y)}{2}\left(v_3 - v_1u_2 + \frac{v_1^3}{4}\right) + \frac{C_2^{\mu}(x,y)v_1}{2}\left(v_2 - \frac{v_1^2}{2}\right) + \frac{1}{8}C_3^{\mu}(x,y)v_1^3. \tag{3.12}$$

From (3.7) and (3.10), it is seen that

$$u_1 + v_1 = 0, \qquad u_1^2 - v_1^2 = 0, \qquad u_1^3 + v_1^3 = 0$$

and

$$\frac{a_1^2}{\kappa^2} = \frac{\left[C_1^{\mu}(x,y)\right]^2\left(u_1^2 + v_1^2\right)}{8K_2^2}.$$

We need to find the estimate on $|u_1|$ for further use in the bounds of $\left|\frac{a_1}{\kappa}\right|$, $\left|\frac{a_2}{\kappa}\right|$ and $\left|\frac{a_3}{\kappa}\right|$ by adding (3.8) and (3.11) which yields

$$2K_3\frac{a_1^2}{\kappa^2} = \frac{1}{2}\left[C_1^{\mu}(x,y)\right](u_2+v_2) - \frac{1}{2}\left[C_1^{\mu}(x,y)\right]u_1^2 + \frac{1}{2}\left[C_2^{\mu}(x,y)\right]u_1^2. \tag{3.13}$$

Using $\frac{a_1}{\kappa} = \frac{\left[C_1^{\mu}(x,y)\right]u_1}{2K}$ from relation (3.13), we have after simple computation that

$$u_1^2 = \frac{K_2^2\left[C_1^{\mu}(x,y)\right](u_2+v_2)}{K_3\left[C_1^{\mu}(x,y)\right]^2 + K_2^2[[C_1^{\mu}(x,y)] - [C_2^{\mu}(x,y)]]}.$$

Hence, for $|u_2| \leq 2$ and $|v_2| \leq 2$, we have

$$|u_1| \leq \frac{2K_2\sqrt{[C_1^\mu(x,y)]}}{\sqrt{K_3\left[C_1^\mu(x,y)\right]^2 + K_2^2[[C_1^\mu(x,y)] - [C_2^\mu(x,y)]]}},$$

which implies

$$\left|\frac{a_1}{\kappa}\right| \leq \frac{x\sqrt{2\mu}}{\sqrt{\left|2K_3x^2\mu + K_2^2[(x+y) - 2x^2(\mu+1) + 1]\right|}}.$$

In order to find an estimate on $\left|\frac{a_2}{\kappa}\right|$ we may express $\left|\frac{a_2}{\kappa}\right|$ in terms of u_1, u_2, v_1 and v_2. To this end, we first subtract (3.9) from (3.12) to obtain

$$2K_3\left(\frac{a_2}{\kappa} - \frac{a_1^2}{\kappa}\right) = \frac{C_1^\mu(x,y)(u_2 - v_2)}{2} - \frac{C_1^\mu(x,y)u_1^2}{2} - \frac{C_2^\mu(x,y)(u_1^2 - v_1^2)}{2}$$

and by $u_1 + v_1 = 0$ in the last expressions, we noticed that

$$2K_3\frac{a_2}{\kappa} = 2K_3\frac{a_1^2}{\kappa^2} + \frac{1}{2}\left[C_1^\mu(x,y)\right](u_2 - v_2). \tag{3.14}$$

Substituting for $\frac{a_1}{\kappa} = \frac{[C_1^\mu(x,y)]u_1}{2K}$ in (3.14), we obtain

$$\begin{aligned}
2K_3\frac{a_2}{\kappa} &= 2K_3\left(\frac{\left[C_1^\mu(x,y)\right]u_1}{2K_2}\right)^2 + \frac{C_1^\mu(x,y)(u_2 - v_2)}{2} \\
&= \frac{2K_3\left[C_1^\mu(x,y)\right]^3(u_2 + v_2)}{4\left\{K - 3\left[C_1^\mu(x,y)\right]^2 + K_2^2[C_1^\mu(x,y) - [C_2^\mu(x,y)]]\right\}} \\
&\quad + \frac{C_1^\mu(x,y)](u_2 - v_2)}{2}. \\
&= [C_1^\mu(x,y)] \times \left\{\frac{\left\{8K_3\left[C_2^\mu(x,y)\right]^2 + 4K_2^2[C_1^\mu(x,y) - C_2^\mu(x,y)]\right\}u_2}{8K_3\left[C_2^\mu(x,y)\right]^2 + 8K_2^2[C_1^\mu(x,y) - C_2^\mu(x,y))}\right. \\
&\quad \left. - \frac{\left\{4K_2^2\left(C_1^\mu(x,y) - C_2^\mu(x,y)\right)\right\}v_2}{8K_3\left[C_2^\mu(x,y)\right]^2 + 8K_2^2\left(C_1^\mu(x,y) - C_2^\mu(x,y)\right)}\right\}.
\end{aligned}$$

