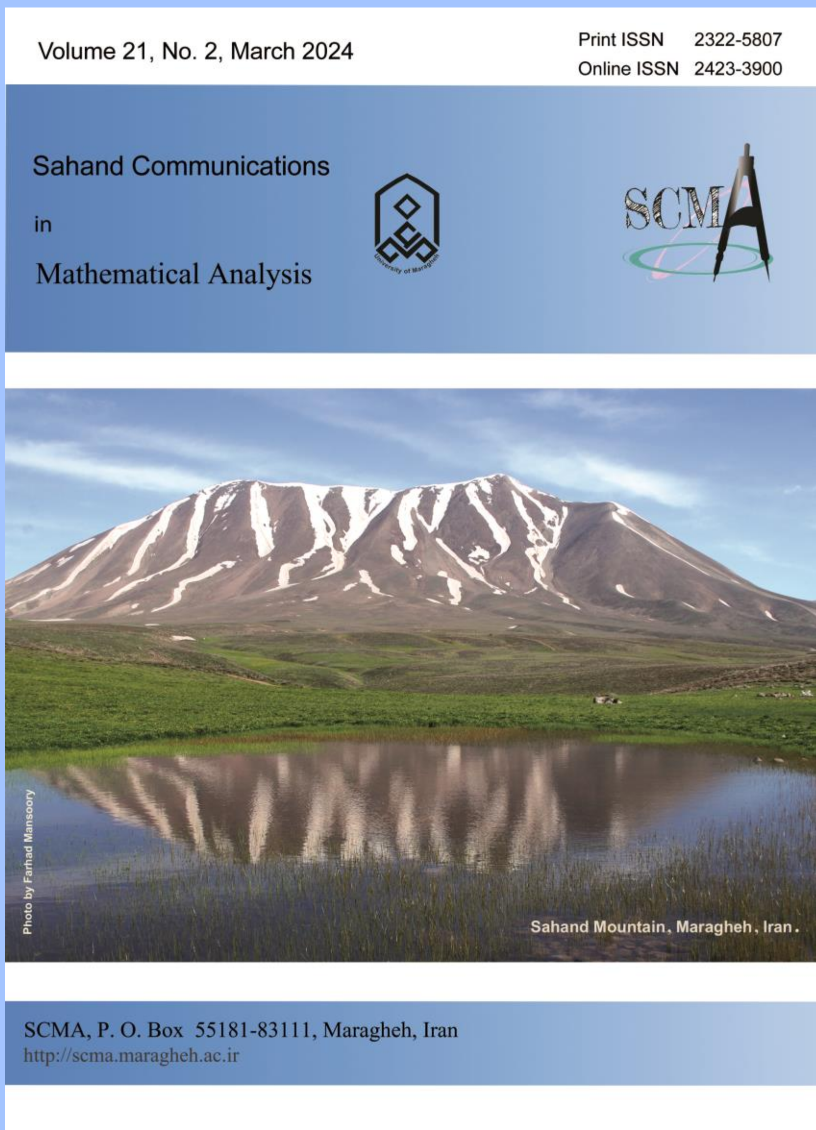
On a General Conditional Cauchy Functional Equation

Elham Mohammadi, Abbas Najati and Yavar Khedmati Yengejeh

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 2
Pages: 315-326

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.2006802.1386



Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 2 (2024), 315-326
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.2006802.1386

On a General Conditional Cauchy Functional Equation

Elham Mohammadi[1], Abbas Najati[2*] and Yavar Khedmati Yengejeh[3]

Abstract. Let $(G,+)$ be an abelian group and Y a linear space over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. In this paper, we investigate the conditional Cauchy functional equation

$$f(x+y) \neq af(x)+bf(y) \quad \Rightarrow \quad f(x+y) = f(x)+f(y), \quad x, y \in G,$$

for functions $f : G \to Y$, where $a, b \in \mathbb{F}$ are fixed constants. The general solution and stability of this functional equation are described.

1. Introduction

Understanding functional equations is essential for solving problems in a wide range of disciplines, including mathematics, physics, engineering, economics, and biology [1]. The field of functional equations constitutes a modern mathematical discipline that has undergone significant and swift development over the past fifty years. Functional equations are investigated and solved without making any assumptions about regularity conditions. Let $(G,+)$ be an abelian group and $(R,+,.)$ be a commutative integral domain with identity and characteristic of zero. Pl. Kannappan and M. Kuczma [15] investigated and solved the functional equation

$$[f(x+y) - af(x) - bf(y)][f(x+y) - f(x) - f(y)] = 0, \tag{1.1}$$

for functions $f : G \to R$, where $a, b \in R$ are constants.

