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The Hölder and Minkowski Inequalities Utilizing a Fractional Operator Involvement of Pseudo-Operator

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Abstract. A generalized integral operator of order α of a real function f including a parameter set P, namely $K_P^\alpha f(t)$ has been introduced by O. P. Agrawal (Computers and Mathematics with Applications, 59 (2010) 1852–1864), which is a generalization of some important fractional integrals such as the Riemann-Liouville fractional integral. Using pseudo-analysis, this paper introduces a pseudo-operator integral of order α including a parameter set P in a semiring $([a,b],\oplus,\odot)$, which is a generalization of $K_P^\alpha f(t)$. We also discuss some particular cases and we obtain the well-known Hölder's and Minkowski's inequalities for this kind of pseudo-operator integral. The results given in this paper provide a generalization of several inequalities obtained in earlier studies.

1. Introduction

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers, a semiring is defined on a real interval $[a,b] \subset [-\infty,\infty]$ with pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ (see, e.g., [17, 19–22, 26]). In 2017, Babakhani et al. [9] generalized some concepts of pseudo-analysis using fractional calculus. Recently, several classical inequalities have been proved for pseudo-integrals (see, e.g., [1–3, 6, 10, 16, 21, 25, 31, 32]). In recent years, some authors have also dealt with the study of pseudo-analysis using fractional calculus (see, e.g., [7, 18, 29, 30]).

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On the other hand, an important part of the classical analysis is the well-known Hölder's and Minkowski's [24] inequalities, which is mentioned in the next theorem.

Definition 1.1. If r and s are positive real numbers such that $\frac{1}{r}+\frac{1}{s}=1$, then we call r and s a pair of conjugate exponents.

Theorem 1.2. *Let r and s be conjugate exponents, $1<r<\infty$. Let X be a measure space with measure μ. Let u and v be measurable functions from X into $[0,\infty]$. Then*

(i) (Hölder's inequality)

$$\int_X uv\,\mathrm{d}\mu \leq \left(\int_X u^p \mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_X v^q \mathrm{d}\mu\right)^{\frac{1}{q}},$$

(ii) (Minkowski's inequality)

$$\left(\int_X (u+v)^p \mathrm{d}\mu\right)^{\frac{1}{p}} \leq \left(\int_X u^p \mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X v^p \mathrm{d}\mu\right)^{\frac{1}{p}}.$$

In recent years, there has been a significant development in classical inequalities using fractional calculus (see, e.g., [5, 8, 11, 12, 27, 28]). Motivated by the previous works, this paper generalizes Agarwal's suggested integral operator using pseudo-analysis. Through this new generalization, we prove Hölder's and Minkowski's inequalities.

This paper is organized as follows: Section 2.2 contains some basic concepts and definitions about pseudo-analysis. In Section 3, we define the pseudo-operator $K^{\alpha}_{P,\oplus,\odot}$ and present some of its properties. Finally, Section 4 provides a generalization of Hölder's and Minkowski's inequalities utilizing the pseudo-operator $K^{\alpha}_{P,\oplus,\odot}$.

2. Preliminaries

The main results of this manuscript require some basic preliminaries such as the pseudo-analysis and the definition of the operator K^{α}_{P} (K^{α}_{P}–op), which are collected in this section. We recall K^{α}_{P}–op and describe some of its properties below.

2.1. The Operator K of Order α and a P-Set.

In 2010, Agrawal [4] proposed a new operator K^{α}_{P}, which is called K–op (or operator K) of order α and a P-set (or parameter set) P, and $K^{\alpha}_{P}f(t)$ as K–opn (or operation K) of $f(t)$ (or a function $f(t)$ of order α and P-set P). In special cases, $K^{\alpha}_{P}f(t)$ reduces to other fractional operators such as the left and the right fractional Riemann-Liouville integrals, the Riesz fractional integral and the Weyl fractional integral.

Definition 2.1 ([4])**.** The operation K of function $f(t)$ of order α and P-set P is defined by

$$\begin{aligned} K^{\alpha}_{\langle a,t,b,p,q\rangle} f(t) &= p\int_a^t k_\alpha(t,s)f(s)\mathrm{d}s + q\int_t^b k_\alpha(s,t)f(s)\mathrm{d}s \\ &= K^{\alpha}_{P} f(t), \end{aligned} \tag{2.1}$$

where $t \in (a,b)$, $p,q \in \mathbb{R}$, $P = \langle a,t,b,p,q\rangle$ and $k_\alpha(t,s)$ is a kernel that is nonnegative and depends upon a parameter $\alpha > 0$.

Remark 2.2. Let $k_\alpha(t,s) := \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}$ and $P = \langle a,t,b,p,q\rangle$.

(i) If $P = P_1 = \langle a,t,b,1,0\rangle$, then the left Riemann-Liouville fractional integral [14] is yielded, i.e.,

$$\begin{aligned} K^{\alpha}_{P_1} f(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\mathrm{d}s \\ &= \mathbb{J}^{\alpha}_{a+} f(t). \end{aligned}$$

(ii) If $P = P_2 = \langle a,t,b,0,1\rangle$, then the right Riemann-Liouville fractional integral [14] is concluded as

$$\begin{aligned} K^{\alpha}_{P_2} f(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1} f(s)\mathrm{d}s \\ &= \mathbb{J}^{\alpha}_{b-} f(t). \end{aligned}$$