Thus by virtue of the inequality $|u_2| \leq 2$ and $|v_2| \leq 2$ and by simple computations, we conclude that

$$\left|\frac{a_2}{\kappa}\right| \leq \frac{[C_1^\mu(x,y)]}{K_3}.$$

Next, in order to find an estimate on $\left|\frac{a_3}{\kappa}\right|$, we first take away (3.12), from (3.14), to get

$$2K_4\frac{a_3}{\kappa} = 5\frac{a_1}{\kappa}\frac{a_2}{\kappa} - 5\frac{a_1^3}{\kappa^3}$$

$$+\frac{C_1^\mu(x,y)(u_3-v_3)-C_1^\mu(x,y)(u_1u_2-v_1v_2)+C_1^\mu(x,y)(u_1^3-v_1^3)}{2}$$
$$+\frac{C_2^\mu(x,y)(u_1u_2-v_1v_2)-C_2^\mu(x,y)(u_1^2-v_1^2)}{2}+\frac{C_3^\mu(x,y)(9u_1^3-v_1^3)}{8}.$$

By (3.9) and (3.10) and the relation above, we have

$$\begin{aligned}\frac{a_3}{\kappa}&=\frac{5\left[C_1^\mu(x,y)\right]^3u_1(u_1^2+v_1^2)-10\left[C_1^\mu(x,y)\right]^3u_1^3}{16K_2^3}\\&+\frac{5\left[C_1^\mu(x,y)\right]^2u_1(u_2-v_2)}{8K_2K_3}\\&+\frac{C_1^\mu(x,y)(u_3-v_3)-C_1^\mu(x,y)u_1(u_2+v_2)}{4K_4}\\&+\frac{C_1^\mu(x,y)u_1^3}{8K_4}-\frac{C_2^\mu(x,y)u_1^3}{4K_4}.\end{aligned}\tag{3.15}$$

Using Lemma 2.2, relations above readily yields

$$\left|\frac{a_3}{\kappa}\right|\leq\frac{5K_4\left[C_1^\mu(x,y)\right]^2+K_2K_3\left[C_3^\mu(x,y)\right]}{K_2K_3K_4}.$$

From (3.15), we write

$$\begin{aligned}\frac{a_2}{\kappa}-\eta\frac{a_1^2}{\kappa^2}&=\frac{a_1^2}{\kappa^2}+\frac{\left[C_1^\mu(x,y)\right](u_2-v_2)}{4K_3}-\eta\frac{a_1^2}{\kappa^2}\\&=\frac{x\mu}{2K_3}+\frac{(1-\eta)2x^3\mu^3}{4K_3x^2\mu^2+K_2^2[2x\mu-2\mu(\mu+1)x^2+\mu(1+y)]}.\end{aligned}$$

Using triangular inequality, we can write that

$$\left|\frac{a_2}{\kappa}-\eta\frac{a_1^2}{\kappa^2}\right|\leq\frac{x\mu}{2K_3}+|1-\eta|\,M_x(y,\mu).$$

Suppose $|1-\eta|\,M_x(y,\mu)\geq\frac{x\mu}{2K_3}$, then

$$\left|\frac{a_2}{\kappa}-\eta\frac{a_1^2}{\kappa^2}\right|\geq|1-\eta|\,M_x(y,\mu),$$

for $|1-\eta|\geq M_x(y,\mu)$. Let $|1-\eta|\,M_x(y,\mu)\leq\frac{x\mu}{2K_3}$, then, we conclude that

$$\left|\frac{a_2}{\kappa}-\eta\frac{a_1^2}{\kappa^2}\right|\leq\frac{x\mu}{K_3},$$

where $|1-\eta|\leq\frac{x\mu}{2K_3M_x(y,\mu)}$ and

$$M_x(y,\mu)=\frac{8x^3\mu^3}{4K_3x^2\mu^2+K_2^2[2x\mu-2\mu(\mu+1)x^2+\mu(1+y)]}.\qquad\square$$

