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Abstract. In this article, we delve into the concept of convexity within fuzzy metric spaces and delve into their structural characteristics. We present several theorems concerning the existence of coincidence points in fuzzy convex metric spaces. Furthermore, we introduce the notion of star-shaped subsets within fuzzy convex metric spaces. Within these star-shaped subsets, we showcase various fixed point theorems for mappings of the non-expansive type that commute. In the final section, we expand the definition of fuzzy convex metric spaces and provide a significant example of such a space. Additionally, we uncover specific fixed point theorems applicable to multi-valued mappings that are non-expansive.

1. Introduction

The concept of fuzzy sets, initially introduced by Zadeh [38] in 1965, has had a profound and transformative impact on nearly every scientific and engineering discipline, including the realm of mathematics. A fuzzy set is characterized by assigning a membership grade within the range of [0, 1] to each element within a set. Fuzzy logic finds relevance in the domain of probabilistic metric spaces, driven by the realization that uncertainty in certain scenarios often arises from fuzziness rather than randomness. In response to this ambiguity, Kramosil and Michalek [16] expanded the concept of probabilistic metric spaces in 1975. It is worth noting that the proof of the fixed point theorem in fuzzy metric spaces relies significantly on the concept of fuzzy contraction mappings.

Several authors have recently introduced the concept of fuzzy metric spaces in various ways, as evidenced by works such as those by [3, 4, 15], among others. Furthermore, they have provided demonstrations of the fuzzy Banach contraction theorem, as exemplified by [8, 23, 24, 26–32].

A natural question arises: can we adapt the principle of fuzzy Banach's contraction, which stipulates that global fuzzy contraction applies only to points

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in close proximity, to maintain its validity? It becomes intriguing to define the concept of locally fuzzy contraction mapping to explore this possibility and ascertain that any locally fuzzy contraction mapping retains the property of being globally fuzzy contractive.

Conversely, fixed point theory, a renowned mathematical theory, finds its utility across a multitude of scientific domains, including chaos theory, game theory, and the theory of differential equations, among others. In the context of non-expansive mappings, the initial theorem of fixed point theory was independently established by Browder [2] and Göhde [7], with a more general proof offered by Kirk [14]. In 1948, Brodskii and Milman introduced the concept of normal structure, as documented in [1]. Takahashi, in 1970 [37], was the first to define convex and normal structures for sets within metric spaces. The introduction of convex structures for sets in probabilistic metric spaces came about in 1987 through the work of Hadžić, as presented in [9].

Within the realm of fixed point theory in probabilistic metric spaces, the initial findings were made by Sehgal and Bharucha-Reid, as referenced in [33]. Hadžić, on the other hand, has put forth the fixed point theorem for mappings in convex probabilistic metric spaces.

For further insights into convexity and fixed point results, one can refer to the sources cited, including [1, 2, 4, 5, 7, 9–13, 18–22, 34–37]. These sources provide extensive information on the topic, covering mappings defined within metric spaces, probabilistic metric spaces, and fuzzy metric spaces.

In the context of strictly convex intuitionistic fuzzy metric spaces, Jesic [12] has established the existence of a fixed point for the category of non-expansive mappings. He made this significant discovery after recently identifying a wide-ranging class of non-expansive mappings within intuitionistic fuzzy metric spaces characterized by their convex, strictly convex, and normal structural properties.

Furthermore, Ješić, Nikolic, and Babačev [13] have introduced the concepts of strictly convex and normal structure within Menger PM-spaces. In addition to these developments, they have provided evidence of a common fixed point theorem for two self-mappings defined within strictly Menger PM-spaces.

In a more recent context, Rashid [25] has contributed to the exploration of fixed point existence and uniqueness for non-self mappings. His work focuses on leveraging the concept of strictly convex structure within the framework of fuzzy cone metric spaces.

In this study, our focus lies in exploring the structural aspects of fuzzy metric spaces, with the primary objective of introducing the notion of convexity within this domain. Additionally, we endeavor to establish a set of theorems that pertain to coincidence points within fuzzy convex metric spaces. We will proceed to delineate the concept of a star-shaped subset within the context of fuzzy convex metric spaces, followed by the presentation of various fixed point theorems specifically applicable to commuting mappings of the non-expansive category when applied to such star-shaped subsets.

Furthermore, we will extend the concept of fuzzy convex metric spaces in a broader sense and present a complex illustrative example of such a space.

Lastly, we will derive a set of fixed point theorems that are applicable to multivalued mappings characterized by their non-expansive nature.

2. Preliminaries and Notations

In this section, we introduce certain notations and definitions that will serve to consolidate the content of the paper.

Definition 2.1 ([38])**.** Let's take an arbitrary set $\mathscr{X}$ as our starting point. Within this set, we establish the concept of a fuzzy set denoted as $\mathscr{A}$. This fuzzy set is essentially a function that associates elements from the set $\mathscr{X}$ with values falling within the range of $[0, 1]$.

Definition 2.2 ([31])**.** A continuous τ-norm can be described as a function $\Theta : [0, 1] \times [0, 1] \to [0, 1]$, which exhibits the following properties: it is continuous, associative, commutative, non-decreasing in each of its coordinates, and satisfies the condition $\Theta(a, 1) = a$ for all $a \in [0, 1]$.

Definition 2.3 ([16])**.** A triple $(\mathscr{X}, \mathscr{M}, \Theta)$ is called a fuzzy metric space (briefly, a FM-space) if $\mathscr{X}$ is an arbitrary (non-empty) set, Θ is a continuous t-norm and $\mathscr{M}$ is a fuzzy set on $\mathscr{X} \times \mathscr{X} \times [0, \infty)$ such that the following axioms hold:

(FM-1) $\mathscr{M}(u, v, 0) = 0$ for all $u, v \in \mathscr{X}$,
(FM-2) $\mathscr{M}(u, v, \tau) = 1$ for every $\tau > 0$ if and only if $u = v$,
(FM-3) $\mathscr{M}(u, v, \tau) = M(v, u, \tau)$ for all $u, v \in \mathscr{X}$ and $\tau > 0$,
(FM-4) $\mathscr{M}(u, v, .) : [0, \infty) \to [0, 1]$ is left continuous for all $u, v \in \mathscr{X}$,
(FM-5) $\mathscr{M}(u, w, \tau + \nu) \geq \Theta(\mathscr{M}(u, v, \tau), \mathscr{M}(v, w, \nu))$ for all $u, v, w \in \mathscr{X}$ and for all $\tau, \nu \in [0, \infty)$.

The fuzzy metric spaces as defined by Kramosil and Michalek will be denoted as KM-fuzzy metric spaces. In the event that the condition (FM-5) in the previously mentioned definition undergoes a modification and is replaced by the condition:

(FM-5A) $\mathscr{M}(u, w, \max\{\tau, \nu\}) \geq \Theta(\mathscr{M}(u, v, \tau), \mathscr{M}(v, w, \nu))$ for all $u, v, w \in \mathscr{X}$ and for all $\tau, \nu \in [0, \infty)$.

Then, the structure $(X, M, *)$ is termed a non-Archimedean fuzzy metric space. It is straightforward to confirm that the fulfillment of condition (FM-5A) implies condition (FM-5), thereby establishing that each non-Archimedean fuzzy metric space inherently qualifies as a fuzzy metric space.

Definition 2.4 ([6])**.** A triple $(\mathscr{X}, \mathscr{M}, \Theta)$ is called a non-Archimedean fuzzy metric space (briefly, a FM-space) if $\mathscr{X}$ is an arbitrary (non-empty) set, Θ is a continuous τ-norm and $\mathscr{M}$ is a fuzzy set on $\mathscr{X} \times \mathscr{X} \times [0, \infty)$ such that the following axioms hold:

(FM-1) $\mathscr{M}(u, v, \tau) > 0$ for all $u, v, w \in \mathscr{X}$,
(FM-2) $\mathscr{M}(u, v, \tau) = 1$ for every $\tau > 0$ if and only if $u = v$,
(FM-3) $\mathscr{M}(u, v, \tau) = M(v, u, \tau)$ for all $u, v \in \mathscr{X}$ and $\tau > 0$,
(FM-4) $\mathscr{M}(u, v, .) : [0, \infty) \to [0, 1]$ is left continuous for all $u, v \in \mathscr{X}$,

(FM-5) $\mathscr{M}(u,w,\max\{\tau,\nu\}) \geq \Theta\left(\mathscr{M}(u,v,\tau),\mathscr{M}(v,w,\nu)\right)$ for all $u,v,w \in \mathscr{X}$ and for all $\tau,\nu \in [0,\infty)$.

The fuzzy metric spaces defined by George and Veeramani shall be referred to as GV-fuzzy metric spaces.

Example 2.5 ([6])**.** Let $(\mathscr{X},\varrho)$ be a metric space.

(i) Define a τ-norm by $\Theta(a,b) = ab$, and set

$$\mathscr{M}_{\varrho}(u,v,\tau) = \frac{\tau}{\tau+\varrho(u,v)}, \quad \text{for all } u,v \in \mathscr{X} \text{ and } \tau > 0.$$

Then $(\mathscr{X},\mathscr{M}_{\varrho},\Theta)$ is a non-Archimedean fuzzy metric space; $\mathscr{M}_{\varrho}$ is called the standard fuzzy metric induced by ϱ. It is interesting to note that the topology induced by the $\mathscr{M}_{\varrho}$ and the corresponding metric ϱ coincide.

(ii) Let $(\mathscr{X},\varrho)$ be a metric space. Define a τ-norm by $\Theta(a,b) = ab$, and set

$$\mathscr{M}(u,v,\Theta) = exp\left[-\frac{\varrho(u,v)}{\tau}\right], \quad \text{for all } u,v \in \mathscr{X} \text{ and } \tau > 0.$$

Then $(\mathscr{X},\mathscr{M},\Theta)$ is a non-Archimedean fuzzy metric space.

Remark 2.6 ([6])**.** Let $(\mathscr{X},\mathscr{M},\Theta)$ be a FM-space. For all $t \in (0,\infty)$, the open ball $\mathscr{B}(u,r,\tau)$ with center $x \in \mathscr{X}$ and radius $r \in (0,1)$ is defined by

$$\mathscr{B}(u,r,\tau) = \{v \in \mathscr{X} : \mathscr{M}(u,v,\tau) > 1-r\}. \tag{2.1}$$

Let $(\mathscr{X},\mathscr{M},\Theta)$ be a FM-space and let $\mathscr{T}_{\mathscr{M}}$ be the set of all $\mathscr{A} \subset \mathscr{X}$ with $x \in \mathscr{A}$ if and only if there exists $\tau > 0$ and $r \in (0,1)$ such that $\mathscr{B}(u,r,\tau) \subset \mathscr{A}$. Then $\mathscr{T}_{\mathscr{M}}$ is a topology on $\mathscr{X}$ (induced by the fuzzy metric $\mathscr{M}$). This topology is Hausdorff and first countable.