In this paper, we treat the conditional Cauchy functional equation

$$f(x+y) \neq af(x) + bf(y) \quad \Rightarrow \quad f(x+y) = f(x) + f(y), \tag{1.2}$$

2020 *Mathematics Subject Classification.* 39B55, 39B82.
Key words and phrases. Abelian group, Conditional functional equation, Mikusiński's functional equation, stability.
Received: 12 July 2023, Accepted: 13 December 2023.
* Corresponding author.

for functions $f : G \to Y$, where $(G, +)$ is an abelian group, Y is a linear space over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $a, b \in \mathbb{F}$ are constants. Obviously, (1.1) and (1.2) are not equivalent. Particular cases of (1.2) are, among others, the following conditional Cauchy functional equations:

$$f(x+y) \neq f(x) \quad \Rightarrow \quad f(x+y) = f(x) + f(y),$$

$$f(x+y) + f(x) + f(y) \neq 0 \quad \Rightarrow \quad f(x+y) = f(x) + f(y), \tag{1.3}$$

$$f(x+y) \neq 0 \quad \Rightarrow \quad f(x+y) = f(x) + f(y). \tag{1.4}$$

Indeed, (1.3) comes from

$$\big[f(x+y)\big]^2 = \big[f(x) + f(y)\big]^2,$$

for real functions f, and then studied in the form $|f(x+y)| = |f(x) + f(y)|$ which admits other generalizations from the real case to further general structures. A natural generalization is $\|f(x+y)\| = \|f(x) + f(y)\|$ for normed spaces (see [7, 12–14]). Moreover, there are various interesting results which deal with the stability of functional equations in restricted domains [8–10, 16].

The conditional Cauchy functional equation (1.4) is called Mikusiński's functional equation.

2. General Solution of (1.2)

In this section, Y denotes a linear space over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $(G, +)$ is an abelian group. We deal with the conditional Cauchy functional equation (1.2), where $f : G \to Y$ and $a, b \in \mathbb{F}$. First we notice that $f \equiv 0$ is a trivial solution of (1.2). We also note that if f is a nonzero and constant solution of (1.2), say $f(x) = c$, then $a + b = 1$. Thus in the sequel we will consider only non-constant solution of (1.2). Under some conditions, we show (1.2) has solutions which are not additive.

The general solution of Mikusiński's functional equation is exactly expressed in [11]. We recall that the index of a subgroup K in a group G, is equal to the number of left (right) cosets of K in G.

Theorem 2.1 ([11])**.** *Let $(G, +)$ and $(H, +)$ be groups. If $f : G \to H$ is a non-additive solution of the conditional Cauchy functional equation*

$$f(x+y) \neq 0 \quad \Rightarrow \quad f(x+y) = f(x) + f(y),$$

then $K := f^{-1}(0)$ is a subgroup of index two, and f is given by

$$f(x) = \begin{cases} 0, & x \in K, \\ c, & x \notin K, \end{cases}$$

where $c \neq 0$ is an arbitrary element of H.

To reach the main result, we start with some lemmas.

Lemma 2.2. *Suppose that $f : G \to Y$ is a non-constant function satisfies* (1.2). *Then $K := f^{-1}(0)$ is a subgroup of G.*

Proof. First we show $f(0) = 0$. Let $f(0) \neq 0$. Put $y = 0$ in (1.2) to abtain

$$(1-a)f(x) = bf(0), \quad x \in G. \tag{2.1}$$

Letting $y = x = 0$ in (1.2), we infer $f(0) = (a+b)f(0)$ and consequently $a + b = 1$. Hence (2.1) yields $bf(x) = bf(0)$ for all $x \in G$. Since f is non-constant, we get $b = 0$, and consequently $a = 1$. Letting $x = 0$ in (1.2), we have $f(y) = af(0) = f(0)$ for all $y \in G$. This implies that f is constant which is a contradiction. So $f(0) = 0$.