(iii) If $P = P_3 = \left\langle a,t,b,\frac{1}{2},\frac{1}{2}\right\rangle$, then we have the Riesz fractional integral [14] as follows:

$$\begin{aligned} K^{\alpha}_{P_3} f(t) &= \frac{1}{2}K^{\alpha}_{P_1} f(t) + \frac{1}{2}K^{\alpha}_{P_2} f(t) \\ &= \frac{1}{2}\mathbb{J}^{\alpha}_{a+} f(t) + \frac{1}{2}\mathbb{J}^{\alpha}_{b-} f(t). \end{aligned}$$

(iv) If $P = P_4 = \langle a,t,\infty,0,1\rangle$, then $K^{\alpha}_{P_4} f(t)$ reduces to the Weyl fractional integral $W^{\alpha} f(t)$ ([14], p. 456) as follows:

$$W^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_t^{\infty} (s-t)^{\alpha-1} f(s)ds$$

The operator K^{α}_{P} is linear, i.e., if $\lambda \in \mathbb{R}$ and $f(t)$ and $g(t)$ are two functions, then

$$\begin{aligned} K^{\alpha}_{P}(f(t)+h(t)) &= K^{\alpha}_{P} f(t) + K^{\alpha}_{P} h(t) \text{ and } K^{\alpha}_{P}(\lambda f(t)) \\ &= \lambda K^{\alpha}_{P}\left(f(t)\right), \end{aligned}$$

and also, it satisfies the following properties:

Theorem 2.3. *The operator K^{α}_{P} satisfies the following properties:*

(i) $K^{\alpha}_{P} f(t) = K^{\alpha}_{P_1} f(t) + K^{\alpha}_{P_2} f(t)$, *where* $P = \langle a,t,b,p,q\rangle, P_1 = \langle a,t,b,1,0\rangle$ *and* $P_2 = \langle a,t,b,0,1\rangle$.

(ii) *The operator K_P^α satisfies the integration by parts formula, i.e.,*

$$\int_a^b g(t)K_P^\alpha f(t)dt = \int_a^b f(t)K_{P^*}^\alpha g(t)dt,$$

where $P = \langle a, t, b, p, q\rangle$ and $P^* = \langle a, t, b, q, p\rangle$.

2.2. **Pseudo-Analysis.** We begin this subsection with definitions and some other basic properties of pseudo-analysis. More details can be seen in [19, 21–23].

Definition 2.4 ([19, 21])**.** A binary operation $\oplus$ on $[a, b]$ is pseudo-addition if it is commutative, non-decreasing (with respect to $\preceq$), continuous, associative and with a zero (neutral) element denoted by $\mathbf{0}$. Let $[a, b]_+ = \{x \,|\, x \in [a, b], \mathbf{0} \preceq x\}$. A binary operation $\odot$ on $[a, b]$ is pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and with a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b], \mathbf{1} \odot x = x$. Also, $\mathbf{0} \odot x = \mathbf{0}$ and $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a, b], \oplus, \odot)$ is a semiring (see [15]).

Definition 2.5 ([22])**.** An important class of pseudo-operations $\oplus$ and $\odot$ is in the case that these pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations $\oplus$ and $\odot$ are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \quad \text{and} \quad x \odot y = g^{-1}(g(x)\, g(y)).$$

Definition 2.6 ([22, 23])**.** Let X be a non-empty set and $\mathcal{A}$ be a σ-algebra of subsets of X. A set $\mu : \mathcal{A} \to [a, b]$ is called a σ-$\oplus$-measure if it satisfies the following conditions:

(i) $\mu(\emptyset) = \mathbf{0}$;

(ii) The equation $\mu\left(\bigcup_{i=1}^{\infty} \mathcal{A}_i\right) = \bigoplus_{i=1}^{\infty} \mu(\mathcal{A}_i)$ holds for any sequence $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{A}$.

Definition 2.7 ([22, 23])**.** Let the pseudo-operations $\oplus$ and $\odot$ be defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$. The g-integral for a measurable function $f : [c, d] \to [a, b]$ is given by

$$\int_{[c,d]}^{\oplus} f \odot \mathrm{d}x = g^{-1}\left(\int_c^d g(f(x))\mathrm{d}x\right).$$

This operator is linear, i.e., if $\lambda \in \mathbb{R}$, $f(t)$ and $h(t)$ are two functions, then

$$\int_{[c,d]}^{\oplus} (\lambda \odot f) \odot \mathrm{d}x = \lambda \odot \int_{[c,d]}^{\oplus} f \odot \mathrm{d}x, \quad \text{and}$$
$$\int_{[c,d]}^{\oplus} (f + h) \odot \mathrm{d}x = \int_{[c,d]}^{\oplus} f \odot \mathrm{d}x + \int_{[c,d]}^{\oplus} h \odot \mathrm{d}x.$$

Definition 2.8 ([19, 23])**.** Let g be an additive generator of the strict pseudo-addition $\oplus$ on $[a, b]$ such that g is continuously differentiable on (a, b). The corresponding pseudo-multiplication $\odot$ will be always defined by $u \odot v = g^{-1}(g(u)g(v))$. If the function f is differentiable on (c, d) and has the same monotonicity as g, then the g-derivative of f at a point $x \in (c, d)$ is defined by

$$\frac{\mathrm{d}^{\oplus} f(x)}{\mathrm{d}x} = g^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}x} g(f(x))\right).$$

Also, if the n-g-derivative of f exists, then

$$\frac{\mathrm{d}^{(n)\oplus} f(x)}{\mathrm{d}x} = g^{-1}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} g(f(x))\right).$$

Definition 2.9 ([17])**.** Let g be a generator of the pseudo-addition $\oplus$ on the interval $[-\infty, +\infty]$. The binary operations $\ominus$ and $\oslash$ on $[-\infty, +\infty]$ are defined by the formulae

$$x \ominus y = g^{-1}(g(x) - g(y)), \quad x \oslash y = g^{-1}\left(\frac{g(x)}{g(y)}\right)$$

whenever the expressions $g(x) - g(y)$ and $\frac{g(x)}{g(y)}$ have sense. These operations are said to be the pseudo-subtraction and the pseudo-division consistent with the pseudo-addition $\oplus$, respectively.

