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# Bound Associated with Certain Hankel Determinants and Zalcman Conjecture for Multivalent Functions of Bounded Turning

Deekonda Vamshee Krishna[1*], Nallamothu Vani[2], Biswajit Rath[3] and Karri Sanjay Kumar[4]

Abstract. In this paper, we investigate for a sharp upper bound to certain generalized second Hankel determinant, the Zalcman conjecture and an upper bound for the third, fourth Hankel determinants for the class of multivalent analytic bounded turning functions. Further, we estimate an upper bound for third and fourth Hankel determinants with respect to two-fold and three-fold symmetric functions belongs to the same class. The practical tools applied in deriving of our main results are the coefficient inequalities of the Carathéodory class $\mathcal{P}$.

## 1. Introduction

Let $\mathcal{A}_p$, with $p \in \mathbb{N} = \{1, 2, 3, \dots\}$ represent family of all $p$-valent mappings $f$ of the form

$$f(z) = z^p \sum_{n=0}^{\infty} a_{p+n} z^n, \quad a_p := 1, \tag{1.1}$$

in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}_1 = \mathcal{A}$, be the subfamily of $f$ normalized by $f(0) = f'(0) - 1 = 0$, i.e., of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1, z \in \mathbb{D}. \tag{1.2}$$

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By $S$, we denote subfamily of $\mathcal{A}$ of univalent functions. Pommerenke [20] characterized the $r^{th}$- Hankel determinant of degree $t$, for $f$ given in (1.2) with $r,\ t \in \mathbb{N}$, as

$$H_{r,t}(f) = \begin{vmatrix} a_t & a_{t+1} & \cdots & a_{t+r-1} \\ a_{t+1} & a_{t+2} & \cdots & a_{t+r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{t+r-1} & a_{t+r} & \cdots & a_{t+2r-2} \end{vmatrix}. \tag{1.3}$$

For a subclass $\mathcal{F}$ of $\mathcal{A}$, with specific values of $r$ and $t$, estimation of an upper bound of $H_{r,t}(f)$ represents an interesting and significant problem. The Fekete-Szegö functional is obtained for $r = 2$ and $t = 1$ in (1.3), given by $H_{2,1}(f) = a_3 - a_2^2$. The sharp estimates of $|H_{2,2}(f)|$, where $H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$, referred to as second Hankel determinant, obtained for $r = 2$ and $t = 2$ in (1.3), computed for the subfamilies of $S$, namely bounded turning, starlike and convex functions, whose respective classes are symbolized as $\Re$, $S^*$ and $\mathcal{K}$, fulfilling the analytic conditions $\mathrm{Re} f'(z) > 0$, $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$ and $\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$ in the unit disc $\mathbb{D}$, were derived by Janteng et al. [12, 13] as $4/9$, $1$, and $1/8$, respectively. For recent results on the second Hankel determinants (see [9, 18, 23, 24, 31]). Similarly, by taking $r = 2$ and $t = 3$ in (1.3), we have $H_{2,3}(f) = a_3a_5 - a_4^2$, represents a second Hankel determinant, for which Zaprawa [29] derived sharp bounds as $|H_{2,3}(f)| \leq 1$, for the class $S^*$ and $|H_{2,3}(f)| \leq \frac{1}{15}$ for the class $\mathcal{K}$, assuming $a_2 = 0$ in $f$, as given in (1.1). By the motivation of the results derived by Zaprawa in [29], recently, Andy Liew Pik Hern et al. [2], have shown that $|H_{2,3}(f)| \leq \dfrac{13}{16}$, for $f \in S_s^*$ and $|H_{2,3}(f)| \leq \frac{13}{240}$, for $f \in \mathcal{K}_s$, respectively, where $S_s^*$ and $\mathcal{K}_s$ denotes the families of starlike and convex functions with respect to symmetric points, analytically defined as

$$f \in S_s^* \quad \Leftrightarrow \quad \mathrm{Re}\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in \mathbb{D}, \tag{1.4}$$

and

$$f \in \mathcal{K}_s \quad \Leftrightarrow \quad \mathrm{Re}\left\{\frac{2\left\{zf'(z)\right\}'}{zf'(z) + zf'(-z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{1.5}$$

Choosing $r = 2$ and $t = p + 1$ in (1.3), we obtain second Hankel determinant for the $p$-valent function (see [27]), given by

$$\begin{aligned} H_{2,p+1}(f) &= \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} \\ &= a_{p+1}a_{p+3} - a_{p+2}^2. \end{aligned}$$

In recent years, many authors have focused on the estimation of an upper bound (in particular sharp) of the third Hankel determinant denoted by $H_{3,1}(f)$, obtained for $r = 3$ and $t = 1$ in (1.3), though it is technically much more difficult for the functions belong to certain subclasses of $\mathcal{A}$ compare to $H_{2,2}(f)$. Babalola [6] initiated to estimate an upper bound to $|H_{3,1}(f)|$ for the classes $\Re$, $S^*$ and $\mathcal{K}$. As a consequence of this paper, many papers containing results associated with third (see [4, 14, 21, 22, 25, 28]) and very few papers on fourth Hankel determinants (see [3, 5, 8]) were obtained. In the 1960s, Zalcman, conjectured that if $f \in S$ then $|a_n^2 - a_{2n-1}| \leq (n-1)^2, \quad \text{for} \quad n \geq 2$. For $f \in S$, Ma [17] proposed a generalized Zalcman conjecture $|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1), \quad \text{for} \quad m, n \geq 2$. Motivated by the results mentioned in this paper, we derive the bounds specified in the abstract for the class of functions defined as follows.