Theorem 3.2. *Let $f \in ZT^{\delta}(\alpha, \beta, \mu, x, y)$. Then*

$$\left|\frac{a_1}{\kappa}\frac{a_3}{\kappa} - \frac{a_2^2}{\kappa^2}\right| \leq \begin{cases} \theta(2), & \text{for } (A_1 \geq 0 \text{ and } A_2 \geq 0), \\ \max\left\{\frac{\left[C_1^{\mu}(x,y)\right]^3}{K_3^2}, \theta(2)\right\}, & \text{for } (A_1 > 0 \text{ and } A_2 < 0), \\ \frac{\left[C_1^{\mu}(x,y)\right]^3}{K_3^2}, & \text{for } (A_1 \leq 0 \text{ and } A_2 \leq 0), \\ \max\left\{\theta(f_0), \theta(2)\right\}, & \text{for } (A_1 < 0 \text{ and } A_2 > 0), \end{cases}$$

where

$$\begin{aligned} [C_1^{\mu}(x,y)] &= 2\mu x, \\ [C_2^{\mu}(x,y)] &= \frac{(\mu)_2}{2!}(2x)^2 - \mu(1+y), \\ [C_3^{\mu}(x,y)] &= \frac{(\mu)_3}{2!}(2x)^3 - \frac{(\mu)_2}{1!}(2x)(1+y), \end{aligned}$$

and

$$\begin{aligned} &A_1(\mu, x, y, K_2, K_3, K_4) \\ &\quad = K_2^3 K_3^2 \left\{ \left[C_1^{\mu}(x,y)\right]^4 + \left[C_1^{\mu}(x,y)\right]\left[C_3^{\mu}(x,y)\right] \right. \\ &\qquad \left. -2\left[C_1^{\mu}(x,y)\right]^2\left[C_1^{\mu}(x,y)\right]\left[C_2^{\mu}(x,y)\right] \right\} \\ &\qquad + K_4 \left\{ K_3^2\left[C_1^{\mu}(x,y)\right]^4 - 7K_2^2 K_3\left[C_1^{\mu}(x,y)\right]^3 + K_2^4\left[C_1^{\mu}(x,y)\right]^2 \right\}, \end{aligned}$$

$$\begin{aligned} &A_2(\mu, x, y, K_2, K_3, K_4) \\ &\quad = 7K_3K_4\left[C_1^{\mu}(x,y)\right]^3 - 2K_2^2\left[C_1^{\mu}(x,y)\right]^2 \\ &\qquad + K_2K_3^2\left\{4\left[C_1^{\mu}(x,y)\right]^2 + 2\left[C_1^{\mu}(x,y)\right]\left[C_2^{\mu}(x,y)\right] + \left[C_1^{\mu}(x,y)\right]^3\right\}, \end{aligned}$$

also

$$\theta(2) = \frac{\left[C_1^{\mu}(x,y)\right]^3}{K_3^2} + \frac{A_1(\mu, x, y, K_2, K_3, K_4)}{K_2^4 K_3^2 K_4} + \frac{A_2(\mu, x, y, K_2, K_3, K_4)}{K_2^2 K_3^2 K_4},$$

and

$$\theta(f_0) = \frac{\left[C_1^{\mu}(x,y)\right]^3}{K_3^2} - \frac{3[A_2(\mu, x, y, K_2, K_3, K_4)]^2}{4K_3^2 K_4 A_1(\mu, x, y, K_2, K_3, K_4)}.$$

and

$$K_n = (1 - \alpha + 2\beta)\left(\frac{j^{\delta} m^{n-1}}{(e^m - 1)(n-1)!}\right) + (\alpha - 2\beta)\left(\frac{j^{\delta+1} m^{n-1}}{(e^m - 1)(n-1)!}\right)$$

$$+\beta\left(\frac{j^{\delta+2}m^{n-1}}{(e^m-1)(n-1)!}\right),$$

for $n=2,3,4,\ldots$.