Lemma 2.7 ([8])**.** *Let $(\mathscr{X},\mathscr{M},\Theta)$ be a FM-spaces. Then $\mathscr{M}(u,v,\tau)$ is non-decreasing with respect to τ, for all $u,v \in \mathscr{X}$.*

Definition 2.8 ([8])**.** Let $(\mathscr{X},\mathscr{M},\Theta)$ be a FM-spaces.

(i) A sequence $\{\kappa_n\}$ in $\mathscr{X}$ is said to be convergent to $\xi \in \mathscr{X}$ if

$$\lim_{n\to\infty} \mathscr{M}(\kappa_n,\xi,\tau) = 1 \quad \text{for all } \tau > 0.$$

(ii) The sequence $\{\kappa_n\}$ in $\mathscr{X}$ is said to be Cauchy if

$$\lim_{n,m\to\infty} \mathscr{M}(\kappa_n,\kappa_m,\tau) = 1 \quad \text{for all } \tau > 0.$$

or equivalently, if for any $\varepsilon \in (0,1)$ and $\tau > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\mathscr{M}(\kappa_n,\kappa_m,t) > 1-\varepsilon \quad \text{for all } n,m \geq n_0.$$

(iii) The FM-space $(\mathscr{X},\mathscr{M},\Theta)$ is said to be complete if every Cauchy sequence in $\mathscr{X}$ is convergent.

Lemma 2.9 ([17]). *Let $(\mathscr{X}, \mathscr{M}, \Theta)$ be a FM-spaces. If there exists $\gamma \in (0,1)$ such that*

$$\mathscr{M}(u, v, \gamma\tau) \geq \mathscr{M}(u, v, \tau)$$

for all $u, v \in \mathscr{X}$ and $\tau > 0$, then $u = v$.

Definition 2.10. Let $(\mathscr{X}, \mathscr{M}, \Theta)$ be a FM-spaces with a continuous τ-norm Θ and $\mathscr{K}$ be a non-empty subset of $\mathscr{X}$. If $\sup_{\tau>0} \mathscr{D}_{\mathscr{K}}(\tau) = 1$, where

$$\mathscr{D}_{\mathscr{K}}(\tau) = \sup_{\nu<\tau} \inf_{p,q\in\mathscr{K}} \mathscr{M}(p, q, \nu),$$

then $\mathscr{K}$ is called a fuzzy bounded subset of $\mathscr{X}$ and $\mathscr{D}_{\mathscr{K}}(\tau)$ the fuzzy diameter of $\mathscr{K}$.

Definition 2.11. Let $(\mathscr{X}, \mathscr{M}, \Theta)$ be a FM-spaces with a continuous τ-norm Θ and $\mathscr{K}$ be a non-empty subset of $\mathscr{X}$. Then $\mathscr{K}$ is bounded if

$$\sup_{\tau} \mathscr{D}_{\mathscr{K}}(\tau) = 1.$$

Definition 2.12. Let $(\mathscr{X}, \mathscr{M}, \Theta)$ be a FM-spaces with a continuous τ-norm Θ and $\mathscr{K}$ be a non-empty subset of $\mathscr{X}$. A Kuratowski's function for a fuzzy bounded subset $\mathscr{K}$ of $\mathscr{X}$ is the function $\alpha_{\mathscr{K}}$ on $\mathbb{R}$ defined by

$$\begin{aligned}\alpha_{\mathscr{K}}(\tau) = \sup\{&\varepsilon > 0, \quad \text{there is a finite cover } \mathbb{K} \text{ of } \mathscr{K} \text{ such that}\\ &\mathscr{D}_{\mathscr{B}}(\tau) \geq \varepsilon \text{ for all } \mathscr{B} \in \mathbb{K}\},\end{aligned}$$

where $\mathscr{D}_{\mathscr{K}}(\tau) = \sup_{\nu<\tau} \inf_{p,q\in\mathscr{K}} \mathscr{M}(p, q, \nu)$, $\tau \in \mathbb{R}^+$.

Definition 2.13. Let $(\mathscr{X}, \mathscr{M}, \Theta)$ be a FM-spaces with a continuous τ-norm Θ and $\mathscr{K}$ be a non-empty subset of $\mathscr{X}$. If $\mathscr{K}$ is a fuzzy bounded subset of $\mathscr{X}$, then $\mathscr{K}$ is a fuzzy precompact if $\alpha_{\mathscr{K}}(\tau) = 1$ for all $\tau > 0$.

3. Coincidence Points in Fuzzy Convex Metric Spaces

In this section, we introduce the concept of convexity within fuzzy metric spaces and conduct an analysis of their structural properties. Furthermore, we will present several theorems concerning the existence of coincidence points specifically within fuzzy convex metric spaces. This discussion is particularly relevant to GV-fuzzy metric spaces as described by George and Veeramani.

Definition 3.1. Let $(\mathscr{X}, \mathscr{M}, \Theta)$ be a fuzzy metric space. A mapping $\mathcal{W} : \mathscr{X}^2 \times [0,1] \to (0,1)$ is said to be a convex structure if for every $(z, u, v, \lambda) \in \mathscr{X}^3 \times (0,1)$,

$$\mathscr{M}(z, \mathcal{W}(u, v, \lambda), 2\varepsilon) \geq \Theta\left(\mathscr{M}\left(z, u, \frac{\varepsilon}{\lambda}\right), \mathscr{M}\left(z, v, \frac{\varepsilon}{1-\lambda}\right)\right)$$

for every $\varepsilon \in \mathbb{R}^+$, and $\mathcal{W}(u, v, 0) = v$, $\mathcal{W}(u, v, 1) = u$.

Definition 3.2. A fuzzy metric space $(\mathscr{X}, \mathscr{M}, \Theta)$ with convex structure $\mathcal{W}$ satisfies the condition (MII) if for all $(u, v, w, \lambda) \in \mathscr{X}^3 \times (0,1)$,

$$\mathscr{M}(\mathcal{W}(u, w, \lambda), \mathcal{W}(v, w, \lambda), 2\varepsilon) \geq \mathscr{M}(u, v, \varepsilon) \quad \text{for every } \varepsilon \in \mathbb{R}^+.$$

Definition 3.3. If $(\mathscr{X}, \mathscr{M}, \Theta)$ is fuzzy metric space with a convex structure $\mathcal{W}$ and $\zeta_0 \in \mathscr{X}$, we say that $T : \mathscr{X} \to \mathscr{X}$ is $(\mathcal{W}, \zeta_0)$-convex if $\mathcal{W}(Tz, \zeta_0, \lambda) = T(\mathcal{W}(z, \zeta_0, \lambda))$, for every $\lambda \in (0, 1)$ and every $z \in \mathscr{X}$.

Theorem 3.4. *Let $(\mathscr{X}, \mathscr{M}, \Theta)$ be a fuzzy metric space with a convex structure $\mathcal{W}$ satisfy condition (MII) and a continuous t-norm Θ, $\mathscr{A}$ and $\mathcal{S}$ be continuous, commutative mappings from $\mathscr{X}$ into $\mathscr{X}$ such that $\mathscr{A}(\mathscr{X})$ is fuzzy bounded subset of $S(\mathscr{X})$, $\zeta_0 \in \mathscr{X}$ and S be $(\mathcal{W}, \zeta_0)$-convex so that*

$$\mathscr{M}(\mathscr{A}x, \mathscr{A}y, \varepsilon) \geq \mathscr{M}(\mathcal{S}x, \mathcal{S}y, \varepsilon),$$

for every $x, y \in \mathscr{X}$ and every $\varepsilon \in \mathbb{R}^+$. If there exists a compact set $\mathscr{K} \subseteq \mathscr{X}$ such that $\mathcal{S}(\mathscr{K})$ is an attractor for $\mathscr{A}$, then there exists $z \in \mathscr{X}$ such that $\mathscr{A}z = \mathcal{S}z$, that is, z is a coincidence point of $\mathcal{S}$ and $\mathscr{A}$.

Proof. Let $\gamma_n \in (0, 1)$, $n = 1, 2, \ldots$, $\lim_{n\to\infty} \gamma_n = 1$ and for every $x \in \mathscr{X}$, $\mathscr{A}_n x = \mathcal{W}(\mathscr{A}x, \zeta_0, \gamma_n)$, $n \in \mathbb{N}$. It follows that all the conditions of [10, Theorem 2] are satisfied for $\mathscr{A}_n$, $\mathcal{S}$ and $T = I$ (: the identity on $\mathscr{X}$) and so there exists $\kappa_n \in \mathscr{X}$ such that $\kappa_n = \mathscr{A}_n\kappa_n = \mathcal{S}\kappa_n$ for every $n \in \mathbb{Z}^+$. Further, we have, for every $n \in \mathbb{Z}^+$ and every $\varepsilon > 0$,

$$\begin{aligned}
\mathscr{M}(\kappa_n, \mathscr{A}\kappa_n, \varepsilon) &= \mathscr{M}(\mathscr{A}_n\kappa_n, \mathscr{A}\kappa_n, \varepsilon) \\
&= \mathscr{M}(\mathcal{W}(\mathscr{A}\gamma_n, \zeta_0, \gamma_n), A\kappa_n, \varepsilon) \\
&\geq \Theta\left(\mathscr{M}\left(\mathscr{A}_n\kappa_n, \mathscr{A}\kappa_n, \frac{\varepsilon}{2\gamma_n}\right), \mathscr{M}\left(\mathscr{A}\kappa_n, \zeta_0, \frac{\varepsilon}{2(1-\gamma_n)}\right)\right) \\
&= \Theta\left(1, \mathscr{M}\left(\mathscr{A}\kappa_n, \zeta_0, \frac{\varepsilon}{2(1-\gamma_n)}\right)\right) \\
&= \mathscr{M}\left(\mathscr{A}\kappa_n, \zeta_0, \frac{\varepsilon}{2(1-\gamma_n)}\right).
\end{aligned}$$

Since $\overline{\bigcup_{m\in\mathbb{Z}^+} \mathscr{A}^m(\kappa_n)} \cap \mathcal{S}(\mathscr{K}) \neq \emptyset$ for every $n \in \mathbb{Z}^+$, it follows that there exists $\mu_n \in \mathscr{K}$, $n \in \mathbb{Z}^+$ such that $S\mu_n \in \overline{\bigcup_{m\in\mathbb{Z}^+} \mathscr{A}^m(\kappa_n)}$. Let us prove that for every $\varepsilon > 0$, $\lim_{n\to\infty} \mathscr{M}(\mathcal{S}\mu_n, \mathscr{A}\mu_n, \varepsilon) = 1$. Let $\varepsilon > 0$ and $\lambda \in (0, 1)$. We have to prove that there exists $n(\varepsilon, \lambda) \in \mathbb{Z}^+$ so that

$$\mathscr{M}(\mathcal{S}\mu_n, \mathscr{A}\mu_n, \varepsilon) > 1 - \lambda, \quad n \geq n(\varepsilon, \lambda).$$