**Definition 1.1.** A function $f \in \mathcal{A}_p$, said to be in the class $\Re_p$, consisting of multivalent bounded turning functions, if it satisfies the condition

$$\operatorname{Re}\left\{\frac{f'(z)}{pz^{p-1}}\right\} > 0, \quad z \in \mathbb{D}. \tag{1.6}$$

Choosing $p = 1$, we obtain $\Re_1 = \Re$, consisting of functions whose derivative has a positive real part (also called functions of bounded turning), established by Alexander [1] (cf. [10], Vol. I, p.88), and is one of the fundamental subfamily of $S$. For the functions belong to the class $\Re$, MacGregor [16] carried a consistent study about the properties and proved that $|a_n| \leq \frac{2}{n}, \text{for} \quad n \geq 2$.

For $r = 4 \quad \text{and} \quad t = p$ in (1.3), we have

$$H_{4,p}(f) = \begin{vmatrix} 1 & a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+1} & a_{p+2} & a_{p+3} & a_{p+4} \\ a_{p+2} & a_{p+3} & a_{p+4} & a_{p+5} \\ a_{p+3} & a_{p+4} & a_{p+5} & a_{p+6} \end{vmatrix}.$$

Expanding the determinant in $H_{4,p}(f)$, through the fourth column, we obtain

$$H_{4,p}(f) = a_{p+6} \triangle p_1 - a_{p+5} \triangle p_2 + a_{p+4} \triangle p_3 - a_{p+3} \triangle_{p_4}.$$

Taking modulus on both sides and applying the triangle inequality to the above expression, we get

$$|H_{4,p}(f)| \leq |a_{p+6}||\triangle p_1| + |a_{p+5}||\triangle p_2| + |a_{p+4}||\triangle p_3| + |a_{p+3}||\triangle p_4|, \tag{1.7}$$

where

$$\triangle p_1 = H_{3,p}(f)$$

$$= \begin{vmatrix} 1 & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix}$$

$$\triangle p_2 = \begin{vmatrix} 1 & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+3} & a_{p+4} & a_{p+5} \end{vmatrix}$$

$$\triangle p_3 = \begin{vmatrix} 1 & a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} & a_{p+4} \\ a_{p+3} & a_{p+4} & a_{p+5} \end{vmatrix}$$

and

$$\triangle p_4 = \begin{vmatrix} a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \\ a_{p+3} & a_{p+4} & a_{p+5} \end{vmatrix}. \tag{1.8}$$

The Hypergeometric function ${}_2F_1(a, b; c; z)$ is defined for $|z| \leq 1$ by the power series

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \\ &= 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \cdots . \end{aligned}$$

It is undefined (or infinite) if $c$ equals a non-positive integer. Here $(q)_n$ is the (rising) Pochhammer symbol, which is defined by:

$$(q)_n = \begin{cases} 1 & n = 0 \\ \\ q(q+1)\cdots(q+n-1) & n > 0 \end{cases}$$

Some examples on Hypergeometric functions.

**Example 1.2.** For $p, m, n \in \mathbb{N}$, then

$$\begin{aligned} {}_2F_1\left[1, \frac{p}{m+n}; 1 + \frac{p}{m+n}; z^{m+n}\right] &= \sum_{k=0}^{\infty} \frac{(1)_k \left(\frac{p}{m+n}\right)_k}{\left(1 + \frac{p}{m+n}\right)_k} \frac{z^{(m+n)k}}{k!} \\ &= 1 + \frac{pz^{m+n}}{m+n+p} + \frac{pz^{2(m+n)}}{2(m+n)+p} \\ &\quad + \frac{pz^{3(m+n)}}{3(m+n)+p} + \cdots . \end{aligned}$$

**Example 1.3.** For $p \in \mathbb{N}$, then

$$
\begin{aligned}
&{}_2F_1\left[1, \frac{2+p}{2}; 1+\frac{2+p}{2}; z^2\right] \\
&= \sum_{k=0}^{\infty} \frac{(1)_k \left(\frac{2+p}{2}\right)_k}{\left(1+\frac{2+p}{2}\right)_k} \frac{z^{2k}}{k!} \\
&= 1 + \frac{(2+p)z^2}{4+p} + \frac{(2+p)z^4}{6+p} + \frac{(2+p)z^6}{8+p} + \cdots .
\end{aligned}
$$

In deriving our results, we make use of the following sharp estimates presented as lemmas, which are applicable to functions with a positive real part.

Define the collection $\mathcal{P}$, of all functions $g$, each one called as Carathéodory function [7] of the form,

$$
g(z) = 1 + \sum_{t=1}^{\infty} c_t z^t, \tag{1.9}
$$

analytic in $\mathbb{D}$ and $\mathrm{Re} g(z) > 0$, for $z \in \mathbb{D}$.