Proof. By (3.7), (3.14) and (3.15) in Theorem 3.1, we can write

$$\begin{aligned}\frac{a_1}{\kappa}\frac{a_3}{\kappa}-\frac{a_2^2}{\kappa^2}&=\frac{5\left[C_1^{\mu}(x,y)\right]^3u_1^2(u_2-v_2)}{16K_2^2K_3}+\frac{\left[C_1^{\mu}(x,y)\right]^2u_1(u_3-v_3)}{8K_2K_4}\\&\quad-\frac{\left[C_1^{\mu}(x,y)\right]^2u_1^2(u_2+v_2)}{8K_2K_4}+\frac{\left[C_1^{\mu}(x,y)\right]^4u_1^4}{16K_2K_4}\\&\quad+\frac{C_1^{\mu}(x,y)C_2^{\mu}(x,y)u_1^2(u_2+v_2)}{8K_2K_4}\\&\quad+\frac{C_1^{\mu}(x,y)C_3^{\mu}(x,y)u_1^4}{16K_2K_4}-\frac{\left[C_1^{\mu}(x,y)\right]^4u_1^4}{16K_2^4}\\&\quad-\frac{\left[C_1^{\mu}(x,y)\right]^2(u_2-v_2)^2}{16K_2^3}\\&\quad-\frac{\left[C_1^{\mu}(x,y)\right]^3u_1^2(u_2-v_2)}{8K_2^2K_3}-\frac{C_1^{\mu}(x,y)C_2^{\mu}(x,y)u_1^4}{8K_4K_2}\\&\quad+\frac{C_1^{\mu}(x,y)C_2^{\mu}(x,y)u_1^2}{8K_4K_2}.\end{aligned}$$

Application of Proposition 2.4 in the relation above gives

$$\begin{aligned}\frac{a_1}{\kappa}\frac{a_3}{\kappa}-\frac{a_2^2}{\kappa^2}&=\frac{5\left[C_1^{\mu}(x,y)\right]^3u_1^2\left(4-u_1^2\right)(x-y)}{32K_2^2K_3}+\frac{\left[C_1^{\mu}(x,y)\right]^2u_1^4}{16K_2K_4}\\&\quad+\frac{\left[C_1^{\mu}(x,y)\right]^2u_1^2\left(4-u_1^2\right)(x+y)}{16K_2K_4}\\&\quad-\frac{\left[C_1^{\mu}(x,y)\right]^2u_1^2\left(4-u_1^2\right)(x^2+y^2)}{32K_2K_4}\\&\quad+\frac{\left[C_1^{\mu}(x,y)\right]^2u_1\left(4-u_1^2\right)[\left(1-|x|^2\right)z-(1-|y|^2)w]}{16K_2K_4}\\&\quad-\frac{\left[C_1^{\mu}(x,y)\right]^2u_1^4}{8K_2K_4}\\&\quad-\frac{\left[C_1^{\mu}(x,y)\right]^2u_1^2\left(4-u_1^2\right)(x+y)}{16K_2K_4}+\frac{\left[C_1^{\mu}(x,y)\right]^4u_1^4}{16K_2K_4}\\&\quad+\frac{C_1^{\mu}(x,y)\left[C_2^{\mu}(x,y)\right]^2u_1^2\left(4-u_1^2\right)(x+y)}{16K_2K_4}\end{aligned}$$

$$+\frac{C_1^{\mu}(x,y)\left[C_3^{\mu}(x,y)\right]u_1^4}{16K_2K_4}-\frac{\left[C_1^{\mu}(x,y)\right]^4u_1^4}{16K_2^4}$$
$$-\frac{\left[C_1^{\mu}(x,y)\right]^2\left(4-u_1^2\right)^2(x-y)^2}{64K_3^2}$$
$$-\frac{\left[C_1^{\mu}(x,y)\right]^3u_1^2\left(4-u_1^2\right)(x-y)}{16K_2^2K_3}.$$

Assuming that $f=u_1$ and $f\in[0,2]$, $m_1=|x|\leq 1$, $m_2=|y|$, without any limitation, we obtain

$$\left|\frac{a_1}{\kappa}\frac{a_3}{\kappa}-\frac{a_2^2}{\kappa^2}\right|\leq V_1(f)+V_2(f)(m_1+m_2)+V_3(f)\left(m_1^2+m_2^2\right)+V_4(f)(m_1+m_2)^2=F(m_1,m_2), \tag{3.16}$$