As in [11], it can proved that, for every n and s in $\mathbb{Z}^+$,

$$\mathscr{M}(\mathscr{A}^s\kappa_n, \mathscr{A}^{s+1}\kappa_n, \varepsilon) \geq \mathscr{M}(\kappa_n, \mathscr{A}\kappa_n, \varepsilon), \quad \varepsilon > 0.$$

Since Θ is continuous, there exists $\eta \in (0, 1)$ so that

$$x, y, z \geq \eta \quad \Rightarrow \quad \Theta(x, \Theta(y, z)) > 1 - \lambda.$$

Further, for every $\varepsilon > 0$ and $s \in \mathbb{Z}^+$, we have

$$\mathscr{M}(\mathcal{S}\mu_n, \mathscr{A}\mu_n, \varepsilon) \geq \Theta\left(\mathscr{M}\left(\mathcal{S}\mu_n, \mathscr{A}^s\kappa_n, \frac{\varepsilon}{3}\right), \Theta\left(\mathscr{M}\left(A^s\kappa_n, A^{s+1}\kappa_n, \frac{\varepsilon}{3}\right),\right.\right.$$

$$\mathscr{M}\left(\mathscr{A}^{s+1}\kappa_n, \mathscr{A}\mu_n, \frac{\varepsilon}{3}\right)\Big)\Big)$$
$$\geq \Theta\left(\mathscr{M}\left(\mathcal{S}\mu_n, \mathscr{A}^s\kappa_n, \frac{\varepsilon}{3}\right), \Theta\left(\mathscr{M}\left(\mathscr{A}\kappa_n, \zeta_0, \frac{\varepsilon}{6(1-\gamma_n)}\right)\right.\right.$$
$$\left.\left.* \ \mathscr{M}\left(\mathscr{A}^s\kappa_n, S\mu_n, \frac{\varepsilon}{3}\right)\right)\right).$$

Since $\mathcal{S}\mu_n \in \overline{\bigcup_{m\in\mathbb{Z}^+} \mathscr{A}^m(\kappa_n)}$, for every $n \in \mathbb{Z}^+$, there exists $m_n \in \mathbb{Z}^+$ such that

$$\mathscr{M}\left(\mathcal{S}\mu_n, \mathscr{A}^{m_n}, \frac{\varepsilon}{3}\right) > \eta, \quad n \in \mathbb{Z}^+.$$

Further, the set $\mathscr{A}(\mathscr{X})$ is fuzzy bounded, which means that

$$\sup_x \mathscr{D}_{\mathscr{A}(\mathscr{X})}(x) = 1,$$

where

$$\mathscr{D}_{\mathscr{A}(\mathscr{X})}(x) = \sup_{t<x} \sup_{p,q\in\mathscr{A}(\mathscr{X})} \mathscr{M}(p,q,t) = 1.$$

Let us prove that there exists $n_0(\varepsilon,\eta) \in \mathbb{Z}^+$ such that

$$\mathscr{M}\left(\mathscr{A}\kappa_n, \zeta_0, \frac{\varepsilon}{6(1-\gamma_n)}\right) > \eta, \quad n \geq n_0(\varepsilon,\eta). \tag{3.1}$$

For every $z \in \mathscr{X}$, we have

$$\mathscr{M}\left(A\kappa_n, \zeta_0, \frac{\varepsilon}{6(1-\gamma_n)}\right) \geq \Theta\left(\mathscr{M}\left(\mathscr{A}\kappa_n, \mathscr{A}z, \frac{\varepsilon}{12(1-\gamma_n)}\right),\right.$$
$$\left.\mathscr{M}\left(\mathscr{A}z, \zeta_0, \frac{\varepsilon}{12(1-\gamma_n)}\right)\right)$$

and, since $\lim_{n\to\infty} \gamma_n = 1$ and $\mathscr{A}(\mathscr{X})$ is fuzzy bounded, it follows that there exists $n_0(\varepsilon,\eta) \in \mathbb{Z}^+$ so that (3.1) holds. Then the following:

$$\mathscr{M}(\mathcal{S}\mu_n, \mathscr{A}\mu_n, \varepsilon) \geq \Theta\left(\mathscr{M}\left(\mathcal{S}\mu_n, \mathscr{A}^{m_n}\kappa_n, \frac{\varepsilon}{3}\right), \Theta\left(\mathscr{M}\left(\mathscr{A}\kappa_n, \zeta_0, \frac{\varepsilon}{6(1-\gamma_n)}\right),\right.\right.$$
$$\left.\left.\mathscr{M}\left(\mathscr{A}^{m_n}\kappa_n, \mathcal{S}\mu_n, \frac{\varepsilon}{3}\right)\right)\right)$$
$$\geq \Theta\left(\eta, \Theta\left(\eta,\eta\right)\right), \quad n \geq n(\varepsilon,\eta),$$

we obtain

$$\mathscr{M}(\mathcal{S}\mu_n, \mathscr{A}\mu_n, \varepsilon) > 1-\lambda, \quad n \geq n_0(\varepsilon,\eta),$$

and so

$$\lim_{n\to\infty} \mathscr{M}(\mathcal{S}\mu_n, \mathscr{A}\mu_n, \varepsilon) > 1-\varepsilon.$$

Since $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathscr{K}$ and the set $\mathscr{K}$ is compact, there exists a convergent subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ and let $\lim_{k\to\infty} \mu_{n_k} = z$. We shall prove that $\mathscr{A}z = \mathcal{S}z$. This follows from the inequality

$$\mathscr{M}(\mathscr{A}z, \mathcal{S}z, \varepsilon) \geq \Theta\left(\mathscr{M}\left(\mathscr{A}z, \mathscr{A}\mu_{n_k}, \frac{\varepsilon}{3}\right), \Theta\left(\mathscr{M}\left(\mathscr{A}\mu_{n_k}, \mathcal{S}\mu_{n_k}, \frac{\varepsilon}{3}\right),\right.\right.$$
$$\left.\left.\mathscr{M}\left(\mathcal{S}\mu_{n_k}, \mathcal{S}z, \frac{\varepsilon}{3}\right)\right)\right)$$

for every $k \in \mathbb{Z}^+$, and $\varepsilon > 0$. Since $\mathscr{A}$ and $\mathcal{S}$ are continuous and $\Theta(1, \Theta(1,1)) = 1$. This achieves the proof. □

4. Common Fixed Points for Non-expansive Type Mappings in Fuzzy Convex Metric Spaces

In this section, we will introduce the concept of star-shaped subsets within the context of fuzzy convex metric spaces. Additionally, we will illustrate specific fixed point theorems applicable to commuting non-expansive mappings on a star-shaped subset of fuzzy convex metric spaces.

Definition 4.1. A continuous mapping $\psi : \mathscr{K} \to \mathscr{K}$ is said to be μ-densifying if for any bounded set $\mathscr{D} \subset \mathscr{K}$ with $\mu(\mathscr{D}) > 0$,

$$\mu(\psi(\mathscr{D})) < \mu(\mathscr{D}),$$

where μ is the Kuratowski measure of non-compactness.

Definition 4.2. Let $(\mathscr{X}, \mathscr{M}, \Theta)$ be a fuzzy metric space and let ψ be a self-mapping on $\mathscr{X}$. We say that ψ is a non-expansive mapping if

$$\mathscr{M}(\psi x, \psi y, \tau) \geq \mathscr{M}(x, y, \tau)$$

for all $x, y \in \mathscr{X}$ and $\tau > 0$.

Definition 4.3. A non-empty set $\mathscr{K}$ of $\mathscr{X}$ is said to be a star-shaped subset of a fuzzy metric space $(\mathscr{X}, \mathscr{M}, \Theta)$ with a convex structure $\mathcal{W}$ if there exists a point $\zeta_0 \in \mathscr{K}$ such that the set $\{\mathcal{W}(x, \zeta_0, \lambda) : x \in \mathscr{K}, \lambda \in [0,1]\} \subset \mathscr{K}$. The point ζ_0 is called a star-center of $\mathscr{K}$.

Definition 4.4. A star-shaped subset $\mathscr{K}$ of a fuzzy metric space $(\mathscr{X}, \mathscr{M}, \Theta)$ with a convex structure $\mathcal{W}$ satisfies the condition (β) if for all $(x, y, \lambda) \in \mathscr{K} \times \mathscr{K} \times (0,1)$,

$$\mathscr{M}(\mathcal{W}(x, \zeta_0, \lambda), \mathcal{W}(y, \zeta_0, \lambda), \lambda\varepsilon) \geq \mathscr{M}(x, y, \varepsilon)$$

for every $\varepsilon \geq 0$.

Definition 4.5. Let $(\mathscr{X}, \mathscr{M}, \Theta)$ be a fuzzy metric with convex structure $\mathcal{W}$. A mapping $\mathcal{S} : \mathscr{X} \to \mathscr{X}$ is said to be $(\mathcal{W}, \zeta_0)$-convex if for each $(x, \lambda) \in \mathscr{X} \times [0,1]$, $\mathcal{W}(\mathcal{S}x, \zeta_0, \lambda) = \mathcal{S}(\mathcal{W}(x, \zeta_0, \lambda))$.