**Lemma 1.4** ([19])**.** *For $g \in \mathcal{P}$, then $|c_i| \leq 2$, for $i \in \mathbb{N}$, equality occurs for the function $h(z) = \frac{1+z}{1-z}$, $z \in \mathbb{D}$.*

**Lemma 1.5** ([11])**.** *If $g \in \mathcal{P}$, then the following estimate holds for the values $i, j \in \mathbb{N}$, with $i > j$.*

$$
|c_i - \mu c_j c_{i-j}| \leq \begin{cases} 2, & \mu \in [0,1], \\ 2|2\mu - 1|, & \mu > 1. \end{cases}
$$

**Lemma 1.6** ([29])**.** *If $g \in \mathcal{P}$, then for $\mu \in \mathbb{R}$ and $i \in \mathbb{N}$ then*

$$
\left|c_{2i} - \mu c_i^2\right| \leq \begin{cases} 2 - \mu|c_i|^2, & \mu < \frac{1}{2}, \\ 2 - (1-\mu)|c_i|^2, & \mu > \frac{1}{2}. \end{cases}
$$

**Lemma 1.7** ([4])**.** *For $g \in \mathcal{P}$, then $|c_r c_s - c_u c_v| \leq 4$, for $r,\ s,\ u,\ v \in \mathbb{N}$ satisfying the condition that $r + s = u + v$.*

**Lemma 1.8.** *For $g \in \mathcal{P}$, then $|c_r c_s - \lambda c_u c_v| \leq 4$, for $r + s = u + v$ with $r, s, u, v \in \mathbb{N}$, where $\lambda \in [0, 1]$.*

*Proof.* Consider

$$
c_r c_s - \lambda c_u c_v = c_r c_s - c_{r+s} + c_{r+s} - \lambda c_u c_v.
$$

Using the condition $r + s = u + v$ in the above expression, we have

$$
c_r c_s - \lambda c_u c_v = c_r c_s - c_{r+s} + c_{u+v} - \lambda c_u c_v.
$$

Taking modulus on both sides in the above expression and then applying the triangle inequality, with the help of Lemmas 1.5 and 1.6, we obtain

$$\begin{aligned}|c_r c_s - \lambda c_u c_v| &\leq |c_r c_s - c_{r+s}| + |c_{u+v} - \lambda c_u c_v| \\ &\leq 2 + 2 = 4,\end{aligned}$$

which completes the proof of Lemma. □

To procure our results, we adopt the procedure framed through Libera and Zlotkiewicz [15].

## 2. Important Outcomes

**Theorem 2.1.** *If $f \in \Re_p$, with $k,\ p \in \mathbb{N}$, then $|a_{p+k}| \leq \frac{2p}{p+k}$ and the inequality is sharp for the function*

$$\begin{aligned}f_0(z) &= z^p + \frac{2p}{p+1} \times {}_2F_1\left[1, 1+p; 2+p; z\right] z^{p+1} \\ &= z^p + \sum_{k=1}^{\infty} \frac{2p}{p+k} z^{p+k}.\end{aligned}$$

*Proof.* For $f \in \Re_p$, there exists an analytic function $g$, such that

$$f'(z) = pz^{p-1} g(z), \quad z \in \mathbb{D}. \tag{2.1}$$

Substitute the values for $f$ and $g$ in (2.1), after simplifying, we get

$$a_{p+k} = \frac{pc_k}{p+k}. \tag{2.2}$$

Taking modulus on both sides in (2.2), upon execution of Lemma 1.4, we obtain

$$|a_{p+k}| \leq \frac{2p}{p+k}. \tag{2.3}$$

□

**Theorem 2.2.** *If $f \in \Re_p$ and $m, n, p \in \mathbb{N}$, then*

$$|a_{p+m} a_{p+n} - a_{p+m+n}| \leq \frac{2p}{p+m+n}$$

*and the inequality is sharp for the function*

$$\begin{aligned}f_1(z) &= -z^p \left(1 - 2 \times {}_2F_1\left[1, \frac{p}{m+n}; 1 + \frac{p}{m+n}; z^{m+n}\right]\right) \\ &= z^p + \frac{2pz^{p+m+n}}{p+m+n} + \frac{2pz^{p+2(m+n)}}{p+2(m+n)} + \cdots.\end{aligned}$$

*Proof.* Using the values of $a_{p+m},\ a_{p+n}$ and $a_{p+m+n}$ from (2.2), we have

$$a_{p+m} a_{p+n} - a_{p+m+n} = \frac{p^2}{(p+m)(p+n)} c_n c_m - \frac{p}{p+m+n} c_{n+m}$$

$$= -\frac{p}{p+m+n}\left(c_{n+m} - \frac{p(p+m+n)}{(p+m)(p+n)}c_n c_m\right).$$

Taking modulus on either side of the above relation and applying Lemma 1.5, we obtain

$$|a_{p+m}a_{p+n} - a_{p+m+n}| \leq \frac{2p}{p+m+n}. \tag{2.4}$$

The above inequality is sharp for $f_1(z)$, for which $a_{p+m} = 0$, $a_{p+n} = 0$ and $a_{p+m+n} = 2p/(p+m+n)$. □

**Remark 2.3.** For $p = 1$ in (2.4), we have

$$|a_{m+1}a_{n+1} - a_{m+n+1}| \leq \frac{2}{m+n+1},$$

which is sharp for the function $f_1(z)$ for $p = 1$, given by

$$f_2(z) = -z\left(1 - 2 \times {}_2F_1\left[1, \frac{1}{m+n}; 1 + \frac{1}{m+n}; z^{m+n}\right]\right)$$

$$= z + \frac{2}{1+m+n}z^{1+m+n} + \frac{2}{1+2(m+n)}z^{1+2(m+n)} + \cdots.$$

**Remark 2.4.** Under Remark 2.3, choosing $m = m-1, n = n-1$, with $m, n \geq 2$, we obtain

$$\begin{aligned} |a_m a_n - a_{m+n-1}| &\leq \frac{2}{m+n-1} \\ &< (m-1)(n-1), \end{aligned}$$

which proves the generalized Zalcman conjecture.