Definition 2.10 ([17])**.** Let $g : [-\infty, +\infty] \to [-\infty, +\infty]$ be a continuous, strictly increasing and odd function such that $g(0) = \mathbf{0}$, $g(1) = \mathbf{1}$, and $g(+\infty) = +\infty$. Then the system $\{\oplus, \ominus, \odot, \oslash\}$ consisting of pseudo-arithmetical operations generated by g is said to be a consistent system.

3. The Pseudo-Operator $K^{\alpha}_{P,\oplus,\odot}$

In this section, we define the pseudo-operator $K^{\alpha}_{P,\oplus,\odot}$ ($K^{\alpha}_{P,\oplus,\odot^-}$ ps–op), which reduces to the left and the right Riemann-Liouville pseudo-fractional integrals [6, 7, 9] and the operator K^{α}_{P} in some special cases.

Definition 3.1. Suppose that a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is an increasing

function. The $K^{\alpha}_{P,\oplus,\odot}$– ps–op of a measurable function $f:[a,b]\to[a,b]$ is defined by

$$K^{\alpha}_{P,\oplus,\odot}f(t)=\left(\int_{[a,\,t]}^{\oplus}g^{-1}\left(pk_{\alpha}(t,s)\right)\odot\ f(s)\mathrm{d}s\right)\oplus\left(\int_{[t,\,b]}^{\oplus}g^{-1}\left(qk_{\alpha}(s,t)\right)\odot\ f(s)\mathrm{d}s\right), \tag{3.1}$$

where $t\in(a,b)$, $p,q\in\mathbb{R}$, $P=\langle a,t,b,p,q\rangle$ and $k_{\alpha}(t,s)$ is a kernel that is nonnegative and depends upon a parameter $\alpha>0$.

Theorem 3.2. *Suppose that a generator $g:[a,b]\to[0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is an increasing function. Then*

(i)

$$K^{\alpha}_{P,\oplus,\odot}f(t)=g^{-1}\left(p\int_{a}^{t}k_{\alpha}(t,s)g(f(s))\mathrm{d}s+q\int_{t}^{b}k_{\alpha}(s,t)g(f(s))\mathrm{d}s\right)$$

$$=g^{-1}\left\{K^{\alpha}_{P}\left(g\left(f(t)\right)\right)\right\}. \tag{3.2}$$

(ii) *If a generator g is linear, then $K^{\alpha}_{P,\oplus,\odot}$ – ps– op is also linear and*

$$K^{\alpha}_{P,\oplus,\odot}\left(\lambda_1f_1(t)+\lambda_2f_2(t)\right)=K^{\alpha}_{P,\oplus,\odot}\left(\lambda_1f_1(t)\oplus\lambda_2f_2(t)\right),$$

where $t\in(a,b)$, $p,q\in\mathbb{R}$, $P=\langle a,t,b,p,q\rangle$ and $k_{\alpha}(t,s)$ is a kernel that is nonnegative and depends upon a parameter $\alpha>0$.

Proof. (i) In view of Definitions 2.5 and 3.1, we have

(3.3)

$$\begin{aligned}K^{\alpha}_{P,\oplus,\odot}f(t)&=g^{-1}\left(p\int_{a}^{t}k_{\alpha}(t,s)g(f(s))\mathrm{d}s\right)\\&\quad\oplus g^{-1}\left(q\int_{t}^{b}k_{\alpha}(s,t)g\left(f(s)\right)\mathrm{d}s\right)\\&=g^{-1}\left[g\left(g^{-1}\left(p\int_{a}^{t}k_{\alpha}(t,s)g\left(f(s)\right)\mathrm{d}s\right)\right)\right.\\&\quad\left.+g\left(g^{-1}\left(q\int_{t}^{b}k_{\alpha}(s,t)g\left(f(s)\right)\mathrm{d}s\right)\right)\right]\\&=g^{-1}\left(p\int_{a}^{t}k_{\alpha}(t,s)g\left(f(s)\right)\mathrm{d}s+q\int_{t}^{b}k_{\alpha}(s,t)g\left(f(s)\right)\mathrm{d}s\right)\\&=g^{-1}\left\{K^{\alpha}_{P}\left(g\left(f(t)\right)\right)\right\}.\end{aligned}$$

(ii) Suppose that f_1 and f_2 are two functions and λ_1 and λ_2 are two real numbers. Then using (3.2) and the linearity of $K-op$, we have

$$\begin{aligned}
&K^{\alpha}_{P,\oplus,\odot}\left(\lambda_1 f_1(t)+\lambda_2 f_2(t)\right)\\
&\quad= g^{-1}\left\{K^{\alpha}_{P}\left(g\left(\lambda_1(f_1(t)+\lambda_2 f_2(t))\right)\right)\right\}\\
&\quad= g^{-1}\left\{\lambda_1 K^{\alpha}_{P}\left(g\left((f_1(t)\right)\right\}+g^{-1}\left\{\lambda_2 K^{\alpha}_{P} g\left(f_2(t)\right)\right)\right\}\\
&\quad= \lambda_1 K^{\alpha}_{P,\oplus,\odot}\left(f_1(t)\right)+\lambda_2 K^{\alpha}_{P,\oplus,\odot}\left(f_2(t)\right).
\end{aligned}$$