Lemma 4.6. *Let $(\mathscr{X}, \mathscr{M}, \Theta)$ be a fuzzy metric with convex structure $\mathcal{W}$ and let $\mathscr{K}$ be a star-shaped subset of $\mathscr{X}$ with a star-center ζ_0. If $\mathcal{S} : \mathscr{K} \to \mathscr{K}$ is $(\mathcal{W}, \zeta_0)$-convex mapping, then the fixed point set $\mathscr{F}(\mathcal{S})$ of $\mathcal{S}$ is a star-shaped subset with a star-center ζ_0.*

Proof. By the definition of convex structure $\mathcal{W}$, we have $\zeta_0 = \mathcal{W}(\zeta_0, \zeta_0, \lambda)$, $\lambda \in [0,1]$, and $\mathcal{W}(S\zeta_0, \zeta_0, \lambda) = \zeta_0$. Since $\mathcal{S}$ is $(\mathcal{W}, \zeta_0)$-convex mapping,

$$\mathcal{W}(\mathcal{S}\zeta_0, \zeta_0, \lambda) = \mathcal{S}(\mathcal{W}(\zeta_0, \zeta_0, \lambda)) = \mathcal{S}\zeta_0$$

for all $\lambda \in [0,1]$. Putting $\lambda = 0$, we obtain $\zeta_0 = S\zeta_0$, and so $\zeta_0 \in F(S)$. Hence for any $x \in \mathscr{F}(\mathcal{S})$ and $\lambda \in [0,1]$, we have

$$\mathcal{W}(x, \zeta_0, \lambda) = \mathcal{W}(\mathcal{S}x, \zeta_0, \lambda) = \mathcal{S}(\mathcal{W}(x, \zeta_0, \lambda)).$$

Thus the set $\{\mathcal{W}(x,\zeta_0,\lambda) : x \in \mathscr{F}(\mathcal{S}), \lambda \in [0,1]\} \subset \mathscr{F}(\mathcal{S})$, i.e., $\mathscr{F}(\mathcal{S})$ is a star-shaped set with the star-center ζ_0. This achieves the proof. □

Theorem 4.7. *Let $\mathscr{K}$ be a closed star-shaped subset of a complete fuzzy metric space $(\mathscr{X},\mathscr{M},\Theta)$ with a convex structure $\mathcal{W}$ which satisfies the condition (β). Suppose that $\psi : \mathscr{K} \to \mathscr{K}$ is a mapping such that $\psi(\mathscr{K})$ is a fuzzy bounded and for some $m \in \mathbb{Z}^+$, ψ^m is precompact on the set $\{\mathcal{W}(x,\zeta_0,\lambda) : x \in \psi(\mathscr{K}), \lambda \in [0,1]\}$, where ζ_0 is a star-center of $\mathscr{K}$ and for all $x,y \in \mathscr{K}$ and $\varepsilon \geq 0$*

$$\mathscr{M}(\psi x,\psi y,\varepsilon) \geq \mathscr{M}(x,y,\varepsilon). \tag{4.1}$$

Then ψ has a fixed point in $\mathscr{K}$.

Proof. Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence satisfying $\gamma_n \in (0,1)$ and $\gamma_n \to 1$. For each $n \in \mathbb{N}$, define $\psi_n x = \mathcal{W}(\psi x,\zeta_0,\gamma_n)$ for all $x \in \mathscr{K}$. Since $\mathscr{K}$ satisfies the condition (β), from (4.1), we have

$$\begin{aligned}\mathscr{M}(\psi x,\psi y,\gamma_n\varepsilon) &= \mathscr{M}(\mathcal{W}(\psi x,\zeta_0,\gamma_n),\mathcal{W}(\psi y,\zeta_0,\gamma_n),\gamma_n\varepsilon)\\ &\geq \mathscr{M}(\psi,x,\psi,y,\varepsilon)\\ &\geq M(x,y,\varepsilon)\end{aligned}$$

for all $x,y \in \mathscr{K}$ and $n \in \mathbb{N}$. Since $\psi(\mathscr{K})$ is fuzzy bounded it is easy to see that the set $\{\mathcal{W}(\psi,x,\zeta_0,\gamma_n) : x \in \mathscr{K}\}$ is also fuzzy bounded for each $n \in \mathbb{N}$. From [31, Theorem 11.2.3], it follows that for each $n \in \mathbb{N}$, there exists $\kappa_n \in \mathscr{K}$ such that $\kappa_n = \psi_n\kappa_n$. Furthermore, we have

$$\begin{aligned}\mathscr{M}(\kappa_n,\psi\kappa_n,2\varepsilon) &= \mathscr{M}(\psi_n\kappa_n,\psi\kappa_n,2\varepsilon)\\ &= \mathscr{M}(\mathcal{W}(\psi\kappa_n,\zeta_0,\gamma_n),\psi\kappa_n,2\varepsilon)\\ &\geq \Theta\left(\mathscr{M}\left(\psi\kappa_n,\psi\kappa_n,\frac{\varepsilon}{\gamma_n}\right),\mathscr{M}\left(\psi\kappa_n,\zeta_0,\frac{\varepsilon}{1-\gamma_n}\right)\right)\\ &= \Theta\left(1,\mathscr{M}\left(\psi\kappa_n,\zeta_0,\frac{\varepsilon}{1-\gamma_n}\right)\right)\\ &= \mathscr{M}\left(\psi\kappa_n,\zeta_0,\frac{\varepsilon}{1-\gamma_n}\right).\end{aligned}$$

Since $\psi(\mathscr{K})$ is fuzzy bounded, for each $z \in \mathscr{K}$, we have

$$\begin{aligned}\inf_{n\in\mathbb{N}}\mathscr{M}(\psi\kappa_n,\psi z,\varepsilon) &\geq \sup_{r<\varepsilon}\inf_{p,q\,\mathrm{in}(\mathscr{K})}\mathscr{M}(p,q,r)\\ &= \mathscr{D}_{\psi(\mathscr{K})}(\varepsilon)\end{aligned}$$

and so

$$\sup_{\varepsilon\geq 0}\inf_{n\in\mathbb{N}}\mathscr{M}(\psi\kappa_n,\psi\, z,\varepsilon) = \sup_{\varepsilon\geq 0}\mathscr{D}_{\psi(\mathscr{K})}(\varepsilon) = 1 \tag{4.2}$$

Since

$$\mathscr{M}\left(\psi\kappa_n,\zeta_0,\frac{\varepsilon}{1-\gamma_n}\right) \geq \Theta\left(\mathscr{M}\left(\psi\kappa_n,\psi\, z,\frac{\varepsilon}{2(1-\gamma_n)}\right),\mathscr{M}\left(\psi\, z,\zeta_0,\frac{\varepsilon}{2(1-\gamma_n)}\right)\right), \tag{4.3}$$

using the continuity of Θ, (4.2) and (4.3), and $\lim_{n\to\infty}\gamma_n = 1$, we obtain

$$\lim_{n\to\infty} \mathscr{M}\left(\psi\kappa_n, \zeta_0, \frac{\varepsilon}{1-\gamma_n}\right) = 1$$

and so, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} M(\kappa_n, \psi\kappa_n, \varepsilon) = 1. \tag{4.4}$$

Since, for each $n \in \mathbb{N}$,

$$\begin{aligned}\kappa_n &= \psi_n\kappa_n \\ &= \mathcal{W}(\psi\kappa_n, \zeta_0, \gamma_n) \in \{\mathcal{W}(x, \zeta_0, \lambda) : x \in \psi(\mathscr{K}), \lambda \in (0,1)\}\end{aligned}$$

and ψ^m is precompact on

$$\{\mathcal{W}(x, \zeta_0, \lambda) : x \in \psi(\mathscr{K}), \lambda \in (0,1)\},$$

there exists a subsequence $\{\kappa_{n_k}\}_{k\in\mathbb{N}}$ of $\{\kappa_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \psi^m\kappa_{n_k} = \xi^* \in K.$$

By (4.1), for each $\varepsilon > 0$ and $n, k \in \mathbb{N}$, we have

$$\mathscr{M}(\psi^k\kappa_n, \psi^{k+1}\kappa_n, \varepsilon) \geq \mathscr{M}(\kappa_n, \psi\kappa_n, \varepsilon)$$

and so

$$\begin{aligned}\mathscr{M}(\kappa_n, \psi^m\kappa_n, \varepsilon) \geq \Theta\left(\mathscr{M}\left(\kappa_n, \psi\kappa_n, \frac{\varepsilon}{2}\right), \Theta\left(\mathscr{M}\left(\kappa_n, \psi\kappa_n, \frac{\varepsilon}{2^2}\right), \ldots,\right.\right. \\ \left.\left.\mathscr{M}\left(\kappa_n, \psi\kappa_n, \frac{\varepsilon}{2^{m-1}}\right), \ldots\right)\right).\end{aligned}$$

Hence, from $\Theta(1,1) = 1$, the continuity of Θ and (4.4), we obtain

$$\lim_{k\to\infty} \mathscr{M}(\kappa_n, \psi^m\kappa_n, \varepsilon) = 1, \quad \varepsilon > 0. \tag{4.5}$$

The continuity of Θ, (4.5) and the inequality:

$$\mathscr{M}(\kappa_{n_k}, \xi^*, \varepsilon) \geq \Theta\left(\mathscr{M}\left(\kappa_{n_k}, \psi^m\kappa_{n_k}, \frac{\varepsilon}{2}\right), \mathscr{M}\left(\xi^*, \psi^m\kappa_{n_k}, \frac{\varepsilon}{2}\right)\right)$$

imply that $\lim_{n\to\infty} \kappa_{n_k} = \xi^*$. From the inequality

$$\begin{aligned}\mathscr{M}(\xi^*, \psi, \xi^*, \varepsilon) &\geq \Theta\left(\mathscr{M}\left(\xi^*, \kappa_{n_k}, \frac{\varepsilon}{2}\right), \Theta\left(\mathscr{M}\left(\kappa_{n_k}, \psi\kappa_{n_k}, \frac{\varepsilon}{4}\right), \mathscr{M}\left(\kappa_{n_k}, \psi\kappa_{n_k}, \frac{\varepsilon}{4}\right)\right)\right) \\ &\geq \Theta\left(\mathscr{M}\left(\xi^*, \kappa_{n_k}, \frac{\varepsilon}{2}\right), \Theta\left(\mathscr{M}\left(\kappa_{n_k}, \psi\kappa_{n_k}, \frac{\varepsilon}{4}\right), \mathscr{M}\left(\kappa_{n_k}, \psi, \xi^*, \frac{\varepsilon}{4}\right)\right)\right),\end{aligned}$$

it follows that $\mathscr{M}(\xi^*, \psi, \xi^*, \varepsilon) = 1$ for each $\varepsilon > 0$ and so $\xi^* = \psi\xi^*$. This achieves the proof. □

Theorem 4.8. *Let $\mathscr{K}$ be a closed star-shaped subset of a complete fuzzy metric space $(\mathscr{X}, \mathscr{M}, \Theta)$ with a convex structure $\mathcal{W}$ which satisfies the condition (β). Suppose that $\psi, \phi, \mathcal{S}, \mathcal{T} : \mathscr{K} \to \mathscr{K}$ are mappings such that $\mathcal{S}$ and $\mathcal{T}$ are commute with ψ (or ϕ), $\psi(\mathscr{K})$ (or $\phi(\mathscr{K})$) is a fuzzy bounded and the following conditions are satisfied:*

(i) *There exists* $m \in \mathbb{N}$ *such that* ψ^m *(or* ϕ^m*) is precompact on the set* $\{\mathcal{W}(x, \zeta_0, \lambda) : x \in \psi(\mathscr{K}), \lambda \in (0,1)\}$ *(or* $\{\mathcal{W}(x, \zeta_0, \lambda) : x \in \phi(\mathscr{K}), \lambda \in (0,1)\}$ *and for all* $x, y \in \mathscr{K}$*,*

$$\mathscr{M}(\psi x, \phi y, \varepsilon) \geq \mathscr{M}(\mathcal{S}x, \mathcal{T}y, \varepsilon), \tag{4.6}$$

(ii) $\mathcal{S}$ *and* $\mathcal{T}$ *are continuous and* $(\mathcal{W}, \zeta_0)$*-convex.*

Then there exist $\xi^* \in \mathscr{K}$ *such that* $\xi^* = \psi\xi^* = \phi\xi^* = \mathcal{T}\xi^* = \mathcal{S}\xi^*$.