**Remark 2.5.** For $p = 1$, $m = 1$ and $n = 1$ in (2.4), we get

$$\begin{aligned} \left|H_{2,1}(f)\right| &= \left|a_2^2 - a_3\right| \\ &\leq \frac{2}{3}. \end{aligned}$$

This inequality is sharp for the function $f_1(z)$ (by taking $p = m = n = 1$), which is same as

$$\begin{aligned} f_3(z) &= \log\left[\frac{1+z}{1-z}\right] - z \\ &= z + \frac{2}{3}z^3 + \frac{2}{5}z^5 + \frac{2}{7}z^7 + \frac{2}{9}z^9 + \cdots. \end{aligned}$$

**Theorem 2.6.** *If $f \in \Re_p$, $n, p \in \mathbb{N}$ then*

$$|H_{2,p+2n}(f)| \leq \frac{4p^2}{(p+2n)(p+2(n+1))}$$

*and the inequality is sharp for the function*

$$\begin{aligned} f_4(z) &= z^p + \frac{2p}{2+p} \times {}_2F_1\left[1, \frac{2+p}{2}; 1+\frac{2+p}{2}; z^2\right] z^{p+2} \\ &= z^p + \sum_{k=1}^{\infty} \frac{2p}{p+2k} z^{p+2k}. \end{aligned}$$

*Proof.* For $f \in \Re_p$, by taking $r = 2$ and $t = p + 2n$ in (1.3), we have

$$\begin{aligned} H_{2,p+2n}(f) &= \begin{vmatrix} a_{p+2n} & a_{p+2n+1} \\ a_{p+2n+1} & a_{p+2(n+1)} \end{vmatrix} \\ &= a_{p+2n}a_{p+2(n+1)} - a_{p+2n+1}^2. \end{aligned} \tag{2.5}$$

Putting the values of $a_{p+2n}$, $a_{p+2n+1}$ and $a_{p+2n+2}$ from (2.2) in (2.5), after simplifying, we get

$$\begin{aligned} H_{2,p+2n}(f) = &\frac{p^2}{(p+2n)\,(p+2\,(n+1))} \\ &\times \left(c_{2n}c_{2n+2} - \frac{(p+2n)\,(p+2\,(n+1))}{(p+2n+1)^2} c_{2n+1}^2\right). \end{aligned}$$

Taking modulus on both sides of the above expression and execution of Lemma 1.8, we obtain

$$|H_{2,p+2n}(f)| \leq \frac{4p^2}{(p+2n)(p+2(n+1))}. \tag{2.6}$$

The inequality obtained in (2.6) is sharp for the function $f_4(z)$, from which, we have $a_{p+2n} = 2p/(p+2n)$, $a_{p+2n+1} = 0$ and $a_{p+2(n+1)} = 2p/(p+2(n+1))$. □

**Remark 2.7.** For $p = 1$ and $n = 1$ in (2.6), we get

$$|H_{2,3}(f)| \leq \frac{4}{15},$$

which is sharp for the function $f_3(z)$ given under Remark 2.5, it coincides with the result of Zaprawa [29].

**Theorem 2.8.** *If $f \in \Re_p$, with $p \in \mathbb{N}$, then*

$$|\triangle p_2| \leq \left[\frac{8p^2(2p^3+18p^2+45p+22)}{(p+2)^2(p+3)(p+4)(p+5)}\right].$$

*Proof.* Expanding the first column of the determinant $\triangle p_2$, given in (1.8), we obtain

$$\begin{aligned}(2.7)\quad \triangle p_2 = [&a_p(a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) + a_{p+1}(a_{p+2}a_{p+4} - a_{p+1}a_{p+5}) \\ &+ a_{p+3}(a_{p+1}a_{p+3} - a_{p+2}^2)].\end{aligned}$$

Substitute the values of $a_{p+j}$, for $j \in \{0, 1, 2, 3, 4, 5\}$ from (2.2) in (2.7), after simplifying, we get

(2.8)

$$\begin{aligned}\triangle p_2 = p^2\Bigg[&\frac{c_2c_5}{(p+2)(p+5)} - \frac{c_3c_4}{(p+3)(p+4)} - \frac{pc_1^2c_5}{(p+1)^2(p+5)} \\ &+ \frac{pc_1c_3^2}{(p+1)(p+3)^2} - \frac{pc_3c_2^2}{(p+3)(p+2)^2} + \frac{pc_1c_2c_4}{(p+1)(p+2)(p+4)}\Bigg].\end{aligned}$$

On grouping, the terms in (2.8), upon applying the triangle inequality, we obtain

$$\begin{aligned}(2.9)\quad |\triangle p_2| \le p^2\Bigg[&\frac{p|c_5||c_2 - c_1^2|}{(p+1)^2(p+5)} + \frac{p|c_3||c_4 - c_2^2|}{(p+2)^2(p+3)} + \frac{p|c_3|\,|c_4 - c_1c_3|}{(p+1)(p+3)^2} \\ &+ \frac{\left(p^4 + 8p^3 + 22p^2 + 24p + 12\right)|c_4|}{(p+1)(p+2)^2(p+3)^2(p+4)} \\ &\times \left|c_3 - \left(\frac{p(p+2)(p+3)^2}{p^4 + 8p^3 + 22p^2 + 24p + 12}\right)c_1c_2\right| \\ &+ \frac{|c_2||c_5|}{(p+1)^2(p+2)(p+5)}\Bigg].\end{aligned}$$

Execution of appropriate Lemmas 1.4 and 1.5 mentioned in this paper, the inequality in (2.9), simplifies to

$$(2.10)\qquad |\triangle p_2| \le \left[\frac{8p^2(2p^3 + 18p^2 + 45p + 22)}{(p+2)^2(p+3)(p+4)(p+5)}\right]. \qquad \square$$