Obviously, if $x, y \in K$, then (1.2) yields $x + y \in K$. Take $x \in K$ and put $y = -x$ in (1.2), we get either $bf(-x) = 0$ or $f(-x) = 0$ because of $f(0) = 0$. If we had $f(-x) \neq 0$, then $b = 0$. Applying (1.2), we get $af(-x) = 0$ and consequently $a = 0$. Therefore (1.2) reduces to (1.4), i.e.,

$$f(x+y) \neq 0 \quad \Rightarrow \quad f(x+y) = f(x) + f(y).$$

Putting $y = -2x$ in (1.4) and using $f(-x) \neq 0$, we have $f(-x) = f(x) + f(-2x) = f(-2x)$. So $f(-2x) \neq 0$. By (1.4), we acquire $f(-2x) = 2f(-x)$. Hence

$$f(-x) = f(-2x) = 2f(-x).$$

This yields $f(-x) = 0$ which is a contradiction. Hence the proof is complete. □

Lemma 2.3. *Let $r, s \in \mathbb{F}$ with $r + s \neq 0$ and $f : G \to Y$ satisfies* (1.2). *Then f fulfills*

$$(r+s)f(x+y) \neq (ra+sb)f(x)+(sa+rb)f(y) \quad \Rightarrow \quad f(x+y) = f(x)+f(y).$$

Proof. If $r = 0$ or $s = 0$, then the result is obvious. So we suppose that r, s are non zero. Let $(r + s)f(x + y) \neq (ra + sb)f(x) + (sa + rb)f(y)$. Then

$$rf(x+y) \neq raf(x) + rbf(y) \quad \text{or} \quad sf(x+y) \neq saf(y) + sbf(x).$$

In each of these two cases, (1.2) implies that $f(x+y) = f(x)+f(y)$. □

Corollary 2.4. *Let $f : G \to Y$ satisfies* (1.2). *Then f satisfies*

$$2f(x+y) \neq (a+b)f(x) + (a+b)f(y) \quad \Rightarrow \quad f(x+y) = f(x) + f(y).$$

Especially, if $a + b = 0$, then f satisfies (1.4).

Lemma 2.5. *Let $f : G \to Y$ be a non-constant function that satisfies* (1.2). *Then f is odd or $a + b = 0$ and f fulfills* (1.4).

Proof. Suppose that f is not odd. Then there exists $z \in G$ such that $f(z)+f(-z) \neq 0$. It is clear that $z, -z \notin K$. Putting $x = z$ and $y = -z$ in (1.2) and using $f(0) = 0$, we conclude $af(z)+bf(-z) = 0$. Similarly, we get $af(-z) + bf(z) = 0$. Therefore

$$(a + b)[f(z) + f(-z)] = 0.$$

This implies that $a+b = 0$, and consequently f fulfills (1.4) by Corollary 2.4. □

Lemma 2.6. *Suppose that* $f : G \to Y$ *is a solution of* (1.2) *and* $a^2+b^2 > 0$. *Then* f *satisfies*

$$f(x) \neq f(y) \quad \Rightarrow \quad f(x + y) = f(x) + f(y), \quad x, y \in G. \tag{2.2}$$

Proof. Let $a \neq b$ and $x, y \in G$ such that $f(x + y) \neq f(x) + f(y)$. It follows from (1.2) that

$$f(x + y) = af(x) + bf(y) \quad \text{and} \quad f(x + y) = af(y) + bf(x).$$

Then $(a - b)[f(x) - f(y)] = 0$. Since $a \neq b$, we get $f(x) = f(y)$.

If $a = b = 1$, then (1.2) implies that f is additive. Thus (2.2) holds for f. For the case $a = b \neq 0, 1$, we may assume that f is non-constant (every constant function satisfies (2.2)). Then $f(0) = 0$ by Lemma 2.2. Now we prove f is odd. On the contrary, suppose $z \in G$ such that $f(z) + f(-z) \neq 0$. Letting $x = z$ and $y = -z$ in (1.2) and applying $f(0) = 0$, one gets $a[f(-z) + f(z)] = 0$. Since $a \neq 0$, one concludes $f(z) + f(-z) = 0$ which is a contradiction. To prove (2.2), let $x, y \in G$ such that $f(x + y) \neq f(x) + f(y)$. By (1.2), we have