Proof. Let $\mathscr{F}(\mathcal{S}, \mathcal{T})$ be the set of common fixed points of $\mathcal{S}$ and $\mathcal{T}$. Since $\mathcal{S}$ and $\mathcal{T}$ are $(\mathcal{W}, \zeta_0)$-convex, from Lemma 4.6, it follows that $\zeta_0 \in \mathscr{F}(\mathcal{S}, \mathcal{T})$ and $\mathscr{F}(\mathcal{S}, \mathcal{T})$ is a star-shaped subset of $\mathscr{X}$ with a star-center ζ_0. By the continuity of $\mathcal{S}$ and $\mathcal{T}$, $\mathscr{F}(\mathcal{S}, \mathcal{T})$ is also closed. From (4.6), it follows that for all $x, y \in \mathscr{F}(\mathcal{S}, \mathcal{T})$

$$\mathscr{M}(\psi x, \phi y, \varepsilon) \geq \mathscr{M}(x, y, \varepsilon), \quad \varepsilon \geq 0,$$

and hence for all $x \in \mathscr{F}(\mathcal{S}, \mathcal{T})$, $\psi x = \phi x$ and so, for all $x, y \in \mathscr{F}(\mathcal{S}, \mathcal{T})$

$$\mathscr{M}(\psi x, \psi, y, \varepsilon) \geq \mathscr{M}(x, y, \varepsilon), \quad \varepsilon > 0.$$

Since $\mathcal{S}$ and $\mathcal{T}$ commute with ψ, for each $x \in \mathscr{F}(\mathcal{S}, \mathcal{T})$ we have $\psi x = \psi\mathcal{S}x = \mathcal{S}\psi x$ and $\psi x = \psi\mathcal{T}x = \mathcal{T}\psi x$ and so $\psi x \in \mathscr{F}(\mathcal{S}, \mathcal{T})$. Thus, from Theorem 4.7, it follows that there exists $\xi^* \in \mathscr{F}(\mathcal{S}, \mathcal{T})$ such that $\xi^* = \psi\xi^*$ and so $\xi^* = \psi\xi^* = \phi\xi^* = \mathcal{S}\xi^* = \mathcal{T}\xi^*$. This achieves the proof. □

5. Fixed Point Theorems for Multi-values Mappings in Fuzzy Convex Metric Spaces

In this section, we will extend the concept of a fuzzy convex metric space in a more general context. Furthermore, we will provide an intricate example of such a space and derive several fixed point theorems applicable to multi-valued mappings characterized by their non-expansive nature.

Example 5.1. In the following example, we demonstrate that any metric space $(\mathscr{X}, \varrho)$ equipped with a convex structure $\mathcal{W}(x, y, \lambda)$, where $\varrho(u, \mathcal{W}(x, y, \lambda)) \leq \lambda\varrho(u, x) + (1-\lambda)\varrho(u, y)$ for all $(x, y, \lambda) \in \mathscr{X}^2 \times [0,1]$, can be considered a fuzzy metric space with a convex structure.

Let $(\mathscr{X}, \varrho)$ represent a metric space with a convex structure $\mathcal{W}$, defined as follows: $\mathcal{W} : \mathscr{X} \times \mathscr{X} \times [0,1] \to \mathscr{X}$, such that for every $(x, y, \lambda) \in \mathscr{X}^2 \times [0,1]$, the following inequality holds for every $u \in \mathscr{X}$:

$$\varrho(u, \mathcal{W}(x, y, \lambda)) \leq \lambda\varrho(u, x) + (1-\lambda)\varrho(u, y) \tag{5.1}$$

It can be readily observed that $(\mathscr{X}, \mathscr{M}, \min)$ constitutes a fuzzy metric space, with:

$$\mathscr{M}(u, v, t) = \begin{cases} 0, & \text{if } \varrho(u, v) \geq t; \\ \\ 1, & 1 \text{ if } \varrho(u, v) < t. \end{cases}$$

We will now establish the following assertion:

$$\mathscr{M}(u, \mathcal{W}(x, y, \lambda), 2\varepsilon) \geq \min\left\{\mathscr{M}(u, x, \frac{\varepsilon}{\lambda}), \mathscr{M}(u, y, \frac{\varepsilon}{1-\lambda})\right\} \tag{5.2}$$

This assertion holds for all $(u,x,y) \in \mathscr{X}\times\mathscr{X}\times\mathscr{X}$ and for all $(\varepsilon,\lambda) \in \mathbb{R}^+\times(0,1)$. Let's assume that $\mathscr{M}\left(u,x,\frac{\varepsilon}{\lambda}\right)=1$ and $\mathscr{M}\left(u,y,\frac{\varepsilon}{1-\lambda}\right)=1$. Consequently, we have $\varrho(u,x)<\varepsilon/\lambda$ and $\varrho(u,y)<\varepsilon/(1-\lambda)$. Using (5.1), we can deduce that:

$$\varrho(u,\mathcal{W}(x,y,\lambda))<\lambda.\frac{\varepsilon}{\lambda}+(1-\lambda).\frac{\varepsilon}{1-\lambda}=2\varepsilon.$$

Thus, we arrive at:

$$\varrho(u,\mathcal{W}(x,y,\lambda))<\lambda.\frac{\varepsilon}{\lambda}+(1-\lambda).\frac{\varepsilon}{1-\lambda}=2\varepsilon.$$

If $\min\left\{\mathscr{M}(u,x,\frac{\varepsilon}{\lambda}),\mathscr{M}(u,y,\frac{\varepsilon}{1-\lambda})\right\}=0$, then from $\mathscr{M}(u,\mathcal{W}(x,y,\lambda)),2\varepsilon)\geq 0$, we can conclude that (5.2) is satisfied. Furthermore, from (5.1), it follows that $\mathcal{W}(x,y,0)=y$ for $\lambda=0$ and $u=y$, and $\mathcal{W}(x,y,1)=x$ for $\lambda=1$ and $u=x$.

Example 5.2. Suppose that $(\mathscr{K},\varrho)$ is a separable metric space with a convex structure $\mathcal{W}$ so that, for every $\lambda\in[0,1]$, the mapping $(x,y)\mapsto\mathcal{W}(x,y,\lambda)$ is continuous and $(\Omega,\mathscr{A},\mathfrak{P})$ is a probability space.

Let $\mathscr{X}$ be the space of all measurable mappings from Ω into $\mathscr{K}$ (i.e., the space of all equivalence classes). Then the triple $(\mathscr{X},\mathscr{M},\min)$ is a fuzzy metric space if for every $x,y\in\mathscr{X}$, $\varepsilon>0$, $u,v\in[0,1]$,

$$\mathscr{M}(x,y,\varepsilon)=\mathfrak{P}(\{\omega\in\Omega:\varrho(x(\omega),y(\omega))<\varepsilon\}).$$

Let $\overline{\mathcal{W}}:\mathscr{X}\times\mathscr{X}\times[0,1]\to\mathscr{X}$ be defined by

$$\overline{\mathcal{W}}(x,y,\lambda)(\omega)=\mathcal{W}(x(\omega),y(\omega),\lambda)$$

for every $\omega\in\Omega$ and $x,y\in\mathscr{X}$, $\lambda\in[0,1]$.

Since x and y are measurable mappings and $\mathcal{W}$ is, for every fixed $\lambda\in[0,1]$, a continuous mapping, it follows for every $x,y\in\mathscr{X}$, $\overline{\mathcal{W}}(x,y,\lambda)\in\mathscr{X}$.
Now, we prove that for every $u,x,y\in\mathscr{X}$ and $\lambda\in(0,1)$, $\varepsilon>0$,

$$\mathscr{M}(u,\overline{\mathcal{W}}(x,y,\lambda)),2\varepsilon)\geq\Theta_m\left\{\mathscr{M}(u,x,\frac{\varepsilon}{\lambda}),\mathscr{M}(u,y,\frac{\varepsilon}{1-\lambda})\right\},$$

where $\Theta_{\mathscr{M}}(a,b)=\max\{a+b-1,0\}$.

From the definition of $\mathscr{M}$, it follows that

$$\mathscr{M}(u,\overline{\mathcal{W}}(x,y,\lambda)),2\varepsilon)=\mathfrak{P}(\{\omega\in\Omega:\varrho(u(\omega),\mathcal{W}(x(\omega),y(\omega),\lambda))<2\varepsilon\}),$$

Further, from (5.1) it follows that, for every $\omega\in\Omega$,

$$\varrho(u(\omega),\mathcal{W}(x(\omega),y(\omega),\lambda))\leq\lambda\varrho(u(\omega),x(\omega))+(1-\lambda)\varrho(u(\omega),y(\omega)),$$

which implies that

$$\begin{aligned}&\{\omega\in\Omega:\varrho(u(\omega),\mathcal{W}(x(\omega),y(\omega),\lambda))<2\varepsilon\}\\&\quad\supset\left\{\omega\in\Omega:\varrho(u(\omega),x(\omega))<\frac{\varepsilon}{\lambda}\right\}\cap\left\{\omega\in\Omega:\varrho(u(\omega),y(\omega))<\frac{\varepsilon}{1-\lambda}\right\}\end{aligned}$$

and so we obtain

$$\mathfrak{P}\left(\{\omega\in\Omega:\varrho(u(\omega),\mathcal{W}(x(\omega),y(\omega),\lambda))<2\varepsilon\}\right)$$

$$\geq \mathfrak{P}\left(\left\{\omega\in\Omega : \varrho(u(\omega),x(\omega)) < \frac{\varepsilon}{\lambda}\right\}\cap\left\{\omega\in\Omega : \varrho(u(\omega),y(\omega)) < \frac{\varepsilon}{1-\lambda}\right\}\right).$$

Since for every $\mathscr{A},\mathscr{B}\in\mathscr{X}$, $\mathfrak{P}(\mathscr{A}\cap\mathscr{B}) = \mathfrak{P}(\mathscr{A})+\mathfrak{P}(\mathscr{B})-\mathfrak{P}(\mathscr{A}\cup\mathscr{B})$, we obtain

$$\begin{aligned}
&\mathfrak{P}\left(\left\{\omega\in\Omega : \varrho(u(\omega),x(\omega)) < \frac{\varepsilon}{\lambda}\right\}\cap\left\{\omega\in\Omega : \varrho(u(\omega),y(\omega)) < \frac{\varepsilon}{1-\lambda}\right\}\right)\\
&\quad= \mathfrak{P}\left(\left\{\omega\in\Omega : \varrho(u(\omega),x(\omega)) < \frac{\varepsilon}{\lambda}\right\}\right) + \mathfrak{P}\left(\left\{\omega\in\Omega : \varrho(u(\omega),y(\omega)) < \frac{\varepsilon}{1-\lambda}\right\}\right)\\
&\qquad- \mathfrak{P}\left(\mathfrak{P}\left(\left\{\omega\in\Omega : \varrho(u(\omega),x(\omega)) < \frac{\varepsilon}{\lambda}\right\}\cap\left\{\omega\in\Omega : \varrho(u(\omega),y(\omega)) < \frac{\varepsilon}{1-\lambda}\right\}\right)\right)\\
&\quad\geq \mathscr{M}\left(u,x,\frac{\varepsilon}{\lambda}\right) + \mathscr{M}\left(u,x,\frac{\varepsilon}{\lambda}\right) - 1,
\end{aligned}$$

which completes the proof.