**Remark 2.9.** For $p = 1$ in the inequality (2.10), it coincides with that of Arif et al. [5].

**Theorem 2.10.** *If $f \in \Re_p$, with $p \in \mathbb{N}$, then*

$$|\triangle p_3| \le \left[\frac{4p^2(4p^5 + 52p^4 + 239p^3 + 451p^2 + 288p + 4)}{(p+1)(p+2)^2(p+3)(p+4)^2(p+5)}\right].$$

*Proof.* Expanding the first row of the determinant $\triangle p_3$, given in (1.8), we obtain

$$\begin{aligned}(2.11)\quad \triangle p_3 = [&a_p(a_{p+3}a_{p+5} - a_{p+4}^2) + a_{p+1}(a_{p+3}a_{p+4} - a_{p+2}a_{p+5}) \\ &+ a_{p+2}(a_{p+2}a_{p+4} - a_{p+3}^2)].\end{aligned}$$

Putting $a_{p+j}$, for $j \in \{0,1,2,3,4,5\}$ values from (2.2) in (2.11), after simplifying, we get

(2.12)
$$\triangle p_3 = p^2\Bigg[\frac{c_3c_5}{(p+3)(p+5)} - \frac{c_4^2}{(p+4)^2} + \frac{pc_1c_3c_4}{(p+1)(p+3)(p+4)} - \frac{pc_1c_2c_5}{(p+1)(p+2)(p+5)} + \frac{pc_2^2c_4}{(p+2)^2(p+4)} - \frac{pc_2c_3^2}{(p+2)(p+3)^2}\Bigg].$$

Performing the same procedure as we carried in the earlier Theorem 2.8, the expression in (2.12) takes the form

(2.13)
$$|\triangle p_3| \leq p^2\Bigg[\frac{p|c_5||c_3 - c_1c_2|}{(p+1)(p+2)(p+5)} + \frac{p|c_4||c_4 - c_2^2|}{(p+2)^2(p+4)} + \frac{p|c_3|\,|c_5 - c_2c_3|}{(p+2)(p+3)^2} + \frac{4|c_4||c_4 - c_1c_3|}{(p+2)^2(p+4)^2} + \frac{\left(p^4+8p^3+16p^2-12\right)|c_3|\,|c_5 - c_1c_4|}{(p+1)(p+2)^2(p+3)(p+4)^2} + \frac{12|c_3||c_5|}{(p+1)(p+2)^2(p+3)^2(p+4)^2(p+5)}\Bigg].$$

Further simplification of the inequality in (2.13) results

$$|\triangle p_3| \leq \left[\frac{4p^2(4p^5+52p^4+239p^3+451p^2+288p+4)}{(p+1)(p+2)^2(p+3)(p+4)^2(p+5)}\right]. \qquad (2.14)$$ □

**Remark 2.11.** For $p = 1$ in the inequality under (2.14), it coincides with that of Arif et al. [5].

**Theorem 2.12.** *If $f \in \Re_p$, with $p \in \mathbb{N}$, then*

$$|\triangle p_4| \leq \frac{24p^3(p^2+6p+6)}{(p+1)(p+2)(p+3)(p+4)(p+5)}.$$

*Proof.* Upon expanding the first column of the determinant $\triangle p_4$, given in (1.8), we obtain

$$\triangle p_4 = [a_{p+1}(a_{p+3}a_{p+5} - a_{p+4}^2) + a_{p+2}(a_{p+3}a_{p+4} - a_{p+2}a_{p+5}) + a_{p+3}(a_{p+2}a_{p+4} - a_{p+3}^2). \qquad (2.15)$$

Substitute the values of $a_{p+j}$, for $j \in \{1,2,3,4,5\}$ from (2.2) in (2.15), after simplifying, we get

$$\triangle p_4 = p^3\Bigg[\frac{c_1c_3c_5}{(p+1)(p+3)(p+5)} - \frac{c_1c_4^2}{(p+1)(p+4)^2} - \frac{c_2^2c_5}{(p+2)^2(p+5)} + \frac{2c_2c_3c_4}{(p+2)(p+3)(p+4)} - \frac{c_3^3}{(p+3)^3}\Bigg]. \qquad (2.16)$$

Applying the same procedure as we carried in the earlier Theorem 2.8, the expression (2.16) takes the form

$$
\begin{aligned}
|\triangle p_4| \leq p^3 \Bigg[ & \frac{|c_5||c_4 - c_2^2|}{(p+2)^2(p+5)} + \frac{|c_5||c_4 - c_1 c_3|}{(p+1)(p+3)(p+5)} \\
& + \frac{|c_3|\left|c_6 - c_3^2\right|}{(p+3)^3} + \frac{|c_3|\,|c_6 - c_2 c_4|}{(p+3)^3} + \frac{|c_4|\,|c_5 - c_1 c_4|}{(p+1)(p+4)^2} \\
& + \frac{\left(p^2+6p+10\right)|c_4|\,|c_5 - c_2 c_3|}{(p+2)(p+3)^3(p+4)} \\
& + \frac{4|c_4||c_5|}{(p+1)(p+2)^2(p+3)^3(p+4)^2(p+5)} \Bigg].
\end{aligned} \tag{2.17}
$$

Upon simplification, the inequality under (2.17) becomes

$$
|\triangle p_4| \leq \frac{24p^3(p^2+6p+6)}{(p+1)(p+2)(p+3)(p+4)(p+5)}. \tag{2.18}
$$

□

**Remark 2.13.** Choosing $p = 1$ in the inequality given in (2.18), it coincides with that of Arif et al. [5].

**Theorem 2.14** ([26])**.** *If $f \in \Re_p$, with $p \in \mathbb{N}$, then*

$$
\begin{aligned}
|\triangle p_1| &= \left|H_{3,p}(f)\right| \\
&\leq \frac{4p^2(6p^3+30p^2+29p+17)}{(p+1)(p+2)(p+3)^2(p+4)}.
\end{aligned} \tag{2.19}
$$

**Remark 2.15.** For $p = 1$ in (2.19), it coincides with the result obtained by Zaprawa [30], i.e., $|H_{3,1}(f)| \leq \frac{41}{60}$.