$$f(x + y) = a[f(x) + f(y)]. \tag{2.3}$$

Since f is odd, we get by (1.2) either

$$f(x) = f(x + y - y) = a[f(x + y) - f(y)] \tag{2.4}$$

or

$$f(x) = f(x + y - y) = f(x + y) - f(y). \tag{2.5}$$

It is clear that (2.5) yields $f(x + y) = f(x) + f(y)$, contrary to the assumption. Hence we must have (2.4). Adding (2.3) and (2.4), we get $f(x+y)+f(x) = a[f(x+y)+f(x)]$. Since $a \neq 1$, we get $f(x+y) = -f(x)$. Similarly, one obtains $f(x + y) = -f(y)$. Hence $f(x) = f(y)$, and this proves (2.2). □

Lemma 2.7. *Suppose that* $x_0 \notin K$ *and* $f : G \to Y$ *is a non-constant solution of* (1.2). *Then*

$$x_0 + K = \{x \in G : \; f(x) = f(x_0)\}.$$

Proof. One of two the following situations occurs:

Case I. In case of $a^2 + b^2 > 0$, f fulfills (2.2). If $x \in K$, then $f(x) \neq f(x_0)$. Hence (2.2) implies $f(x_0 + x) = f(x_0) + f(x) = f(x_0)$. Thus

$$x_0 + K \subseteq \{x \in G : \ f(x) = f(x_0)\}.$$

If f is odd and $f(x) = f(x_0)$, then $f(x) \neq -f(x_0) = f(-x_0)$. By (2.2), one obtains $f(x - x_0) = f(x) - f(x_0) = 0$. This means $x \in x_0 + K$. If f is not odd and $f(x) = f(x_0)$, then f satisfies (1.4) by Lemma 2.5. By Theorem 2.1, we infer that the index of K is two and $G = K \cup (x_0 + K)$. Since $f(x) = f(x_0)$, we get $x \in x_0 + K$. Therefore

$$\{x \in G : \ f(x) = f(x_0)\} \subseteq x_0 + K.$$

Case II. $a = b = 0$. In this case f satisfies (1.4). By Theorem 2.1, we conclude either f is additive or the index of K is two and

$$f(x) = \begin{cases} 0, & x \in K, \\ f(x_0), & x \in x_0 + K. \end{cases} \tag{2.6}$$

In both cases, it can be easily seen that $x_0 + K = \{x \in G : \ f(x) = f(x_0)\}$. □

Theorem 2.8. *Let $a^2 + b^2 > 0$ and $f : G \to Y$ be a non-constant solution of* (1.2). *Then we have one of the following assertions:*

(i) *f is odd, the index of K is 3, $a + b = -1$ and f is given by*

$$f(x) = \begin{cases} 0, & x \in K, \\ c, & x \in z_0 + K, \\ -c, & x \in -z_0 + K, \end{cases} \tag{2.7}$$

where $z_0 \notin K$ and $c \neq 0$ is an arbitrary element in Y.

(ii) *f is odd, the index of K is infinite and f is additive.*

(iii) *f is not odd, the index of K is 2, $a + b = 0$ and f is given by*

$$f(x) = \begin{cases} 0, & x \in K, \\ c, & x \notin K, \end{cases} \tag{2.8}$$

where $c \neq 0$ is an arbitrary element in Y.

Proof. By Lemma 2.6, f fulfills (2.2). We now consider the following two cases according to Lemma 2.5.

Case I. Let f be odd. We claim that the index of K is greater than two. Since f is not constant, the index of K is not one. We now supposing the contrary, the index of K is two. Let $z \notin K$. Then $z, -z \in z + K$, and Lemma 2.7 yields $f(-z) = f(z)$. Since f is odd, we get $f(z) = 0$ which is a contradiction. So the index of K is greater than two. If the index of K is 3 and $z_0 \notin K$, then $G/K = \{K, z_0 + K, -z_0 + K\}$. Hence f is given by (2.7). So (1.2) with $x = y = z_0$, yields $a + b = -1$.

We now claim that if the index of K is greater than 3, then f is additive and the index of K should be infinite. Let $x, y \in G$.

(*i*) If $f(x) \neq f(y)$, then (2.2) implies $f(x+y) = f(x) + f(y)$.

(*ii*) If $f(x) = f(y) = 0$, then $f(x+y) = 0$ by virtue of Lemma 2.2. So $f(x+y) = f(x) + f(y)$ is concluded.