In the sequel, we shall suppose that $(\mathscr{X},\mathscr{M},\Theta)$ is a fuzzy metric space with a convex structure $\mathcal{W}:\mathscr{X}\times\mathscr{X}\times[0,1]\to\mathscr{X}$ so that the following condition is satisfied:

$$\mathscr{M}(\mathcal{W}(x,z,\lambda),\mathcal{W}(y,z,\lambda),2\varepsilon) \geq \mathscr{M}(x,y,\varepsilon) \tag{5.3}$$

for every $(\varepsilon,\lambda)\in\mathbb{R}^+\times(0,1)$ and $(x,y,z)\in X\times X\times X$.

If $\mathscr{K}$ is a non-empty subset of $\mathscr{X}$, we shall denote the family of all non-empty subsets of $\mathscr{K}$ by $2^{\mathscr{K}}$ and the family of all non-empty closed subsets of $\mathscr{K}$ by $2_c^{\mathscr{K}}$.

Definition 5.3. Let $(\mathscr{X},\mathscr{M},\Theta)$ be a complete FM-space with a convex structure $\mathcal{W}:\mathscr{X}\times\mathscr{X}\times[0,1]\to\mathscr{X}$ and a subset $\mathscr{K}$ of $\mathscr{X}$. The set $\mathscr{K}$ is said to be star-convex if there exists $\zeta_0\in K$ such that

$$\mathcal{W}(x,\zeta_0,\lambda)\in\mathscr{K},\quad (x,\lambda)\in\mathscr{X}\times[0,1].$$

Then ζ_0 is a star-point of the set $\mathscr{K}$.

Definition 5.4. Let $\mathscr{K}\subset\mathscr{X}$ and let $\mathscr{T}:\mathscr{K}\to 2^{\mathscr{X}}$ be multi-valued mapping. The mapping $\mathscr{T}$ is said to be demicompact if for every two sequences $\{\kappa_n\}_{n\in\mathbb{N}}$ and $\{\mu_n\}_{n\in\mathbb{N}}$ such that $\kappa_n\in\mathscr{K}$, $\mu_n\in\mathscr{T}\kappa_n$, $n\in\mathbb{N}$, and $\lim_{n\to\infty}\mathscr{M}(\kappa_n,\mu_n,\varepsilon)=1$ for every $\varepsilon>0$, there exists a convergent subsequence $\{\kappa_{n_k}\}_{n\in\mathbb{N}}$ of $\{\kappa_n\}$.

For multi-valued mappings in fuzzy metric spaces with convex structure, we now present several fixed point theorems.

Theorem 5.5. *Let $(\mathscr{X},\mathscr{M},\Theta)$ be a complete FM-space with continuous τ-norm Θ, $\mathscr{K}$ be a nonempty closed subset of $\mathscr{X}$, $\mathscr{T}:\mathscr{K}\to 2_c^{\mathscr{K}}$ be a multi-valued mapping so that the following conditions are satisfied:*

(i) *For every $u,v\in\mathscr{K}$, every $x\in\mathscr{T}u$ and every $\Theta>0$ there exists $y\in\mathscr{T}v$ such that*

$$\mathscr{M}(x,y,\varepsilon) \geq \mathscr{M}\left(u,v,\frac{\varepsilon-\Theta}{q}\right) \tag{5.4}$$

for every $\varepsilon>0$, where $q\in(0,1)$.

(ii) *$\mathscr{T}$ is demicompact or the family $\{\Phi_n(\Theta,t)\}_{n\in\mathbb{N}}$ is equicontinuous at the point $\tau=1$, where*

$\Phi_1(\Theta,\tau)=\Theta(\tau,\tau)$ *and*

$\Phi_n(\Theta,\tau)=\underbrace{\Theta(\Theta(\cdots\Theta}_{n-times}(\Theta(\tau,\tau),\ldots,\tau))),\quad n=2,3,\ldots,\tau\in[0,1].$

Then there exists at least one element $x\in\mathscr{K}$ such that $x\in\mathscr{T}x$.

Proof. It is easy to see that we can construct a sequence $\{\kappa_n\}_{n\in\mathbb{N}}\subseteq\mathscr{K}$ (ζ_0 is an arbitrary element from $\mathscr{K}$) such that

(i) $x_{n+1}\in\mathscr{T}\kappa_n$ for every $n=0,1,2,\ldots$.

(ii) $\mathscr{M}(x_{n+1},\kappa_n,\varepsilon)\geq\mathscr{M}\left(\kappa_n,x_{n-1},\dfrac{\varepsilon-q^n}{q}\right)$ for every $n\in\mathbb{N}$ and $\varepsilon>0$.

From (ii), it follows that for every $n\in\mathbb{N}$

$$\mathscr{M}(x_{n+1},\kappa_n,\varepsilon)\geq\mathscr{M}\left(x_1,\zeta_0,\frac{\varepsilon-nq^n}{q^n}\right)$$

and so,

$$\lim_{n\to\infty}\mathscr{M}(x_{n+1},\kappa_n,\tau)=1,\quad\text{for all }\tau>0. \tag{5.5}$$

We shall show that there exists a subsequence $\{\kappa_{n_k}\}_{k\in\mathbb{N}}$ which is convergent. If the mapping T is demicompact from (i) and (5.5) it follows that there exists a convergent subsequence $\{\kappa_{n_k}\}_{k\in\mathbb{N}}$.

Suppose now that the family $\{\Phi_n(\Theta,t)\}_{n\in\mathbb{N}}$ is equicontinuous at the point $t=1$. WE shall show that $\{\kappa_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., for every $\varepsilon>0$ and every $\lambda\in(0,1)$ there exists $n(\varepsilon,\lambda)\in\mathbb{N}$ such that

$$\mathscr{M}(x_{n+p},\kappa_n,\varepsilon)>1-\lambda,\quad\text{for every } n\geq n(\varepsilon,\lambda)\text{ and every } p\in\mathbb{N}. \tag{5.6}$$

Using the inequality $\varepsilon>\varepsilon\left(1-\dfrac{1}{p+2}\right)=\varepsilon\left(\dfrac{1}{1.2}+\dfrac{1}{2.3}+\cdots+\dfrac{1}{(p+1)(p+2)}\right)$, we obtain the following inequality:

(5.7)

$$\begin{aligned}
\mathscr{M}(x_{n+p+1},\kappa_n,\varepsilon)&\geq\mathscr{M}\left(x_{n+p+1},\kappa_n,\varepsilon\left(1-\frac{1}{p+2}\right)\right)\\
&\geq\Theta\left(\mathscr{M}\left(x_{n+p+1},\kappa_n,\sum_{r=2}^{p+1}\frac{\varepsilon}{r(r+1)}\right),\mathscr{M}(x_{n+1},\kappa_n,\left(\frac{\varepsilon}{1.2}\right)\right)\\
&\ \ \vdots\\
&\geq\underbrace{\Theta(\Theta(\cdots\Theta}_{p-\text{times}}\left(\mathscr{M}\left(x_{n+p+1},x_{n+p},\left(\frac{\varepsilon}{(p+1)(p+2)}\right)\right),\right.\\
&\quad\mathscr{M}\left(x_{n+p},x_{n+p-1},\left(\frac{\varepsilon}{(p)(p+1)}\right)\right),\ldots,\mathscr{M}\left(x_{n+2},x_{n+1},\left(\frac{\varepsilon}{2.3}\right)\right),\\
&\quad\left.\mathscr{M}\left(x_{n+1},x_n,\left(\frac{\varepsilon}{1.2}\right)\right)\right).
\end{aligned}$$

Since $\mathscr{M}(x_{n+s+1}, x_{n+s}, \varepsilon) \geq \mathscr{M}\left(x_{n+1}, x_n, \left(\frac{\varepsilon - sq^{n+s}}{q^s}\right)\right)$, it follows that

$$\mathscr{M}(x_{n+p+1}, \kappa_n, \varepsilon) \geq \underbrace{\Theta(\Theta(\cdots\Theta}_{p-\text{times}}\left(\mathscr{M}\left(x_{n+1}, x_n, \left(\frac{\frac{\varepsilon}{(p+1)(p+2)} - pq^{n+p}}{q^p}\right)\right)\right.,$$
$$\left.\ldots, \mathscr{M}\left(x_{n+1}, x_n, \left(\frac{\frac{\varepsilon}{2.3} - q^{n+1}}{q}\right)\right), \mathscr{M}\left(x_{n+1}, x_n, \left(\frac{\varepsilon}{1.2}\right)\right)\right).$$

Now we shall show that there exists $\mathscr{M}(\varepsilon) > 0$ such that for every $n \geq n'(\varepsilon)$ for every $s = 0, 1, 2, \ldots$. $\mathscr{A}(n, s, \varepsilon) \geq \mathscr{M}(\varepsilon)$, where

$$\mathscr{A}(n, s, \varepsilon) = \frac{\frac{\varepsilon}{(s+1)(s+2) - sq^{n+s}}}{q^s}, \quad n, s \in \mathbb{N}, \varepsilon > 0.$$

Since

$$\frac{\frac{\varepsilon}{(s+1)(s+2) - sq^{n+s}}}{q^s} \geq \frac{\frac{\varepsilon}{(s+1)(s+2) - sq^s}}{q^s}, \quad \text{for all } n \in \mathbb{N}$$

and if $s \to \infty$, then $\frac{\frac{\varepsilon}{(s+1)(s+2) - sq^s}}{q^s}$ tends to infinity. Then there exists $S(\varepsilon) \in \mathbb{N}$ such that for every $s \geq S(\varepsilon)$ we have

$$\frac{\frac{\varepsilon}{(s+1)(s+2) - sq^s}}{q^s} \geq \frac{\varepsilon}{2}, \quad i.e., \mathscr{A}(n, s, \varepsilon) \geq \frac{\varepsilon}{2}$$

for every $s \geq \mathcal{S}(\varepsilon)$ and every $n \in \mathbb{N}$. Since $q \in (0, 1)$ for every $\varepsilon > 0$, there exists $N'(\varepsilon)$ such that for every $s = 0, 1, \ldots, \mathcal{S}(\varepsilon) - 1$, we have

$$sq^{N'(\varepsilon)+s} \leq \frac{\varepsilon}{2(s+1)(s+2)}$$

and so for every $n \geq N'(\varepsilon)$ and every $s = 0, 1, \ldots, S(\varepsilon) - 1$, we have

$$\mathscr{A}(n, s, \varepsilon) \geq \mathscr{A}(N'(\varepsilon), s, \varepsilon) \geq \frac{\varepsilon}{2(s+1)(s+2)q^s}.$$