**Theorem 2.16.** *If $f \in \Re_p$, with $p \in \mathbb{N}$, then*

$$
\left|H_{4,p}(f)\right| \leq 16p^3 \left[ \frac{\begin{array}{c} 10p^8+264p^7+2922p^6+17536p^5+61525p^4 \\ +126431p^3+143420p^2+77118p+13316 \end{array}}{(p+1)(p+2)^2(p+3)^2(p+4)^3(p+5)^2(p+6)} \right]. \tag{2.20}
$$

*Proof.* Substituting the derived results of Theorems 2.1, 2.8, 2.10, 2.12 and 2.14 in the inequality (1.7), after simplifying, it yields the result of Theorem 2.16. □

**Remark 2.17.** In particular $p = 1$ in the inequality under (2.20), the result coincides with the development obtained by Arif et al. [5].

## 3. Bounds of $|H_{3,p}(f)|$ and $|H_{4,p}(f)|$ for 2-Fold and 3-Fold Symmetric Functions

A domain $\Omega$ is said to $k$-fold symmetric if a rotation of $\Omega$ with respect to origin through an angle $\frac{2\pi}{k}$, where $k$ is a natural number, carries $\Omega$ onto itself. For a given class $\mathfrak{T}_{\mathfrak{p}} \subset \mathcal{A}_p$, with $p \in \mathbb{N}$, a mapping $f \in \mathfrak{T}_{\mathfrak{p}}$ is said to be $k$-fold symmetric, if a rotation of the unit disc $\mathbb{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathbb{D}$ onto it self. The collection of all $k$-fold multivalent symmetric mappings of $\mathfrak{T}_{\mathfrak{p}}$ is denoted by ${\mathfrak{T}_{\mathfrak{p}}}^{(k)}$. If $f \in {\mathfrak{T}_{\mathfrak{p}}}^{(k)}$ then $f$ has the following Taylor series expansion,

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+nk} z^{p+nk}. \tag{3.1}$$

The subfamily ${\Re_p}^{(k)}$ of ${\mathfrak{T}_{\mathfrak{p}}}^{(k)}$ is the set of all $k$-fold multivalent symmetric mappings for the class of functions $\Re_p$. The definition of $k$-fold multivalent symmetric mappings can be extended to the class of caratheödory functions $g$, normalized by $g(0) = 1$. A multivalent holomorphic function of the form (3.1), belongs to ${\Re_p}^{(k)}$, if and only if

$$f'(z) = pz^{p-1} g(z), \quad z \in \mathbb{D},\, g \in \mathscr{P}^{(k)},$$

where the set $\mathscr{P}^{(k)}$ is defined by

$$\mathscr{P}^{(k)} = \left\{ g \in \mathscr{P} : g(z) = 1 + \sum_{n=1}^{\infty} c_{nk} z^{nk},\ z \in \mathbb{D} \right\}. \tag{3.2}$$

Now, we can obtain the following results.

**Theorem 3.1.** *If $f \in \Re_p^{(2)}$, with $p \in \mathbb{N}$, then*

$$\left|H_{3,p}(f)\right| \leq \begin{cases} \left(\frac{2}{3}\right)^{\frac{3}{2}} \left(\frac{p^2}{p+4}\right), & p = 1, 2. \\ \\ \frac{4p^2\left(p^2+4p-4\right)}{(p+2)^3(p+4)}, & p \geq 3. \end{cases}$$

*Proof.* Since $f \in \Re_p^{(2)}$, therefore there exists a function $g \in \mathscr{P}^{(2)}$, such that

$$f'(z) = pz^{p-1} g(z). \tag{3.3}$$

For $f \in \Re_p^{(2)}$, substituting the series expressions from (3.1) and (3.2), when $k = 2$ in (3.3), after simplifying, we get

$$a_{p+2j} = \frac{p c_{2j}}{p + 2j}, \quad \text{with } j \in \mathbb{N}. \tag{3.4}$$

In view of the values obtained in (3.4), for $f \in \Re_p^{(2)}$, we have

$$H_{3,p}(f) = a_{p+2}a_{p+4} - a_{p+2}^3. \tag{3.5}$$

From the expressions (3.4) and (3.5), we obtain

$$H_{3,p}(f) = \frac{p^2 c_2}{(p+2)(p+4)}\left[c_4 - \frac{p(p+4)}{(p+2)^2}c_2^2\right]. \tag{3.6}$$

Now, we discuss the following two cases.