where

$$V_1(f)=\frac{3\left[C_1^{\mu}(x,y)\right]^2f^4}{16K_2K_4}+\frac{\left[C_1^{\mu}(x,y)\right]^4f^4}{16K_2K_4}+\frac{\left[C_1^{\mu}(x,y)\right]\left[C_3^{\mu}(x,y)\right]f_1^4}{16K_2K_4}+\frac{\left[C_1^{\mu}(x,y)\right]^4f^4}{16K_2^4}+\frac{\left[C_1^{\mu}(x,y)\right]^2f(4-f^2)}{18K_2K_4}\geq 0$$

$$V_2(f)=\frac{5\left[C_1^{\mu}(x,y)\right]^3f^2(4-f^2)}{32K_2^2K_4}+\frac{5\left[C_1^{\mu}(x,y)\right]^2f^2(4-f^2)}{32K_2^2K_4}+\frac{C_1^{\mu}(x,y)C_3^{\mu}(x,y)f^2(4-f^2)}{16K_2K_4}+\frac{\left[C_1^{\mu}(x,y)\right]^3f^2(4-f^2)}{16K_2^2K_4}\geq 0$$

$$V_3(f)=\left[\frac{\left[C_1^{\mu}(x,y)\right]^2f^2(4-f^2)}{32K_2^2K_4}-\frac{\left[C_1^{\mu}(x,y)\right]^2f(4-f^2)}{16K_2^2K_4}\right]\leq 0$$

$$V_4(f)=\frac{\left[C_1^{\mu}(x,y)\right]^2(4-f^2)^2}{64K_3^2}\geq 0.$$

The function $F(m_1, m_2)$ must be maximized for $f \in [0, 2]$ in the closed square $S_2 = \{(m_1, m_2) : (m_1, m_2) \in [0, 1]\}$. The coefficients of the function $F(m_1, , m_2)$ in equation (3.16) depend on f for a fixed value of f; thus, the maximum of $F(m_1, m_2)$ with regard to f must be determined for the situations $f = 0$, $f = 2$ and $f \in (0, 2)$.

Case I: Setting $f = 0$, then from (3.16), we have

$$F(m_1, m_2) = \frac{\left[C_1^{\mu}(x, y)\right]^2 (4 - f^2)^2}{64K_3^2}.$$

The maximum value of the function $F(m_1, m_2)$ is evidently reached at (m_1, m_2).

$$\begin{aligned} \max\{F(m_1, m_2) : m_1, m_2 \in [0, 1]\} &= F(1, 1) \\ &= \frac{\left[C_1^{\mu}(x, y)\right]^2}{K_3^2}. \end{aligned}$$

Case II: Let $f = 2$, then from (3.16) we have

$$F(m_1, m_2) = \frac{3\left[C_1^{\mu}(x, y)\right]^2}{K_2K_4} + \frac{\left[C_1^{\mu}(x, y)\right]^4}{K_2K_4} + \frac{C_1^{\mu}(x, y)C_3^{\mu}(x, y)}{K_2K_4} + \frac{\left[C_1^{\mu}(x, y)\right]^4}{K_2^4}$$

as a single constant value.

Case III: When $f \in (0, 2)$, we set $m_1 + m_2 = \varepsilon_1$ and $\delta\tau = \varepsilon_2$, in that case, for $\varepsilon_1 \in [0, 2]$, $\varepsilon_2 \in [0, 1]$, (3.16) can be expressed by

$$\begin{aligned} F(m_1, m_2) &= V_1(f) + V_2(f)\varepsilon_1 + [V_3(f) + V_4(f)]\varepsilon^2 - 2V_3(f)\varepsilon_2 \\ &= G(\varepsilon_1, \varepsilon_2). \end{aligned} \tag{3.17}$$

In order to conduct a complete investigation,

$$G(\varepsilon_1, \varepsilon_2) : (\varepsilon_1, \varepsilon_2) \in T = [0, 2] \times [0, 1]$$

and its partial derivatives are examined as follows:

$$\frac{\partial G}{\partial \varepsilon_1} - (V_2(f) + 2[V_3(f) + V_4(f)])\,\varepsilon^2 = 0$$

and

$$\frac{\partial G}{\partial \varepsilon_2} + V_3(f) = 0.$$

The two observations indicate that the critical point for $F(\varepsilon_1, \varepsilon_2)$ does not exist in T, hence $F(m_1, m_2)$ has critical point on the square S_1.

Now, in order to examine the maximum of $F(\varepsilon_1, \varepsilon_2)$, we make use of following cases.