On the other hand, from the linearity of g and g^{-1}, we have

$$\begin{aligned}
K^{\alpha}_{P,\oplus,\odot}\left(\lambda_1 f_1(t)+\lambda_2 f_2(t)\right) &= K^{\alpha}_{P,\oplus,\odot}\left(g^{-1}\left\{g\left(\lambda_1 f_1(t)\right)+g\left(\lambda_2 f_2(t)\right)\right\}\right)\\
&= K^{\alpha}_{P,\oplus,\odot}\left(\lambda_1 f_1(t)\oplus\lambda_2 f_2(t)\right).
\end{aligned}$$

Therefore, the theorem is proved completely. □

Theorem 3.3. *The operator $K^{\alpha}_{P,\oplus,\odot}$ satisfies the following integration by parts formula:*

$$\int_a^b h(t)g\left(K^{\alpha}_{P,\oplus,\odot}f(t)\right)dt=\int_a^b g\left(f(s)\right)K^{\alpha}_{P^*,\oplus,\odot}\left(g^{-1}\left(h(s)\right)\right)ds, \tag{3.4}$$

where $P=\langle a,t,b,p,q\rangle$ and $P^*=\langle a,t,b,q,p\rangle$.

Proof. Theorem 3.2 yields that

$$K^{\alpha}_{P,\oplus,\odot}f(t)=g^{-1}\left(p\int_a^t k_\alpha(t,s)g\left(f(s)\right)\mathrm{d}s+q\int_t^b k_\alpha(s,t)g\left(f(s)\right)\mathrm{d}s\right).$$

Hence,

$$g\left(K^{\alpha}_{P,\oplus,\odot}f(t)\right)=p\int_a^t k_\alpha(t,s)g\left(f(s)\right)\mathrm{d}s+q\int_t^b k_\alpha(s,t)g\left(f(s)\right)\mathrm{d}s. \tag{3.5}$$

By using (3.5) and changing the order of the integration, we get

$$\begin{aligned}
&\int_a^b h(t)g\left(K^{\alpha}_{P,\oplus,\odot}f(t)\right)dt\\
&\quad= p\int_a^b h(t)\int_a^t k_\alpha(t,s)g\left(f(s)\right)\mathrm{d}s\mathrm{d}t+q\int_a^b h(t)\int_t^b k_\alpha(s,t)g\left(f(s)\right)\mathrm{d}s\mathrm{d}t\\
&\quad= p\int_a^b\int_a^t h(t)k_\alpha(t,s)g\left(f(s)\right)\mathrm{d}s\mathrm{d}t+q\int_a^b\int_t^b h(t)k_\alpha(s,t)g\left(f(s)\right)\mathrm{d}s\mathrm{d}t\\
&\quad= p\int_a^b\int_s^b h(t)k_\alpha(t,s)g\left(f(s)\right)\mathrm{d}t\mathrm{d}s+q\int_a^b\int_a^s h(t)k_\alpha(s,t)g\left(f(s)\right)dt\mathrm{d}s\\
&\quad= \int_a^b g\left(f(s)\right)\left\{p\int_s^b h(t)k_\alpha(t,s)\mathrm{d}t\mathrm{d}s+q\int_a^s h(t)k_\alpha(s,t)dt\right\}ds
\end{aligned} \tag{3.6}$$

$$= \int_a^b g\left(f(s)\right) K^{\alpha}_{P^*,\oplus,\odot}\left(g^{-1}\left(h(s)\right)\right) ds.$$

So the result is obtained. □

In the next theorem, we define the reflection operator Q such that $(Qf)(t) = f(a+b-t)$.

Theorem 3.4. *If $k_\alpha(t,\tau) = k_\alpha(t-\tau)$, then the operators $K^{\alpha}_{P,\oplus,\odot}$ and Q satisfy the following identity:*

$$QK^{\alpha}_{P,\oplus,\odot} f(t) = K^{\alpha}_{P^*,\oplus,\odot} Qf(t),$$

where $P = \langle a,t,b,p,q\rangle$ and $P^* = \langle a,t,b,q,p\rangle$.

Proof. From the definitions of $K^{\alpha}_{P^*,\oplus,\odot} - ps - op$ and Q, we have

(3.7)

$$\begin{aligned}
&QK^{\alpha}_{P^*,\oplus,\odot} f(t)\\
&= Q\left(g^{-1}\left(p\int_a^t k_\alpha(t,s)g(f(s))ds\right) \oplus g^{-1}\left(q\int_t^b k_\alpha(s,t)g(f(s))ds\right)\right)\\
&= Q\left(g^{-1}\left(p\int_a^t k_\alpha(t-s)g(f(s))ds\right) \oplus g^{-1}\left(q\int_t^b k_\alpha(s-t)g(f(s))ds\right)\right)\\
&= g^{-1}\left(p\int_a^{a+b-t} k_\alpha(a+b-t-s)g(f(s))ds\right)\\
&\quad \oplus g^{-1}\left(q\int_{a+b-t}^b k_\alpha(s-a-b+t)g(f(s))ds\right).
\end{aligned}$$

Setting $a+b-s=u$ in (3.7) we find

$$\begin{aligned}
QK^{\alpha}_{P^*,\oplus,\odot} f(t) &= g^{-1}\left\{Q\left(p\int_t^b k_\alpha\left(u-(a+b-t)\right) g(f(a+b-u))du\right)\right\}\\
&\quad \oplus g^{-1}\left\{Q\left(q\int_t^a k_\alpha\left((a+b-t)-u\right) g(f(a+b-u))du\right)\right\}\\
&= K^{\alpha}_{P^*,\oplus,\odot} Qf(t).
\end{aligned}$$

□

We now consider several special cases of $K^{\alpha}_{P,\oplus,\odot}$ with the following fractional integral formulae [6, 7, 10]:

Case 1: If we put $g(t) = t$, $\oplus = +$ and $\odot = \times$ in (3.3), then we get (2.1).