**Case 1:** For $p \in \mathbb{N}$, with $p = 1, 2$.

Taking modulus on both sides of (3.6), it takes the form

$$\left|H_{3,p}(f)\right| \leq \frac{p^2}{(p+2)(p+4)}t_1(x), \tag{3.7}$$

$$\text{where } t_1(x) = |c_2|\left|c_4 - \frac{p(p+4)}{(p+2)^2}c_2^2\right|. \tag{3.8}$$

For $p \in \mathbb{N}$, we have $\frac{p(p+4)}{(p+2)^2} > \frac{1}{2}$, therefore, upon using Lemma 1.6 with $|c_2| = x \in [0, 2]$, $t_1(x)$ takes the form

$$t_1(x) = 2x - \frac{4x^3}{(p+2)^2} \tag{3.9}$$

$$t_1'(x) = 2 - \frac{12x^2}{(p+2)^2} \tag{3.10}$$

$$t_1''(x) = -\frac{24x}{(p+2)^2}. \tag{3.11}$$

For extreme values of $t_1(x)$, consider $t_1'(x) = 0$. From (3.10), upon simplification, we obtain

$$x = \frac{(p+2)}{\sqrt{6}} \in [0, 2], \quad \text{for } p = 1, 2. \tag{3.12}$$

From (3.11) and (3.12), we obtain $t_1''(x) < 0$. By the second derivative test, $t_1(x)$ has maximum at $x = (p+2)/\sqrt{6}$, from (3.9), it is computed as

$$\max\, t_1(x) = \frac{4}{3\sqrt{6}}(p+2). \tag{3.13}$$

From (3.7) and (3.13), after simplifying, we obtain

$$\left|H_{3,p}(f)\right| \leq \left(\frac{2}{3}\right)^{\frac{3}{2}}\left(\frac{p^2}{p+4}\right).$$

**Case 2:** For $p \in \mathbb{N}$ with $p \geq 3$, we observe that $t_1(x)$ obtained in (3.9)

is an increaseing function of $x \in [0,2]$ and hence it attains maximum value at $x = 2$ only.

$$\max t_1(x) = t_1(2) = \frac{4(p^2+4p-4)}{(p+2)^2}. \tag{3.14}$$

From (3.7) and (3.14), upon simplification, we obtain

$$\left|H_{3,p}(f)\right| \leq \frac{4p^2\left(p^2+4p-4\right)}{(p+2)^3(p+4)}.$$

□

**Theorem 3.2.** *If $f \in \Re_p^{(3)}$, with $p \in \mathbb{N}$, then*

$$\left|H_{3,p}(f)\right| \leq \left(\frac{2p}{p+3}\right)^2,$$

*and the inequality is sharp for the function $f_1(z)$ with $m = 2$ and $n = 1$ or $m = 1$ and $n = 2$.*

*Proof.* For $f \in \Re_p^{(3)}$, substituting the series expressions from (3.1) and (3.2), for $k = 3$ in the relation (3.3), after simplifying, we get

$$a_{p+3} = \frac{pc_3}{p+3}. \tag{3.15}$$

For $f \in \Re_p^{(3)}$, we have

$$H_{3,p}(f) = -a_{p+3}^2. \tag{3.16}$$

From the expressions (3.15) and (3.16), upon taking modulus on both sides and using coefficient estimate for the class $\mathcal{P}$, we obtain

$$\left|H_{3,p}(f)\right| \leq \left(\frac{2p}{p+3}\right)^2.$$

□

**Theorem 3.3.** *If $f \in \Re_p^{(2)}$, with $p \in \mathbb{N}$, then*

$$\left|H_{4,p}(f)\right| \leq \frac{8p^3}{(p+2)(p+4)(p+6)}.$$

*Proof.* For $f \in \Re_p^{(2)}$, substituting the series expressions from (3.1) and (3.2), for $k = 3$ in the relation (3.3), after simplifying, we get

$$a_{p+n} = \frac{pc_n}{p+n}, \quad \text{for } n \in \{2,4,6\} \text{ and } a_{p+n} = 0, \text{ for } n \in \{1,3,5\}. \tag{3.17}$$

For $f \in \Re_p^{(2)}$, in view of the obtained values in (3.17), we obtain

$$H_{4,p}(f) = (a_{p+2}a_{p+6} - a_{p+4}^2)(a_{p+4} - a_{p+2}^2). \tag{3.18}$$

Substituting the values of $a_{p+2j}$, for $j \in \{1,2,3\}$ from the expression (3.17) in (3.18), after simplifying, we get

$$H_{4,p}(f) = \left[\frac{p^3 c_2 c_4 c_6}{(p+2)(p+4)(p+6)} - \frac{p^4 c_2^3 c_6}{(p+2)^3(p+6)} + \frac{p^4 c_2^2 c_4^2}{(p+2)^2(p+4)^2} - \frac{p^3 c_4^3}{(p+4)^3}\right].$$

By re-arranging the terms in the above expression, in order to apply Lemmas mentioned in this paper, we obtain

$$H_{4,p}(f) = \frac{p^3}{(p+2)(p+4)(p+6)}\left[c_2 c_6 - \frac{(p+2)(p+6)}{(p+4)^2}c_4^2\right]\left[c_4 - \frac{p(p+4)}{(p+2)^2}c_2^2\right].$$

Taking modulus on both sides and then applying the Lemmas 1.5 and 1.8 in the above expression, it yields

$$\left|H_{4,p}(f)\right| \leq \frac{8p^3}{(p+2)(p+4)(p+6)}. \tag{3.19}$$ □

**Remark 3.4.** For $p = 1$ in (3.19), we get $|H_{4,1}(f)| \leq \frac{8}{105}$, this bound is more refined than the bound developed by Arif et al. [5].