Case IIIa: Put $m_1 = 0$, $m_2 \in [0,1]$ or $m_2 = 0$, $m_1 \in [0,1]$, then from (3.17), we arrive

$$\begin{aligned} F(0,m_2) &= V_1(f) + V_2(f)m_2 + [V_3(f) + V_4(f)]m_2^2 \\ &= \lambda_1(m_2). \end{aligned} \tag{3.18}$$

Then $\lambda_1'(m_2) = V_2(f) + 2[V_3(f) + V_4(f)]m_2$.

In light of the fact that $[V_3(f) + V_4(f)] \geq 0$, we have

$$\begin{aligned} \lambda_1'(m_2) &= V_2(f) + 2[V_3(f) + V_4(f)]m_2 \\ &> 0. \end{aligned}$$

The function $\chi_1(m_2)$ increases for all $m_2 \in [0,1]$ and reaches its maximum at $m_2 = 1$. Therefore, by (3.18), we arrive

$$\begin{aligned} \max\{F(0,m_2) : m_2 \in [0,1]\} &= F(0,m_2) \\ &= V_1(f) + V_2(f) + V_3(f) + V_4(f). \end{aligned} \tag{3.19}$$

Case IIIb: Let $m_1 = 1$, $m_2 \in [0,1]$ or $m_2 = 1$, $m_1 \in [0,1]$. Then from (3.17), we get

$$\begin{aligned} F(1,m_1) &= V_1(f) + V_2(f) + V_3(f) + V_4(f) \\ &\quad + [V_2(f) + 2V_4(f)]m_2 + [V_3(f) + V_4(f)]m_1^2 \\ &= \chi_2(m_2). \end{aligned} \tag{3.20}$$

Indeed,

$$\chi_2' = [V_2(f) + 2V_4(f)] + 2[V_3(f) + V_4(f)]m_2$$

and since $[V_3(f) + V_4(f)] \geq 0$, we get

$$\begin{aligned} \chi_2' &= [V_2(f) + 2V_4(f)] + 2[V_3(f) + V_4(f)]m_2 \\ &> 0. \end{aligned}$$

Thus, $\chi_2(m_2)$ is an increasing function $\forall m_2 \in [0,1]$ and the greatest value occurs at $m_1 = 1$, then by (3.20), we have

$$\begin{aligned} \max\{F(1,m_2) : m_2 \in [0,1]\} &= F(1,1) \\ &= V_1(f) + 2[V_2(f) + V_3(f)] + 4V_4(f) \end{aligned} \tag{3.21}$$

and we can deduce from (3.19) and (3.21) and for $u \in (0,2)$ that

$$V_1(f) + 2[V_2(f) + V_3(f)] + 4V_4(f) > V_1(f) + V_2(f) + V_3(f) + V_4(f)$$

which indicates that

$$\begin{aligned} &\max\{F(m_1,m_2) : m_1 \in [0,1], m_2 \in [0,1]\} \\ &\quad = F(1,1) \\ &\quad = V_1(f) + 2[V_2(f) + V_3(f)] + 4V_4(f). \end{aligned} \tag{3.22}$$

In conclusion, given that $\chi_1(1) \leq \chi_2$ for $u \in [0,2]$,

$$\max\{F(m_1,m_2)\} = F(1,1)$$

on the edge of the square S_1. As a result, on the closed square S_1, the maximum value of F is reached at $m_1 = m_2 = 1$.

In order to evaluate the maximum point (3.22), we represent $\theta : (0,2) \longrightarrow \mathbb{R}$ as:

$$\begin{aligned}\theta(f) &= \max\{F(m_1,m_2)\} \\ &= V_1(f) + 2[V_2(f) + V_3(f)] + 4V_4(f).\end{aligned} \tag{3.23}$$

Substituting $V_1(f), V_2(f), V_3(f)$ and $V_4(f)$ into (3.23) and using straightforward calculations produces

$$\theta(f) = \frac{[C_1^{\mu}(x,y)]^3}{K_3^2} + \frac{A_1(\mu,x,y,K_2,K_3,K_4)}{16K_2^4K_3^2K_4} + \frac{A_2(\mu,x,y,K_2,K_3,K_4)}{4K_2^2K_3^2K_4},$$