Case 2: For $P = P_1 = \langle a,t,b,1,0\rangle$ and $k_\alpha(t,s) = (t-s)^{\alpha-1}/\Gamma(\alpha)$, the left Riemann-Liouville pseudo-fractional integral is yielded, i.e.,

$$\begin{aligned}
K^{\alpha}_{P_1,\oplus,\odot} f(t) &= \mathbb{J}_{\oplus,\odot,a^+} f(t)\\
&= \int_{[a,t]}^{\oplus} g^{-1}\left(\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\right) \odot f(s)ds.
\end{aligned}$$

Case 3: For $P = P_2 = \langle a, t, b, 0, 1\rangle$ and $k_\alpha(t,s) = (s-t)^{\alpha-1}/\Gamma(\alpha)$, the right Riemann-Liouville pseudo-fractional integral is yielded, i.e.,

$$\begin{aligned} K^{\alpha}_{P_2,\oplus,\odot} f(t) &= \mathbb{J}_{\oplus,\odot,b^-} f(t) \\ &= \int_{[t,\,b]}^{\oplus} g^{-1}\left(\frac{1}{\Gamma(\alpha)}(s-t)^{\alpha-1}\right) \odot f(s)ds. \end{aligned}$$

Case 4: If we put $g(t) = t$ and $K^{\alpha}_{P,\oplus,\odot} = K^{\alpha}_{P}$, then (3.4) reduces to Theorem 2.3(ii).

4. Generalized Hölder's and Minkowski's Inequalities

In this section, we develop Hölder's and Minkowski's inequalities utilizing $K^{\alpha}_{P,\oplus,\odot^-}$ ps–op. We first go on the Hölder's inequality.

Theorem 4.1 (Hölder's $K^{\alpha}_{P,\oplus,\odot}$–ps–op inequality)**.** *Let r and s be conjugate exponents with $1 < r < \infty$. Assume that two functions f and h are both measurable for $t > a$. Let $g : [a,b] \to [0,\infty]$ be a generator of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$. Then the following inequality holds:*

$$\left[K^{\alpha}_{P,\oplus,\odot}(f \odot h)(t)\right] \le \left[K^{\alpha}_{P,\odot,\oplus} f^r_{\odot}(t)\right]^{\frac{1}{r}}_{\odot} \odot \left[K^{\alpha}_{P,\odot,\oplus} f^s_{\odot}(t)\right]^{\frac{1}{s}}_{\odot}, \tag{4.1}$$

where $t \in (a,b)$, $p, q \in \mathbb{R}$, $P = \langle a, t, b, p, q\rangle$ and $k_\alpha(t,s)$ is a kernel that is nonnegative and depends upon a parameter $\alpha > 0$.

Proof. Taking into account the Young's inequality

$$ab \le \frac{a^r}{r} + \frac{b^s}{s}$$

with $a = f(\tau)/A$ and $b = h(\tau)/B$ for nonzero values A and B we have

$$f(\tau)h(\tau) \le AB\left(\frac{f^r(\tau)}{rA^r} + \frac{h^s(\tau)}{sB^s}\right). \tag{4.2}$$

Multiplying both sides of (4.2) by $p\,k_\alpha(t,\tau)$ and then integrating the result with respect to τ over $[a,t]$ we obtain

$$\begin{aligned} &p\int_a^t k_\alpha(t,\tau)f(\tau)h(\tau)\mathrm{d}\tau \\ &\quad\le AB\left(\frac{p}{rA^r}\int_a^t k_\alpha(t,\tau)f^r(\tau)\mathrm{d}\tau + \frac{p}{sB^s}\int_a^t k_\alpha(t,\tau)h^s(\tau)\mathrm{d}\tau\right). \end{aligned} \tag{4.3}$$

Similarly, we have

$$q\int_t^b k_\alpha(\tau,t)f(\tau)h(\tau)\mathrm{d}\tau \tag{4.4}$$

$$\leq AB\left(\frac{q}{rA^r}\int_t^b k_\alpha(\tau,t)f^r(\tau)\mathrm{d}\tau+\frac{q}{sB^s}\int_t^b k_\alpha(\tau,t)h^s(\tau)\mathrm{d}\tau\right).$$

By adding (4.3) and (4.4), we obtain

$$\begin{aligned}
&p\int_a^t k_\alpha(t,\tau)f(\tau)h(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)f(\tau)h(\tau)\mathrm{d}\tau\\
&\quad\leq AB\left[\frac{1}{rA^r}\left(p\int_a^t k_\alpha(t,\tau)f^r(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)f^r(\tau)\mathrm{d}\tau\right)\right.\\
&\qquad\left.+\frac{1}{sB^s}\left(p\int_a^t k_\alpha(t,\tau)h^s(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)h^s(\tau)\mathrm{d}\tau\right)\right].
\end{aligned}$$