(*iii*) If $f(x) = f(y) \neq 0$, then there exists a $z \in G$ such that $z \notin K \cup (x+K) \cup (-x+K)$. So $f(z) \neq 0, f(z) \neq f(x)$ and $f(z) \neq f(-x)$. Hence $f(-z) \neq f(y)$. By (2.2), we get

$$f(x+z) = f(x) + f(z), \qquad f(y-z) = f(y) - f(z).$$

Therefore $f(x+z) \neq f(y-z)$, and again (2.2) yields

$$f(x+y) = f(x+z) + f(y-z) = f(x) + f(y).$$

Thus f is additive. Since f is not constant, there exists an $x_0 \notin K$. Then $\{nx_0 + K\}_{n=1}^{\infty}$ is a sequence of disjoint cosets. This means that the index of K is infinite.

Case II. Suppose that f is not odd. By Lemma 2.5, $a + b = 0$ and f fulfills (1.4). Since f is not odd, f is a non-additive solution of (1.4). By Theorem 2.1, the index of K is 2 and f is given by (2.8). ☐

Remark 2.9. It is clear that if $a + b = 1$, then every constant function $f : G \to Y$ is a solution of (1.2). For $a + b \neq 1$, $f \equiv 0$ is the only constant solution of (1.2).

3. Stability

The main goal in this section is to study the stability and hyperstability of the conditional Cauchy functional equation

$$f(x+y) \neq af(x) + bf(y) \quad \Rightarrow \quad f(x+y) = f(x) + f(y).$$

Stability of a special case of this conditional Cauchy functional equation ($a = b = -1$) has been studied and investigated in [4](see also [5]).

In this section, $(G, +)$ denotes an abelian group.

The following lemma is used to prove the main theorem of this section.

Lemma 3.1. *Let X be a linear normed space, $\delta, \varepsilon \geq 0$ and $a \notin \{-\frac{1}{2}, 0, 1\}$. Suppose that $f : G \to X$ is a function fulfilling*

(3.1)
$$\|f(x+y) - af(x) - af(y)\| \leq \delta \quad \text{or} \quad \|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in G$. If $\|f(2z) - 2af(z)\| \leq \delta$ for some $z \in G$, then $\|f(z)\| \leq M$, where M depends only on a, δ, ε.

Proof. Take $\beta := \|f(0)\|$. Letting $y = -z$ and $x = z$ in (3.1), one obtains

$$\|f(0) - a(f(z) + f(-z))\| \leq \delta \quad \text{or} \quad \|f(0) - f(z) - f(-z)\| \leq \varepsilon. \tag{3.2}$$

Then

$$\|f(z)+f(-z)\| \le \frac{\delta+\beta}{|a|} \quad \text{or} \quad \|f(z)+f(-z)\| \le \varepsilon+\beta.$$

Thus

$$\|f(z)+f(-z)\| \le \gamma := \max\left\{\frac{\delta+\beta}{|a|}, \varepsilon+\beta\right\}. \tag{3.3}$$

Letting $y=-z$ and $x=2z$ in (3.1), we acquire

$$\|f(z)-af(2z)-af(-z)\| \le \delta \quad \text{or} \quad \|f(z)-f(2z)-f(-z)\| \le \varepsilon. \tag{3.4}$$

Since $\|f(2z)-2af(z)\| \le \delta$, (3.4) yields

$$\big\|(1-2a^2)f(z)-af(-z)\big\| \le \delta(1+|a|) \quad \text{or} \quad \|(1-2a)f(z)-f(-z)\| \le \delta+\varepsilon.$$

By (3.3) and the above inequalities, we have

$$\|f(z)\| \le \frac{\delta(1+|a|)+\gamma|a|}{|2a^2-a-1|} \quad \text{or} \quad \|f(z)\| \le \frac{\delta+\varepsilon+\gamma}{|2a-2|}.$$

Then

$$\|f(z)\| \le \max\left\{\frac{\delta(1+|a|)+\gamma|a|}{|2a^2-a-1|}, \frac{\delta+\varepsilon+\gamma}{|2a-2|}\right\}. \tag{3.5}$$

□

Throughout the remainder of this section, X represents a Banach space.