If $\mathscr{M}(\varepsilon) = \min\left\{\frac{\varepsilon}{2(s+1)(s+2)q^s} : s = 0, 1, \ldots, \mathcal{S}(\varepsilon) - 1\right\}$, then $\mathscr{A}(n, s, \varepsilon) \geq \mathscr{M}(\varepsilon)$ for every $n \geq N'(\varepsilon)$ and every $s = 0, 1, 2, \ldots$. So, we have for every $n \geq N'(\varepsilon)$ that

$$\mathscr{M}(x_{n+p+1}, \kappa_n, \varepsilon) \geq \Phi_{p-1}\left(\Theta, \mathscr{M}\left(x_{n+1}, \kappa_n, \mathscr{M}(\varepsilon)\right)\right)$$
$$\geq \Phi_{p-1}\left(\Theta, \mathscr{M}\left(x_1, \zeta_0, \left(\frac{\mathscr{M}(\varepsilon) - nq^n}{q^n}\right)\right)\right)$$

and since the family $\{\Phi_n(\Theta, t)\}_{n\in\mathbb{N}}$ is equicontinuous at the point $t = 1$, there exists $N''(\varepsilon, \lambda)$ such that for every $n \geq N''(\varepsilon, \lambda)$ and every $p \in \mathbb{N}$ we have

$$\Phi_{p-1}\left(\Theta, \mathscr{M}\left(x_1, \zeta_0, \left(\frac{\mathscr{M}(\varepsilon) - nq^n}{q^n}\right)\right)\right) \geq 1 - \lambda.$$

So the sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and since X is complete, then $\{\kappa_n\}$ is convergent,say, there exists $\xi^* \in \mathscr{X}$ such that $\lim_{n\to\infty} \kappa_n = \xi^*$. Now, we shall show that if $\lim_{n\to\infty} \kappa_{n_k} = \xi^*$, then $\xi^* \in T\xi^*$, i.e., ξ^* is fixed point of $\mathscr{T}$.

Since $\mathscr{T}\xi^* = \overline{\mathscr{T}\xi^*}$, we shall show that $\xi^* \in \overline{\mathscr{T}\xi^*}$. If $U_{\xi^*}(\varepsilon,\lambda) = \{y \in \mathscr{X} : \mathscr{M}(\xi^*, y, \varepsilon) > 1-\lambda\}$, then we will show that there exists $b \in U_{\xi^*}(\varepsilon,\lambda) \cap T\xi^*$. Suppose that $\eta(\lambda) > 0$ is such that $x > 1-\eta(\lambda)$ implies that $\Theta(x,x) > 1-\lambda$. Such element exists since $\lim_{x\to 1}\Theta(x,x) = \Theta(1,1) = 1$. If we apply the relation (5.4) for $v = \xi^*$ and $u = \kappa_{n_k}$, $x = x_{n_k+1} \in T\kappa_{n_k}$ and $\Theta = \dfrac{\varepsilon}{4}$, it follows that there exists $b_{(k,\varepsilon)} \in \mathscr{T}\xi^*$ such that

$$\mathscr{M}\left(x_{n_k+1}, b_{(k,\varepsilon)}, \frac{\varepsilon}{2}\right) \geq \mathscr{M}\left(\kappa_{n_k}, \xi^*, \frac{\varepsilon}{4q}\right).$$

Since $\lim_{k\to\infty} x_{n_k} = \xi^*$, it follows that there exists $N_1(\varepsilon,\eta(\lambda))$ such that

$$\mathscr{M}\left(x_{n_k+1}, b_{(k,\varepsilon)}, \frac{\varepsilon}{2}\right) \geq 1 - \frac{\eta(\lambda)}{2}$$

for every $k \geq N_1(\varepsilon,\eta(\lambda))$. Since $\lim_{k\to\infty}\mathscr{M}(x_{n_k+1}, \kappa_{n_k}, t) = 1$ for all $t > 0$, we have $\lim_{k\to\infty}\mathscr{M}(x_{n_k+1}, \xi^*, t) = 1$ for all $t > 0$ and so there exists $N_2(\varepsilon,\eta(\lambda))$ such that

$$\mathscr{M}\left(\xi^*, x_{n_k+1}, \frac{\varepsilon}{2}\right) \geq 1 - \frac{\eta(\lambda)}{2}$$

for every $k \geq N_2(\varepsilon,\eta(\lambda))$. Now, we have

$$\begin{aligned}\mathscr{M}(\xi^*, b_{(k,\varepsilon)}, \varepsilon)\Theta\left(\mathscr{M}\left(\xi^*, x_{n_k+1}, \frac{\varepsilon}{2}\right), \mathscr{M}\left(x_{n_k+1}, b_{(k,\varepsilon)}, \frac{\varepsilon}{2}\right)\right)&\\ \geq \Theta\left(1-\frac{\eta(\lambda)}{2}, 1-\frac{\eta(\lambda)}{2}\right)&\\ \geq 1-\lambda&\end{aligned}$$

for every $k \geq N(\varepsilon,\lambda)$, where $N(\varepsilon,\lambda) = \max\{N_1(\varepsilon,\eta(\lambda)), N_2(\varepsilon,\eta(\lambda))\}$. So $k \geq N(\varepsilon,\lambda)$ implies that $b_{(k,\varepsilon)} \in T\xi^* \cap V_{\xi^*}(\varepsilon,\lambda)$ and this completes the proof. □

Using Theorem 5.5, we shall prove a fixed point theorem for multi-valued mappings which are of non-expansive type.

Theorem 5.6. *Let $(\mathscr{X}, \mathscr{M}, \Theta)$ be a complete FM-space with a convex structure $\mathcal{W}$ and a continuous τ-norm Θ, $\mathscr{K}$ be a nonempty, closed star-convex subset of $\mathscr{X}$, $\mathscr{T} : \mathscr{K} \to 2^{\mathscr{K}}_c$ be a multi-valued mapping such that the set $\mathscr{T}(\mathscr{K})$ is fuzzy precompact and the following condition is satisfied:*

(i) *for every $u, v \in \mathscr{K}$, $x \in \mathscr{T}u$ and $\theta > 0$, there exists $y \in \mathscr{T}v$ such that*

$$\mathscr{M}(x, y, \varepsilon) \geq \mathscr{M}(u, v, \varepsilon-\theta), \quad \varepsilon > 0. \tag{5.8}$$

Then there exists at least one element $x \in \mathscr{K}$ such that $x \in \mathscr{T}x$.

Proof. Let ζ_0 be the star-point of the set $\mathscr{K}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers from $(0,1)$ such that $\lim_{n\to\infty}\gamma_n = 1$. For every $n \in \mathbb{N}$ and $x \in \mathscr{K}$, let $\mathscr{T}_n x \in \bigcup_{z\in\mathscr{T}x}\mathcal{W}(z,\zeta_0,\gamma_n)$.

We shall that for every $n \in \mathbb{N}$ there exists $x_n \in \mathscr{K}$ such that $\kappa_n \in \mathscr{T}_n\kappa_n$. Since ζ_0 is a star-point of the set $\mathscr{K}$, it follows that $\bigcup_{z\in\mathscr{T}x}\mathcal{W}(z,\zeta_0,\gamma_n) \subset \mathscr{K}$

and so $\mathscr{T}_n x \subset \mathscr{K}$ for every $n \in \mathbb{N}$ and every $x \in \mathscr{K}$. From (5.3), it follows that the mapping $\mathcal{W}$ is continuous with respect to the first variable. Since $\mathscr{T}x$ is closed, it follows that $\mathscr{T}x$ is compact (as a subset of $\mathscr{T}(\mathscr{K})$) and so the set $\mathcal{W}(\mathscr{T}x, \zeta_0, \gamma_n)$ is closed for every $n \in \mathbb{N}$. This implies that the set $\mathscr{T}_n x$ is closed for every $n \in \mathbb{N}$ and every $x \in \mathscr{K}$.

Next, we shall prove that for every $u, v \in \mathscr{K}$, every $x \in \mathscr{T}_n u$ and every $\Theta > 0$ there exists $y \in \mathscr{T}_n v$ such that

$$\mathscr{M}(x, y, \varepsilon) \geq \mathscr{M}\left(u, v, \frac{\varepsilon - \Theta}{k}\right), \quad \varepsilon > 0.$$

Let $u, v \in K$, $\Theta > 0$ and $x \in \mathscr{T}_n u$. Then there exists $z \in \mathscr{T}u$ such that $x = \mathcal{W}(z, \zeta_0, \gamma_n)$. From (5.8), it follows that there exists $y' \in \mathscr{T}v$ such that

$$\mathscr{M}(x, y', \varepsilon') \geq \mathscr{M}(u, v, \varepsilon' - \Theta/\gamma_n), \quad \varepsilon' > 0.$$

Let $y = \mathcal{W}(y', \zeta_0, \gamma_n) \in \mathscr{T}_n v$. Then we have

$$\begin{aligned}
\mathscr{M}(x, y, \varepsilon) &= \mathscr{M}\left(\mathcal{W}(z, \zeta_0, \gamma_n), \mathcal{W}(y', \zeta_0, \gamma_n), \gamma_n \frac{\varepsilon}{\gamma_n}\right) \\
&\geq \mathscr{M}\left(z, y', \frac{\varepsilon}{\gamma_n}\right) \\
&\geq \mathscr{M}\left(u, v, \frac{\varepsilon - \Theta}{\gamma_n}\right).
\end{aligned}$$

Since the set $\mathscr{T}(\mathscr{K})$ is fuzzy precompact, this means that for every $\varepsilon > 0$ and every $\lambda \in (0, 1)$, there exists a finite cover of $\mathscr{T}(\mathscr{K})$, $\{\mathscr{A}_i\}_{i \in I}$ (I is finite) such that $\mathscr{D}_{\mathscr{A}_i}(\varepsilon) > 1 - \lambda$, $i \in I$, where $\mathscr{D}_{\mathscr{A}}$ is the fuzzy diameter defined by

$$\mathscr{D}_{\mathscr{A}}(x) = \sup_{t < x} \inf_{p, q \in \mathscr{A}} \mathscr{M}(p, q, t).$$

From this, it is obvious that $\mathscr{T}(\mathscr{K})$ is a fuzzy bounded subset of $\mathscr{X}$, $\sup_{t>0} \mathscr{D}_{\mathscr{T}(\mathscr{K})}(t) = 1$, and in [2], it is proved that $\mathscr{T}(\mathscr{K})$ is precompact in respect to the metric ρ which metricizes the uniformly of $\mathscr{X}$ generated by the (ε, λ)-topology. Hence, the set $\overline{\mathscr{T}(\mathscr{K})}$ is compact. From the continuity of the mapping $\mathcal{W}$ in respect to the first variable, it follows that the set $\mathscr{T}_n(\mathscr{K}) = \mathcal{W}(\mathscr{T}(\mathscr{K}), \zeta_0, \gamma_n)$, $n \in \mathbb{N}$, is relatively compact. Let us prove that $\mathscr{T}_m$ is a demicompact mapping.