Since g is increasing, g^{-1} is also increasing and we have

(4.5)

$$\begin{aligned}
&g^{-1}\left\{p\int_a^t k_\alpha(t,\tau)(g\circ f)(\tau)(g\circ h)(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)(g\circ f)(\tau)(g\circ h)(\tau)\mathrm{d}\tau\right\}\\
&\quad\leq g^{-1}\left\{AB\left[\frac{1}{rA^r}\left(p\int_a^t k_\alpha(t,\tau)(g\circ f)^r(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)(g\circ f)^r(\tau)\mathrm{d}\tau\right)\right.\right.\\
&\qquad\left.\left.+\frac{1}{sB^s}\left(p\int_a^t k_\alpha(t,\tau)(g\circ h)^s(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)(g\circ h)^s(\tau)\mathrm{d}\tau\right)\right]\right\}.
\end{aligned}$$

Now, put

$$A^r=p\int_a^t k_\alpha(t,\tau)(g\circ f)^r(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)(g\circ f)^r(\tau)\mathrm{d}\tau,$$

and

$$B^s=p\int_a^t k_\alpha(t,\tau)(g\circ h)^s(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)(g\circ h)^s(\tau)\mathrm{d}\tau.$$

Then the inequality (4.5) becomes

$$\begin{aligned}
&g^{-1}\left\{p\int_a^t k_\alpha(t,\tau)(g\circ f)(\tau)(g\circ h)(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)(g\circ f)(\tau)(g\circ h)(\tau)\mathrm{d}\tau\right\}\\
&\quad\leq g^{-1}\left\{\left[p\int_a^t k_\alpha(t,\tau)(g\circ f)^r(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)(g\circ f)^r(\tau)\mathrm{d}\tau\right]^{\frac{1}{r}}\right.\\
&\qquad\left.\times\left[p\int_a^t k_\alpha(t,\tau)(g\circ f)^s(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)(g\circ f)^s(\tau)\mathrm{d}\tau\right]^{\frac{1}{s}}\right\}.
\end{aligned}$$

Hence,

$$g^{-1}\left\{p\int_a^t k_\alpha(t,\tau)\left[g\left(g^{-1}\left((g\circ f)(\tau)(g\circ h)(\tau)\right)\right)\right]\mathrm{d}\tau\right.$$

$$
\begin{aligned}
&+q\int_t^b k_\alpha(\tau,t)\left[g\left(g^{-1}\left((g\circ f)(\tau)(g\circ h)(\tau)\right)\right)\right]\mathrm{d}\tau\Bigg\}\\
&\leq g^{-1}\left\{g\left[g^{-1}\left(\left[p\int_a^t k_\alpha(t,\tau)\left[g\left(g^{-1}\left((g\circ f)^r(\tau)\right)\right)\right]\mathrm{d}\tau\right.\right.\right.\\
&\quad\left.\left.\left.+q\int_t^b k_\alpha(\tau,t)\left[g\left(g^{-1}\left((g\circ f)^r(\tau)\right)\right)\right](\tau)\mathrm{d}\tau\right]^{\frac{1}{r}}\right)\right]\\
&\quad\times g\left[g^{-1}\left(\left[p\int_a^t k_\alpha(t,\tau)\left[g\left(g^{-1}\left((g\circ f)^s(\tau)\right)\right)\right](\tau)\mathrm{d}\tau\right.\right.\right.\\
&\quad\left.\left.\left.+q\int_t^b k_\alpha(\tau,t)\left[g\left(g^{-1}\left((g\circ f)^s(\tau)\right)\right)\right]\mathrm{d}\tau\right]^{\frac{1}{s}}\right)\right]\Bigg\}.
\end{aligned}
$$

By using the role of the pseudo-operator in the previous inequality, we have

$$
\begin{aligned}
&g^{-1}\left\{p\int_a^t k_\alpha(t,\tau)\left(g(f\odot h)(\tau)\right)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)\left(g(f\odot h)(\tau)\right)\mathrm{d}\tau\right\}\\
&\quad\leq g^{-1}\left\{\left[p\int_a^t k_\alpha(t,\tau)\left((g\circ f)^r_\odot(\tau)\right)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)\left((g\circ f)^r_\odot(\tau)\right)\mathrm{d}\tau\right]^{\frac{1}{r}}\right\}\\
&\qquad\odot\ g^{-1}\left\{\left[p\int_a^t k_\alpha(t,\tau)\left((g\circ f)^s_\odot(\tau)\right)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)\left((g\circ f)^s_\odot(\tau)\right)\mathrm{d}\tau\right]^{\frac{1}{s}}\right\}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&g^{-1}\left\{p\int_a^t k_\alpha(t,\tau)\left(g(f\odot h)(\tau)\right)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)\left(g(f\odot h)(\tau)\right)\mathrm{d}\tau\right\}\\
&\quad\leq g^{-1}\left\{\left[g\left(g^{-1}\left(\left[p\int_a^t k_\alpha(t,\tau)\left((g\circ f)^r_\odot(\tau)\right)\mathrm{d}\tau\right.\right.\right.\right.\\
&\qquad\left.\left.\left.\left.+q\int_t^b k_\alpha(\tau,t)\left((g\circ f)^r_\odot(\tau)\right)\mathrm{d}\tau\right]\right)\right)\right]^{\frac{1}{r}}\Bigg\}\\
&\qquad\odot g^{-1}\left\{\left[g\left(g^{-1}\left(\left[p\int_a^t k_\alpha(t,\tau)\left((g\circ f)^s_\odot(\tau)\right)\mathrm{d}\tau\right.\right.\right.\right.\\
&\qquad\left.\left.\left.\left.+q\int_t^b k_\alpha(\tau,t)\left((g\circ f)^s_\odot(\tau)\right)\mathrm{d}\tau\right]\right)\right)\right]^{\frac{1}{s}}\Bigg\}
\end{aligned}
$$

or equivalently,

$$
\left[K^\alpha_{P,\oplus,\odot}(f\odot h)(t)\right]\leq\left[K^\alpha_{P,\odot,\oplus}f^r_\odot(t)\right]^{\frac{1}{r}}_\odot\odot\left[K^\alpha_{P,\odot,\oplus}f^s_\odot(t)\right]^{\frac{1}{s}}_\odot.\qquad\square
$$