Theorem 3.2. *Let $\delta, \varepsilon \ge 0$, $a \ne -\frac{1}{2}$ and $f: G \to X$ be a function fulfilling (3.1) for all $x, y \in G$. Then there exists an additive function $A: G \to X$ such that $f - A$ is bounded on G.*

Proof. For the case $a=0$, the result follows from [3, Theorem 1].

If $a=1$, then $\|f(2x)-2f(x)\| \le \max\{\delta, \varepsilon\}$ for all $x \in G$. We now assume $a \notin \{-\frac{1}{2}, 0, 1\}$. For each $x \in G$, we have the following two cases:

Case 1. $\|f(2x)-2af(x)\| \le \delta$. By Lemma 3.1, we get $\|f(x)\| \le M$, where M depends only on a, δ, ε. Therefore

$$\begin{aligned}\|f(2x)-2f(x)\| &\le \|f(2x)-2af(x)\| + \|2af(x)-2f(x)\| \\ &\le \delta + |2a-2|M.\end{aligned}$$

Case 2. $\|f(2x)-2af(x)\| > \delta$. By (3.1), we get $\|f(2x)-2f(x)\| \le \varepsilon$.

So according to the above cases, we have

$$\|f(2x)-2f(x)\| \le \theta := \max\{\delta+|2a-2|M, \varepsilon\}, \quad x \in G.$$

Therefore

$$\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right\| \le \sum_{k=m}^{n-1} \frac{\theta}{2^{k+1}}, \quad x \in G,\ n \ge m \ge 0. \tag{3.6}$$

This yields that $\left\{\frac{f(2^n x)}{2^n}\right\}_n$ is a Cauchy sequence for all $x \in G$. Then the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}_n$ converges, since X is a Banach space. Let us define

$$A : G \to X, \qquad A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}.$$

Letting $m = 0$ in (3.6) and allowing $n \to \infty$, one concludes

$$\|A(x) - f(x)\| \le \theta, \quad x \in G.$$

We claim that

$$A(x + y) = aA(x) + aA(y) \quad \text{or} \quad A(x + y) = A(x) + A(y), \quad x, y \in G.$$

Let $x, y \in G$ such that $A(x + y) \neq aA(x) + aA(y)$. Then the sequence $\{f(2^n(x+y)) - af(2^n x) - af(2^n y)\}_n$ is not bounded. So $\|f(2^n(x+y)) - af(2^n x) - af(2^n y)\| > \delta$ for a sufficiently large $n \in \mathbb{N}$. Thus

$$\|f(2^n(x + y)) - f(2^n x) - f(2^n y)\| \le \varepsilon,$$

on account of (3.1). Divide the inequality above by 2^n and allow $n \to \infty$ to obtain $A(x + y) = A(x) + A(y)$.

We now show that A is additive. Let A be nonzero. By the definition of A, one gets $A(2x) = 2A(x)$ for all $x \in G$. This implies that A is a non-constant solution of (1.2). Since $a \neq -\frac{1}{2}, 0$ and $A(2x) = 2A(x)$, we infer that A satisfies only (2) in Theorem 2.8. So, A is additive. □

Corollary 3.3. *Let $\delta, \varepsilon \ge 0$ and $a + b \neq -1$. Assume that $f : G \to X$ is a function satisfying*

(3.7)
$$\|f(x + y) - af(x) - bf(y)\| \le \delta \quad \text{or} \quad \|f(x + y) - f(x) - f(y)\| \le \varepsilon,$$

for all $x, y \in G$. Then there exists an additive function $A : G \to X$ such that $f - A$ is bounded on G.

Proof. We claim that

$$\left\| f(x + y) - \frac{a + b}{2} f(x) - \frac{a + b}{2} f(y) \right\| \le \delta, \tag{3.8}$$

or

$$\|f(x + y) - f(x) - f(y)\| \le \varepsilon,$$

for all $x, y \in G$. To prove (3.8), let $\left\|f(x + y) - \frac{a+b}{2} f(x) - \frac{a+b}{2} f(y)\right\| > \delta$ for some $x, y \in G$. Then

$$\|f(x + y) - af(x) - bf(y)\| > \delta \quad \text{or} \quad \|f(x + y) - af(y) - bf(x)\| > \delta.$$

In both cases, (3.7) implies $\|f(x + y) - f(x) - f(y)\| \le \varepsilon$. This proves (3.8). Hence the result follows from Theorem 3.2. □

Corollary 3.4. *Let $\varepsilon \geq 0$ and $a+b \neq -1$. If a function $f : G \to \mathbb{C}$ fulfills*

$$\Big|[f(x+y)-af(x)-bf(y)][f(x+y)-f(x)-f(y)]\Big| \leq \varepsilon, \quad x,y \in G, \tag{3.9}$$

then there exists an additive function $A : G \to \mathbb{C}$ such that $f - A$ is bounded on G.