Suppose that $\{\kappa_n\}_{n \in \mathbb{N}}$ and $\{\mu_n\}_{n \in \mathbb{N}}$ are two sequences in $\mathscr{K}$ such that $\mu_n \in \mathscr{T}_m \kappa_n$ and $\lim_{n \to \infty} \mathscr{M}(\kappa_n, \mu_n, \varepsilon) = 1$ for every $\varepsilon > 0$. Then there exists a convergent subsequence $\{\mu_{n_k}\}$ of the sequence $\{\mu_n\}_{n \in \mathbb{N}}$. Let $\lim_{n \to \infty} \mu_{n_k} = z$. Then, from the inequality:

$$\mathscr{M}(\kappa_{n_k}, z, \varepsilon) \geq \Theta\left(\mathscr{M}\left(\kappa_{n_k}, \mu_{n_k}, \frac{\varepsilon}{2}\right), \mathscr{M}\left(\mu_{n_k}, z, \frac{\varepsilon}{2}\right)\right),$$

it follows that $\lim_{n \to \infty} \kappa_{n_k} = z$, which means that the mapping $\mathscr{T}_m$ is demicompact. Hence, all conditions of Theorem 5.5 are satisfied and so, for every $n \in \mathbb{N}$, there exists $\kappa_n \in \mathscr{K}$ such that $\kappa_n \in \mathscr{T}_n \kappa_n$. Since $\mathscr{T}_n \kappa_n = \bigcup_{z \in \mathscr{T}x} \mathcal{W}(z, \zeta_0, \gamma_n)$, it

follows that there exists $z_n \in T\kappa_n$ such that $\kappa_n = \mathcal{W}(z_n, \zeta_0, \gamma_n)$. Then we have

$$\begin{aligned}\mathscr{M}(\kappa_n, z_n, \varepsilon) &= \mathscr{M}(z_n, \mathcal{W}(z_n, \zeta_0, \gamma_n), \varepsilon) \\ &\geq \Theta\left(\mathscr{M}\left(z_n, z_n, \frac{\varepsilon}{2\gamma_n}\right), \mathscr{M}\left(z_n, \zeta_0, \frac{\varepsilon}{2(1-\gamma_n)}\right)\right) \\ &= \Theta\left(1, \mathscr{M}\left(z_n, \zeta_0, \frac{\varepsilon}{2(1-\gamma_n)}\right)\right) \\ &= \mathscr{M}\left(z_n, \zeta_0, \frac{\varepsilon}{2(1-\gamma_n)}\right), \quad n \in \mathbb{N}.\end{aligned}$$

Since $\mathscr{T}(\mathscr{K})$ is fuzzy bounded, for every $z \in \mathscr{T}(\mathscr{K})$ and $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathscr{M}\left(z_n, \zeta_0, \frac{\varepsilon}{4(1-\gamma_n)}\right) = 1.$$

Using the inequality

$$\mathscr{M}\left(z_n, \zeta_0, \frac{\varepsilon}{2(1-\gamma_n)}\right) \geq \Theta\left(\mathscr{M}\left(z_n, z, \frac{\varepsilon}{4(1-\gamma_n)}\right), \mathscr{M}\left(z, \zeta_0, \frac{\varepsilon}{4(1-\gamma_n)}\right)\right)$$

and $\lim_{n\to\infty} \mathscr{M}\left(z, \zeta_0, \frac{\varepsilon}{4(1-\gamma_n)}\right) = 1$, it follows that

$$\lim_{n\to\infty} \mathscr{M}\left(z_n, \zeta_0, \frac{\varepsilon}{2(1-\gamma_n)}\right) = 1, \quad \text{for } \varepsilon > 0.$$

Hence we have

$$\lim_{n\to\infty} \mathscr{M}(\kappa_n, z_n, \varepsilon) = 1, \quad \varepsilon > 0. \tag{5.9}$$

Since $z_n \in \mathscr{T}\kappa_n$ and the set $\overline{\mathscr{T}(\mathscr{K})}$ is compact, there exists a convergent subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ of $\{z_n\}$ and let $z = \lim_{n\to\infty} z_{n_k}$. Then (5.9) and the inequality

$$\mathscr{M}(\kappa_{n_k}, z, \varepsilon) \geq \Theta\left(\mathscr{M}\left(\kappa_{n_k}, z_{n_k}, \frac{\varepsilon}{2}\right), \mathscr{M}\left(z_{n_k}, z, \frac{\varepsilon}{2}\right)\right)$$

imply that $\lim_{n\to\infty} \kappa_{n_k} = z$. To prove that $z \in \mathscr{T}z$, we shall prove that $z \in \overline{\mathscr{T}z}$ which implies, since $\mathscr{T}z$ is closed, that $z \in \mathscr{T}z$.

Let $\varepsilon > 0$ and $\lambda \in (0,1)$. We shall prove that there exists $b \in \mathscr{T}z$ such that $b \in U_z(\varepsilon, \lambda)$. Let us take, in (5.8), $\Theta = \frac{\varepsilon}{4}, u = \kappa_{n_k}$ and $v = z$. Then there exists $b_k \in \mathscr{T}z$ such that

$$\mathscr{M}\left(z_{n_k}, b_k, \frac{\varepsilon}{2}\right) \geq \mathscr{M}\left(\kappa_{n_k}, z, \frac{\varepsilon}{4}\right).$$

Suppose that $\eta(\lambda)$ be such an element in the interval $(0,1)$ that the following implication holds:

$$x > \eta(\lambda) \quad \Longrightarrow \quad \Theta(x,x) > 1 - \lambda.$$

If $n_0(\varepsilon, \lambda) \in \mathbb{N}$ is such that

$$\mathscr{M}\left(z, \kappa_{n_k}, \frac{\varepsilon}{4}\right) > 1 - \frac{\eta(\lambda)}{2}, \qquad \mathscr{M}\left(z, \kappa_{n_k}, \frac{\varepsilon}{2}\right) > 1 - \frac{\eta(\lambda)}{2}$$

for every $k > n_0(\varepsilon, \lambda)$, it follows that

$$\begin{aligned}\mathscr{M}(z, b_k, \varepsilon) &\geq \Theta\left(\mathscr{M}\left(z, z_{n_k}, \frac{\varepsilon}{2}\right), \mathscr{M}\left(z_{n_k}, b_k, \frac{\varepsilon}{2}\right)\right) \\ &\geq \Theta\left(1 - \frac{\eta(\lambda)}{2}, 1 - \frac{\eta(\lambda)}{2}\right) \\ &> 1 - \lambda\end{aligned}$$

and so we have $b_k \in U_z(\varepsilon, \lambda) \cap \mathscr{T}z$. Since $\mathscr{T}z$ is closed, we conclude that $z \in \mathscr{T}z$. This achieves the proof. □

Using Theorem 5.6, we can prove the following:

Theorem 5.7. *Let $(\mathscr{X}, \mathscr{M}, \Theta)$ be a complete FM-space with a convex structure W and a continuous τ-norm Θ such that the family $\{\Phi_n(\Theta, \tau)\}_{n\in\mathbb{N}}$ is equicontinuous at the point $\tau = 1$, $\mathscr{K}$ be a nonempty, closed star-convex subset of $\mathscr{X}$, $\mathscr{T}$ be a demicompact mapping from $\mathscr{K}$ into the family of all non-empty and compact subsets of $\mathscr{K}$ such that $\overline{\mathscr{T}(\mathscr{K})}$ is bounded and the condition (5.8) is satisfied. Then there exists $x \in \mathscr{K}$ such that $x \in \mathscr{T}x$.*

Proof. As in the proof of Theorem 5.6, let for every $n \in \mathbb{N}$ and every $x \in \mathscr{K}$, $\mathscr{T}_n x = \bigcup_{z\in\mathscr{T}\mathscr{X}} \mathcal{W}(z, \zeta_0, \gamma_n)$. Since the set $\mathscr{T}x$ is compact for every $x \in \mathscr{K}$, it follows that $\mathscr{T}_n x$ is closed for every $n \in \mathbb{N}$ and every $x \in \mathscr{K}$. From the equicontinuity of the family $\{\Phi_n(\Theta, \tau)\}_{n\in\mathbb{N}}$ at the point $\tau = 1$, we obtain that the mapping $\mathscr{T}_n$ satisfies all the conditions of Theorem 5.6. Hence, for every $n \in \mathbb{N}$, there exists $\kappa_n \in \mathscr{K}$ such that $\kappa_n \in \mathscr{T}_n\kappa_n$. Since $\mathscr{T}(\mathscr{K})$ is bounded, as in Theorem 5.6, it follows that

$$\lim_{n\to\infty} \mathscr{M}(\kappa_n, z_n, \varepsilon) = 1, \quad \varepsilon > 0,$$

where $z_n \in \mathscr{T}\kappa_n$ for every $n \in \mathbb{N}$. Since the mapping $\mathscr{T}$ is demicompact we can show the existence of a convergent subsequence $\{\kappa_{n_k}\}_{k\in\mathbb{N}}$ of $\{\kappa_n\}$. The rest of the proof is as in the Theorem 5.6. This achieves the proof. □

6. Conclusion and Future Work

In conclusion, this article has thoroughly explored the concept of convexity within fuzzy metric spaces and has analyzed their structural properties. We have presented a series of theorems that focus on the existence of coincidence points in fuzzy convex metric spaces. In addition, we've introduced star-shaped subsets in these spaces, showcasing several fixed point theorems for commuting non-expansive mappings within them. The concluding section expands the definition of fuzzy convex metric spaces, offering a notable example. We've also presented specific fixed point theorems applicable to non-expansive multi-valued mappings.

Prospective research in this domain might entail further exploration of these concepts across diverse mathematical and scientific disciplines. Additionally, delving into the practical implications and real-world applications of fuzzy convex metric spaces and their associated theorems presents an intriguing avenue

for future investigation. Furthermore, examining more intricate and generalized scenarios within this framework could yield deeper insights and contribute to novel mathematical advancements.

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