Remark 4.2. Taking $g(t) = t$ in (4.1), we obtain the Hölder's K_P^α–op inequality, i.e.,

$$[K_P^\alpha f(t)h(t)] \leq [K_P^\alpha f^r(t)]^{\frac{1}{r}} [K_P^\alpha f^s(t)]^{\frac{1}{s}}. \tag{4.6}$$

In the next theorem, we discuss the generalization of the Minkowski's inequality using the $K^\alpha_{P,\oplus,\odot}$–pseudo-operator.

Theorem 4.3 (Minkowski's $K^\alpha_{P,\oplus,\odot}$–ps–op inequality)**.** *Let r and s be conjugate exponents with $1 < r < \infty$. Assume that two functions f and h are both measurable for $t > a$. Let $g : [a, b] \to [0, \infty]$ be a generator of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$. Then the following inequality holds:*

$$\left[K^\alpha_{P,\oplus,\odot}(f \oplus h)^r_\odot(t)\right]^{\frac{1}{r}}_\odot \leq \left[K^\alpha_{P,\oplus,\odot} f^r_\odot(t)\right]^{\frac{1}{r}}_\odot \oplus \left[K^\alpha_{P,\oplus,\odot} h^r_\odot(t)\right]^{\frac{1}{r}}_\odot, \tag{4.7}$$

where $t \in (a, b)$, $p, q \in \mathbb{R}$, $P = \langle a, t, b, p, q\rangle$ and $k_\alpha(t, s)$ is a kernel that is nonnegative and depends upon a parameter $\alpha > 0$.

Proof. Let

$$(f + h)^r(t) = f(t)(f + h)^{r-1}(t) + h(t)(f + h)^{r-1}(t). \tag{4.8}$$

Multiplying both sides of (4.8) by $pk_\alpha(t, \tau)$ and then integrating the result with respect to τ on $[a, t]$, we get

$$\begin{aligned} p\int_a^t k_\alpha(t,\tau)(f+h)^r(\tau)\mathrm{d}\tau &= p\int_a^t k_\alpha(t,\tau)f(\tau)(f+h)^{r-1}(\tau)\mathrm{d}\tau \\ &\quad + p\int_a^t k_\alpha(t,\tau)h(\tau)(f+h)^{r-1}(\tau)\mathrm{d}\tau. \end{aligned} \tag{4.9}$$

Similarly, multiplying both sides of (4.8) by $qk_\alpha(\tau, t)$ and then integrating the result with respect to τ on $[t, b]$, we have

$$\begin{aligned} q\int_t^b k_\alpha(\tau,t)(f+h)^r(\tau)\mathrm{d}\tau &= q\int_t^b k_\alpha(\tau,t)f(\tau)(f+h)^{r-1}(\tau)\mathrm{d}\tau \\ &\quad + q\int_t^b k_\alpha(\tau,t)h(\tau)(f+h)^{r-1}(\tau)\mathrm{d}\tau. \end{aligned} \tag{4.10}$$

By adding (4.9) and (4.10), we obtain

$$\begin{aligned} &\left[p\int_a^t k_\alpha(t,\tau)(f+h)^r(\tau)\mathrm{d}\tau + q\int_t^b k_\alpha(\tau,t)(f+h)^r(\tau)\mathrm{d}\tau\right] \\ &= \left[p\int_a^t k_\alpha(t,\tau)f(\tau)(f+h)^{r-1}(\tau)\mathrm{d}\tau + q\int_t^b k_\alpha(\tau,t)f(\tau)(f+h)^{r-1}(\tau)\mathrm{d}\tau\right] \\ &\quad + \left[p\int_a^t k_\alpha(t,\tau)h(\tau)(f+h)^{r-1}(\tau)\mathrm{d}\tau + q\int_t^b k_\alpha(\tau,t)h(\tau)(f+h)^{r-1}(\tau)\mathrm{d}\tau\right]. \end{aligned}$$

Minkowski's K_P^α–op inequality gives

$$[K_P^\alpha(f+h)^r(t)] \leq [K_P^\alpha f^r(t)]^{\frac{1}{r}} \left[K_P^\alpha(f+h)^{(r-1)s}(t)\right]^{\frac{1}{s}} + [K_P^\alpha h^r(t)]^{\frac{1}{r}} \left[K_P^\alpha(f+h)^{(r-1)s}(t)\right]^{\frac{1}{s}}.$$