Corollary 3.5. *Let $\varepsilon \geq 0$ and $(a,b) \neq (1,1)$. Suppose that a function $f : G \to \mathbb{C}$ fulfills* (3.9)*. Let $\{x_n\}_n$ be a sequence in G with*

$$\lim_n \big|f(x_n+x+y) - af(x_n+x)\big| = +\infty,$$

or

$$\lim_n \big|f(x_n+x+y) - bf(x_n+x)\big| = +\infty,$$

for all $x,y \in G$. Then f is additive.

Proof. By the assumption we have

$$\lim_n \big|f(x_n+y) - af(x_n)\big| = +\infty,$$

or

$$\lim_n \big|f(x_n+y) - bf(x_n)\big| = +\infty, \quad y \in G.$$

Then it follows from (3.9) that

$$\begin{aligned}\lim_n [f(x_n+y) - f(x_n)] &= f(y) \\ &= \lim_n [f(x_n+x+y) - f(x_n+x)], \quad x,y \in G.\end{aligned} \tag{3.10}$$

Let $x,y \in G$. Then (3.10) yields

$$\begin{aligned} f(x+y) - f(x) &= \lim_n [f(x_n+x+y) - f(x_n)] - \lim_n [f(x_n+x) - f(x_n)] \\ &= \lim_n [f(x_n+x+y) - f(x_n+x)] = f(y). \end{aligned}$$

This means f is additive on G. □

In the case where $a = b = 0$, we will have the following superstability result. Of course, this result has been proven in [6], but our proof is slightly different.

Corollary 3.6. *Let $\varepsilon \geq 0$ and $f : G \to \mathbb{C}$ be a function fulfills*

$$\big|f(x+y)[f(x+y) - f(x) - f(y)]\big| \leq \varepsilon, \quad x,y \in G. \tag{3.11}$$

Then f is either additive, or bounded on G.

Proof. Replacing x by $x-y$ in (3.11), one gets

$$\big|f(x)[f(x)-f(x-y)-f(y)]\big| \leq \varepsilon, \quad x,y \in G. \tag{3.12}$$

Let f be unbounded. Then we can find a sequence $\{x_n\}_n$ in G with $\lim_n |f(x_n)| = +\infty$. By (3.12), we have

$$\big|f(x_n)[f(x_n)-f(x_n-y)-f(y)]\big| \leq \varepsilon, \quad x,y \in G.$$

So

$$f(y) = \lim_n \left[f(x_n) - f(x_n-y)\right], \quad y \in G. \tag{3.13}$$

Since $|f(x_n+y)| \geq |f(x_n)| - |f(x_n) - f(x_n+y)|$, we obtain $\lim_n |f(x_n+y)| = +\infty$ by (3.13). Using (3.11), we get

$$f(y) = \lim_n \left[f(x_n+y) - f(x_n)\right], \quad y \in G.$$

This equality results that

$$f(y) = \lim_n \left[f(x_n+z+y) - f(x_n+z)\right], \quad y,z \in G.$$

Therefore

$$\begin{aligned} f(y+z) - f(y) &= \lim_n \left[f(x_n+y+z) - f(x_n)\right] \\ &\quad - \lim_n \left[f(x_n+z+y) - f(x_n+z)\right] \\ &= \lim_n \left[f(x_n+z) - f(x_n)\right] \\ &= f(z), \quad y,z \in G. \end{aligned}$$

This proves f is additive. □

Remark 3.7. The multiplicative property of the absolute value in complex numbers is essential to prove the above results. Since the norm of Cayley numbers and quaternions is multiplicative, it can be shown that the results are also true for functions f with values in Cayley numbers or quaternions. In general, the results are valid for functions whose values are in a normed algebra with the multiplicative norm.

Using the idea of [2]'s example, we give an example to show that if the norm is not multiplicative, the result 3.6 does not hold for functions with values in norm algebras.