**Theorem 3.5.** *If $f \in \Re_p^{(3)}$, with $p \in \mathbb{N}$, then*

$$\left|H_{4,p}(f)\right| \leq \begin{cases} \left(\dfrac{p}{p+6}\right)^2, & p = 1. \\ \dfrac{p^3}{9(p+6)}, & p = 2,3. \\ \dfrac{8p^3\left(p^2+6p-9\right)}{(p+3)^4(p+6)}, & p \geq 4. \end{cases}$$

*Proof.* For $f \in \Re_p^{(3)}$, substituting the series expressions from (3.1) and (3.2), for $k = 3$ in the relation (3.3), after simplifying, we get

$$a_{p+1} = a_{p+2} = a_{p+4} = a_{p+5} = 0, \quad a_{p+3} = \frac{pc_3}{p+3}, \quad a_{p+6} = \frac{pc_6}{p+6}, \tag{3.20}$$

consequently, for $f \in \Re_p^{(3)}$, we have

$$H_{4,p}(f) = -a_{p+3}^2 a_{p+6} + a_{p+3}^4. \tag{3.21}$$

Substitute $a_{p+3}$ and $a_{p+6}$ values from (3.20) in (3.21), it simplifies to

$$H_{4,p}(f) = -\frac{p^3 c_3^2}{(p+3)^2(p+6)}\left[c_6 - \frac{p(p+6)}{(p+3)^2}c_3^2\right].$$

Taking modulus on both sides in the above expression, it takes the form

$$\left|H_{4,p}(f)\right| \leq \frac{p^3}{(p+3)^2(p+6)} t_2(x), \tag{3.22}$$

where

$$t_2(x) = \left| c_6 - \frac{p(p+6)}{(p+3)^2} c_3^2 \right| |c_3|^2. \tag{3.23}$$

Now, we discuss the following cases.

**Case 1:** For $p = 1$, we have $\frac{p(p+6)}{(p+3)^2} < \frac{1}{2}$, therefore, upon using Lemma 1.6 with $|c_3| = x \in [0, 2]$, $t_2(x)$ in (3.23) takes the form

$$t_2(x) = \left[2 - \frac{p(p+6)}{(p+3)^2} x^2\right] x^2. \tag{3.24}$$

then

$$t_2'(x) = \left[4x - \frac{4p(p+6)}{(p+3)^2} x^3\right]. \tag{3.25}$$

$$t_2''(x) = \left[4 - \frac{12p(p+6)}{(p+3)^2} x^2\right]. \tag{3.26}$$

For extreme values of $t_2(x)$, consider $t_2'(x) = 0$. From (3.25), we obtain

$$x = \sqrt{\frac{(p+3)^2}{p(p+6)}}. \tag{3.27}$$

From (3.26) and (3.27), we observe that $t_2''(x) < 0$. Therefore, by the second derivative test, $t_2(x)$ has maximum value at $x$, obtained in (3.27), for which, from (3.24), it is computed as

$$\max\, t_2(x) = \frac{(p+3)^2}{p(p+6)}. \tag{3.28}$$

From (3.22) and (3.28), after simplifying, we get

$$\left|H_{4,p}(f)\right| \leq \left(\frac{p}{p+6}\right)^2.$$

**Case 2:** For $p = 2, 3$, we have $\frac{p(p+6)}{(p+3)^2} > \frac{1}{2}$, therefore, upon using Lemma 1.6 with $|c_3| = x \in [0, 2]$, $t_2(x)$ given in (3.23) takes the form

$$t_2(x) = \left[2 - \frac{9}{(p+3)^2} x^2\right] x^2, \tag{3.29}$$

then

$$t_2'(x) = \left[4x - \frac{36}{(p+3)^2} x^3\right]. \tag{3.30}$$

$$t_2''(x) = \left[4 - \frac{108}{(p+3)^2}x^2\right]. \tag{3.31}$$

For extreme values of $t_2(x)$, consider $t_2'(x) = 0$. From (3.30), we get

$$x = \frac{(p+3)}{3} \in [0,2], \quad \text{for } p = 2,3. \tag{3.32}$$

From (3.31) and (3.32), we observe that $t_2''(x) < 0$. Therefore, by the second derivative test, $t_2(x)$, has maximum value at $x$, obtained in (3.32), for which, from (3.29), it is computed as

$$\max\, t_2(x) = \left(\frac{p+3}{3}\right)^2. \tag{3.33}$$

From (3.22) and (3.33), we obtain

$$\left|H_{4,p}(f)\right| \leq \frac{p^3}{9(p+6)}. \tag{3.34}$$

**Case 3:** For $p \in \mathbb{N}$ with $p \geq 4$, we observe that $t_2(x)$ given in (3.29) under **Case 2** is an increaseing function of $x \in [0,2]$ and hence it attains maximum value at $x = 2$ only.

$$\begin{aligned} \max t_2(x) &= t_2(2) \\ &= \frac{8(p^2+6p-9)}{(p+3)^2}. \end{aligned} \tag{3.35}$$

From (3.22) and (3.35), after simplifying, we get

$$\left|H_{4,p}(f)\right| \leq \frac{8p^3\left(p^2+6p-9\right)}{(p+3)^4(p+6)}. \tag{3.36}$$

□

**Remark 3.6.** For $p = 1$ in Case 1, we get $|H_{4,1}(f)| \leq \frac{1}{49}$, it coincides with the bound developed by Arif et al. [5].

## 4. Conclusion

In the present paper, we have found a sharp upper bound to a specific generalized second Hankel determinant, the Zalcman conjecture, and an upper bound to the third and fourth Hankel determinants for multivalent bounded turning functions. Also, we estimate an upper bound for third and fourth Hankel determinants regarding two-fold and three-fold symmetric functions belonging to the class of $p$- valent bounded turning functions. Estimating a sharp upper bound to the third and fourth Hankel determinants for multivalent bounded turning functions remains an open problem.

We conclude our present investigation by drawing the attention of the interested readers toward the potential for further researchers developing

similar results as those reported in [31] for the third and fourth Hankel determinants.

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