where

$$\begin{aligned}&A_1(\mu,x,y,K_2,K_3,K_4) \\ &= K_2^3K_3^2 \times \Big\{[C_1^{\mu}(x,y)]^4 + [C_1^{\mu}(x,y)]\,[C_3^{\mu}(x,y)] \\ &\quad -2\,[C_1^{\mu}(x,y)]^2\,[C_1^{\mu}(x,y)]\,[C_2^{\mu}(x,y)]\Big\} \\ &\quad + K_4\left\{K_3^2\,[C_1^{\mu}(x,y)]^4 - 7K_2^2K_3\,[C_1^{\mu}(x,y)]^3 + K_2^4\,[C_1^{\mu}(x,y)]^2\right\},\end{aligned}$$

$$\begin{aligned}&A_2(\mu,x,y,K_2,K_3,K_4) \\ &= 7K_3K_4\,[C_1^{\mu}(x,y)]^3 - 2K_2^2\,[C_1^{\mu}(x,y)]^2 \\ &\quad + K_2K_3^2\left\{4\,[C_1^{\mu}(x,y)]^2 + 2\,[C_1^{\mu}(x,y)]\,[C_2^{\mu}(x,y)] + [C_1^{\mu}(x,y)]^3\right\}.\end{aligned}$$

Next, we will examine the distinct outcomes of $A_1(\mu,x,y,K_2,K_3,K_4)$ and $A_2(\mu,x,y,K_2,K_3,K_4)$ that follow from (3.16).

Result I: If $A_1(\mu,x,y,K_2,K_3,K_4) \geq 0$ and $A_2(\mu,x,y,K_2,K_3,K_4) \geq 0$, then $\theta'(f) \geq 0$. This shows that $\theta(f)$ is an increasing function on the boundary of $f \in [0,2]$ that is $f = 2$. Hence

$$\begin{aligned}\max\{\theta(f) : f \in (0,2)\} = \frac{[C_1^{\mu}(x,y)]^3}{K_3^2} &+ \frac{A_1(\mu,x,y,K_2,K_3,K_4)}{K_2^4K_3^2K_4} \\ &+ \frac{A_2(\mu,x,y,K_2,K_3,K_4)}{K_2^2K_3^2K_4}.\end{aligned}$$

Result II: If $A_1(\mu,x,y,K_2,K_3,K_4) > 0$ and $A_2(\mu,x,y,K_2,K_3,K_4) <$

0, then

$$\begin{aligned}\theta(f) &= \frac{A_1(\mu,x,y,K_2,K_3,K_4)}{4K_2^4K_3^2K_4}f^3 + \frac{A_2(\mu,x,y,K_2,K_3,K_4)}{2K_2^2K_3^2K_4}f \\ &= 0\end{aligned}$$

which implies that

$$f_0 = \sqrt{\frac{-2K_2^2A_2(\mu,x,y,K_2,K_3,K_4)}{A_1(\mu,x,y,K_2,K_3,K_4)}}$$

is a critical point of function $\theta(f)$ and since

$$\begin{aligned}\theta''(f_0) &= \frac{-3A_2(\mu,x,y,K_2,K_3,K_4)}{2K_2^2K_3^2K_4} \\ &> 0,\end{aligned}$$

by $\theta''(f_0) > 0$, the minimum value of the function $\theta(f)$ occurs at $f = f_0$ can be simplified to

$$\theta(f_0) = \frac{\left[C_1^{\mu}(x,y)\right]^3}{K_3^2} - \frac{3[A_2(\mu,x,y,K_2,K_3,K_4)]^2}{4K_3^2K_4A_1(\mu,x,y,K_2,K_3,K_4)}.$$

By this case

$$\theta(f_0) < \frac{\left[C_1^{\mu}(x,y)\right]^3}{K_3^2}$$

and

$$\begin{aligned}\max\{F(m_1,m_2) : m_1,m_2 \in [0,1]\} &= \theta(0) \\ &= \max\left\{\frac{\left[C_1^{\mu}(x,y)\right]^3}{K_3^2}, \theta(2)\right\}.\end{aligned}$$

Result III: If $A_1(\mu,x,y,K_2,K_3,K_4) \leq 0$ and $A_2(\mu,x,y,K_2,K_3,K_4) \leq 0$, then $\theta'(f) \leq 0$. Hence, $\theta(f)$ is a decreasing function on the interval $(0,2)$. Thus

$$\begin{aligned}\max\{F(m_1,m_2) : m_1,m_2 \in [0,1]\} &= \theta(0) \\ &= \frac{\left[C_1^{\mu}(x,y)\right]^3}{K_3^2}.\end{aligned}$$