Since $\frac{1}{r}+\frac{1}{s}=1$, it follows that $(r-1)s=r$, and we have

$$[K_P^\alpha(f+h)^r(t)]^{\frac{1}{r}} \leq [K_P^\alpha f^r(t)]^{\frac{1}{r}} + [K_P^\alpha h^r(t)]^{\frac{1}{r}}, \tag{4.11}$$

which is equivalent to

$$\left[p\int_a^t k_\alpha(t,\tau)(f+h)^r(\tau)\mathrm{d}\tau + q\int_t^b k_\alpha(\tau,t)(f+h)^r(\tau)\mathrm{d}\tau\right]^{\frac{1}{r}} \leq \left[p\int_a^t k_\alpha(t,\tau)\, f^r(\tau)\mathrm{d}\tau + q\int_t^b k_\alpha(\tau,t)\, f^r(\tau)\mathrm{d}\tau\right]^{\frac{1}{r}} + \left[p\int_a^t k_\alpha(t,\tau)\, h^r(\tau)\mathrm{d}\tau + q\int_t^b k_\alpha(\tau,t)\, h^r(\tau)\mathrm{d}\tau\right]^{\frac{1}{r}}.$$

Since g is increasing, g^{-1} is also increasing and we have

$$g^{-1}\left\{\left[p\int_a^t k_\alpha(t,\tau)\,(g\circ f+g\circ h)^r(\tau)\mathrm{d}\tau + q\int_t^b k_\alpha(\tau,t)\,(g\circ f+g\circ h)^r(\tau)\mathrm{d}\tau\right]^{\frac{1}{r}}\right\} \leq g^{-1}\left\{\left[p\int_a^t k_\alpha(t,\tau)\,(g\circ f)^r(\tau)\mathrm{d}\tau + q\int_t^b k_\alpha(\tau,t)\,(g\circ f)^r(\tau)\mathrm{d}\tau\right]^{\frac{1}{r}} + \left[p\int_a^t k_\alpha(t,\tau)\,(g\circ h)^r(\tau)\mathrm{d}\tau q\int_t^b k_\alpha(\tau,t)\,(g\circ h)^r(\tau)\mathrm{d}\tau\right]^{\frac{1}{r}}\right\}.$$

By involving g and g^{-1} in the previous inequality, we obtain

$$g^{-1}\left\{\left[p\int_a^t k_\alpha(t,\tau)\left(g\left(g^{-1}\left(g\circ f+g\circ h\right)^r\right)\right)(\tau)\mathrm{d}\tau + q\int_t^b k_\alpha(\tau,t)\left(g\left(g^{-1}\left(g\circ f+g\circ h\right)^r\right)\right)(\tau)\mathrm{d}\tau\right]^{\frac{1}{r}}\right\} \leq g^{-1}\left\{\left[p\int_a^t k_\alpha(t,\tau)\,\left(g\left(g^{-1}(g\circ f)^r\right)\right)(\tau)\mathrm{d}\tau + q\int_t^b k_\alpha(\tau,t)\,\left(g\left(g^{-1}(g\circ f)^r\right)\right)(\tau)\mathrm{d}\tau\right]^{\frac{1}{r}}\right.$$

$$+\left[p\int_a^t k_\alpha(t,\tau)\,\left(g\left(g^{-1}(g\circ h)^r\right)\right)(\tau)\mathrm{d}\tau \right.$$
$$\left. +q\int_t^b k_\alpha(\tau,t)\,\left(g\left(g^{-1}(g\circ h)^r\right)\right)(\tau)\mathrm{d}\tau\right]^{\frac{1}{r}}\Bigg\}$$

or equivalently,

(4.12)

$$g^{-1}\left\{\left[g\left(g^{-1}\left(\left[p\int_a^t k_\alpha(t,\tau)(f\oplus h)^r_\odot(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)(f\oplus h)^r_\odot(\tau)\mathrm{d}\tau\right]\right)\right)\right]^{\frac{1}{r}}\right\}$$
$$\leq g^{-1}\left\{\left[g\left(g^{-1}\left(\left[p\int_a^t k_\alpha(t,\tau)\,f^r_\odot(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)\,f^r_\odot(\tau)\mathrm{d}\tau\right]\right)\right)\right]^{\frac{1}{r}}\right.$$
$$\left.+\left[g\left(g^{-1}\left(\left[p\int_a^t k_\alpha(t,\tau)\,h^r_\odot(\tau)\mathrm{d}\tau+q\int_t^b k_\alpha(\tau,t)\,h^r_\odot(\tau)\mathrm{d}\tau\right]\right)\right)\right]^{\frac{1}{r}}\right\}.$$

Hence (4.12) yields (4.7), which completes the proof. □

Remark 4.4. Taking $g(t)=t$ in (4.7), we obtain the Minkowski's K^α_P–op inequality [8], i.e.,

$$[K^\alpha_P(f+h)^r(t)]^{\frac{1}{r}}\leq[K^\alpha_P f^r(t)]^{\frac{1}{r}}+[K^\alpha_P h^r(t)]^{\frac{1}{r}}.$$

5. Concluding Remarks

The operator $K^\alpha_{P,\oplus,\odot}$, which is a generalization of K^α_P has been introduced and some of its properties have been investigated. It has been demonstrated that for a special kernel, this operator reduces to various fractional integrals and pseudo-fractional integrals. Utilizing $K^\alpha_{P,\oplus,\odot}$, the Hölder's and Minkowski's inequalities have been presented. It is shown that for a special kernel, these inequalities reduce to the generalized inequalities, which are previously presented in the literature. This research may be interposed and developed in other subjects, especially in variational problems.

Conflict of interest statement. The authors have no conflicts of interest to declare. All co-authors have seen and agreed with the contents of the manuscript and there is no financial interest to report. We certify that the submission is original work and it is not under review at any other publication.

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