Example 3.8. Let $M_2(\mathbb{C})$ be the normed algebra of 2×2 matrices with complex entries, which is equipped with the usual norm. Let $\varepsilon > 0$ and $f : \mathbb{R} \to M_2(\mathbb{C})$ be a function given by

$$f(x) = \begin{pmatrix} x & 0 \\ 0 & \varepsilon \end{pmatrix}.$$

Obviously, f is unbounded and $\big\|f(x+y)[f(x+y)-f(x)-f(y)]\big\| = \varepsilon^2$. But f is not additive.

4. Conclusion

Let $(G, +)$ be an abelian group and $(R, +, .)$ a commutative integral domain with identity and of characteristic zero. Pl. Kannappan and M. Kuczma [15] investigated the functional equation (1.1) and obtained its general solution for functions $f : G \to R$, where $a, b \in R$ are constants. We investigated and derived the general solution for the conditional Cauchy functional equation (1.2) for functions $f : G \to Y$, where Y is a linear space over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $a, b \in \mathbb{F}$ are constants. However, in equation (1.2), the multiplication does not take place, and a, b are scalars. The final section of this paper is devoted to the stability of equation (1.2). An open problem pertains to the stability of equation (1.2) under the condition $a+b = -1$. In addition, studying the functional equation (1.2) on a restricted domain will be interesting and such an investigation has the potential to yield valuable results.

Statements and Declarations

Competing interests. The authors declare that they have no competing interests.

Funding. No funds, grants, or other support was received.

References

1. J. Aczél and J. Dhombres, *Functional equations in several variables*, Cambridge University Press, Cambridge, 1989.
2. J. Baker, *The stability of the cosine equation*, Proc. Amer. Math. Soc., 80 (1980), pp. 411-416.
3. B. Batko, *On the stability of Mikusiński's equation*, Publ. Math. Debrecen, 66 (2005), pp. 17-24.
4. B. Batko, *On the stability of an alternative functional equation*, Math. Inequal. Appl., 8 (2005), pp. 685-691.
5. B. Batko, *Stability of an alternative functional equation*, J. Math. Anal. Appl., 339 (2008), pp. 303-311.
6. B. Batko, *Note on superstability of Mikusiński's functional equation*, Functional equations in mathematical analysis, 15-17, Springer Optim. Appl., 52, Springer, New York, 2012.
7. B. Batko and J. Tabor, *Stability of an alternative Cauchy equation on a restricted domain*, Aequationes Math., 57 (1999), pp. 221-232.
8. A. Bahyrycz and J. Brzdek, *On solutions of the d'Alembert equation on a restricted domain*, Aequationes Math., 85 (2013), pp. 169-183.

9. J. Brzdek and J. Sikorska, *A conditional exponential functional equation and its stability*, Nonlinear Anal., 72 (2010), pp. 2929-2934.
10. J. Chung, *Conditional functional equations on restricted domains of measure zero*, J. Math. Anal. Appl., 430 (2015), pp. 1074-1087.
11. L. Dubikajtis, C. Ferens, R. Ger and M. Kuczma, *On Mikusiński's functional equation,* Ann. Polon. Math., 28 (1973), pp. 39-47.
12. R. Ger,*On a characterization of strictly convex spaces*, Atti Accad. Sci. Torino, Cl. Sci. Fis. Mat. Natur., 127 (1993), pp. 131-138.
13. A. Najati and Y. Khedmati Yengejeh, *Functional inequalities associated with additive, quadratic and Drygas functional equations*, Acta Math. Hungar., 168 (2022), pp. 572-586.
14. A. Najati and M.A. Tareeghee, *Drygas functional inequality on restricted domains*, Acta Math. Hungar., 166 (2022), pp. 115-123.
15. Pl. Kannappan and M. Kuczma, *On a functional equation related to the Cauchy equation,* Ann. Polon. Math., 30 (1974), pp. 49-55.
16. M. A. Tareeghee, A. Najati, M.R. Abdollahpour and B. Noori, *On restricted functional inequalities associated with quadratic functional equations*, Aequationes Math., 96 (2022), pp. 763-772.

---

[1]Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran.

Email address: elhammkhanghah@gmail.com

[2]Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran.

Email address: a.nejati@yahoo.com

[3]Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran.

Email address: khedmati.y@uma.ac.ir