Result IV: If $A_1(\mu,x,y,K_2,K_3,K_4) \leq 0$ and $A_2(\mu,x,y,K_2,K_3,K_4) \leq 0$, then

$$\begin{aligned}\theta'(f) &= \frac{A_1(\mu,x,y,K_2,K_3,K_4)}{4K_2^4K_3^2K_4}f^3 + \frac{A_2(\mu,x,y,K_2,K_3,K_4)}{2K_2^2K_3^2K_4}f \\ &= 0\end{aligned}$$

at a critical point

$$f_1 = \sqrt{\frac{-2K_2^2 A_2(\mu, x, y, K_2, K_3, K_4)}{A_1(\mu, x, y, K_2, K_3, K_4)}}.$$

The critical point of function $\theta(f)$ is

$$\begin{aligned}\theta''(f_1) &= \frac{-3A_2(\mu, x, y, K_2, K_3, K_4)}{2K_2^2 K_3^2 K_4} \\ &< 0\end{aligned}$$

hence for $\theta''(f_1) < 0$, the maximum value of $\theta(f)$ attains at $f = f_1$ and from

$$\theta(f_0) = \frac{[C_1^\mu(x,y)]^3}{K_3^2} - \frac{3[A_2(\mu, x, y, K_2, K_3, K_4)]^2}{4K_3^2 K_4 A_1(\mu, x, y, K_2, K_3, K_4)},$$

we have

$$\theta(f_0) > \frac{[C_1^\mu(x,y)]^3}{K_3^2}$$

and

$$\begin{aligned}\max\{F(m_1, m_2) : m_1, m_2 \in [0,1]\} &= \theta(0) \\ &= \max\left\{\frac{[C_1^\mu(x,y)]^3}{K_3^2}, \theta(2)\right\}\end{aligned}$$

and the prove is complete. □

4. Toeplitz Determinant

Theorem 4.1. *Let $f \in ZT^\delta(\alpha, \beta, \mu, x, y)$. Then*

$$\left|\frac{a_2^2}{\kappa} - \frac{a_1^2}{\kappa^2}\right| \leq \begin{cases} \Theta(2), & \text{for } (A_3 \geq 0 \text{ and } A_4 \geq 0), \\ \max\left\{\frac{[C_1^\mu(x,y)]^2}{K_3^2}, \Theta(2)\right\}, & \text{for } (A_3 > 0 \text{ and } A_4 < 0), \\ \frac{[C_1^\mu(x,y)]^2}{K_3^2}, & \text{for } (A_3 \leq 0 \text{ and } A_4 \leq 0), \\ \max\{\Theta(f_0), \Theta(2)\}, & \text{for } (A_3 < 0 \text{ and } A_4 > 0), \end{cases}$$

where

$$A_3(x, y, \mu, K_2, K_3) = K_3^2 [C_1^\mu(x,y)]^4 + K_2^4 [C_1^\mu(x,y)]^2 - 2K_2^2 K_3 [C_1^\mu(x,y)]^3,$$
$$A_4(x, y, \mu, K_2, K_3) = 2K_3 [C_1^\mu(x,y)]^3 + K_3^2 [C_1^\mu(x,y)] - 2K_2^2 [C_1^\mu(x,y)],$$

and

$$\Theta(2) = \frac{\left[C_1^{\mu}(x,y)\right]^2}{K_3^2} + \frac{A_3(x,y,\mu,K_2,K_3)}{K_2^4 K_3^3} + \frac{A_4(x,y,\mu,K_2,K_3)}{K_2^2 K_3^3},$$

$$\Theta(f_0) = \frac{\left[C_1^{\mu}(x,y)\right]^2}{K_3^2} - \frac{A_4^2(x,y,\mu,K_2,K_3)}{4K_3^2 A_3(x,y,\mu,K_2,K_3)}.$$

Proof. The proof follows the same method as in Theorem 3.2. □

5. Conclusion

With various choices of $a_{n-1}s$ in the defined class, some discrete probability distributions can be obtained, such as Yule-Simeon, Zeta, geometric, Bernoulli, Beta-Bernoulli, logarithm and Binomial. They can be used to investigate a gambler's ruin. Furthermore, with special choices of the parameters involved, many existing and new classes of bi-univalent functions can be defined. Our results provide a new stairway of investigations in GFT and further consolidate the relationship between geometric function theory and statistics in general and discrete probability in particular.

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