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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 3
Pages: 165-196

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.2005194.1360



Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 3 (2024), 165-196
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.2005194.1360

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Abstract. This paper studies a fractional biological population model involving the Caputo-Fabrizio fractional derivative. We establish the existence and uniqueness of the solution using Banach's fixed point theorem. Furthermore, we propose a new numerical algorithm called $\mathbb{J}$-decomposition method ($\mathbb{J}$-DM) which is a combined form of the $\mathbb{J}$-transform method and a new decomposition method to solve the proposed model. After the convergence analysis of the $\mathbb{J}$-DM, we provide three numerical examples to illustrate the results obtained. The numerical examples show that this method is easy to use and can give accurate results.

1. Introduction

Historically, fractional calculus has been used successfully to model many complex real-world problems in various engineering and scientific disciplines such as physics, viscoelasticity, fluid mechanics, electrochemistry, chemistry, economics, electromagnetic, control theory, mathematical biology and processing of signals, etc. See [5, 12, 13, 15, 17, 20, 23, 25, 28]. Several fractional derivative operators have recently been defined and developed, and many mathematicians and physicists have studied various differential equations of fractional order. In [16], the author investigated the existence and uniqueness of the solution of the fractional Bratu-type initial value problem involving the Caputo-Fabrizio derivative. In [2], Afshari et al. studied the existence and uniqueness results of fractional differential equations within the Caputo-Fabrizio derivative. In [3], new contraction mappings on existence and uniqueness

2020 *Mathematics Subject Classification.* 35R11, 26A33, 74G10, 34K28.
Key words and phrases. Fractional biological population model, Caputo-Fabrizio fractional derivative, Banach space, $\mathbb{J}$-transform, Decomposition method.
Received: 20 June 2023, Accepted: 12 November 2023.

results for implicit $\phi-$Hilfer fractional pantograph differential equations are presented. In [4], fractional boundary value problems are presented, and the fractional derivative is defined in the generalized Caputo sense.

In 2015, Michele Caputo and Mauro Fabrizio presented a fractional derivative operator based on the exponential function to overcome the singular kernel problem [8]. They demonstrated that their derivative operator was suitable for solving some physical issues. Due to the novelty of this derivative operator, many researchers have used distinct methods to solve some different equations, including the Caputo-Fabrizio fractional derivative.

Generally, it is difficult to determine exact solutions for fractional differential equations with Caputo-Fabrizio fractional derivatives by applying analytical techniques, so numerical and semi-numerical techniques are employed to determine approximate solutions of such types of equations. The finite difference method (FDM) [10], the Adams-Bashforth method (ABM) [22], the Adomian decomposition method (ADM) [24], the homotopy analysis method (HAM) [26] and the variational homotopic perturbation method (VHPM) [6] are some classical numerical and semi-numerical techniques used to determine approximate solutions of fractional differential equations with Caputo-Fabrizio fractional derivative.

In this paper, we study a fractional biological population model of the form

$$\mathbb{D}^{\alpha}u = \left(u^2\right)_{xx} + \left(u^2\right)_{yy} + F(u), \tag{1.1}$$

subject to the initial condition

$$u(x, y, 0) = u_0(x, y), \tag{1.2}$$

where $\mathbb{D}^{\alpha}$ is the Caputo-Fabrizio fractional derivative operator of the function u of order $0 < \alpha \leq 1, u = \left\{u(x, y, t), (x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+\right\}$ denotes the population density, F represents the population supply due to birth and death. α is a parameter describing the order of the fractional derivative. It should be noted that problem (1.1) leads to Malthusian law [11] for $F(u) = c_1 u$ and Verhulst law [11] for $F(u) = c_2 u - c_3 u^2$, where c_1, c_2, c_3 are positive constants. Moreover, (1.1) with $F(u) = -cu^p$, where $c \geq 0, 0 < p \leq 1$, appears in the study of flow through porous media [7, 19].

The main objective of this paper is to apply the Banach fixed point theorem to establish the existence and uniqueness results of the fractional biological population model (1.1) and to propose a new numerical algorithm called the $\mathbb{J}$-decomposition method (J-DM) via the Caputo-Fabrizio fractional derivative for solving the proposed model.

The rest of the paper is organized as follows. Section 2 gives basic concepts and results on the Caputo-Fabrizio fractional derivative. In Section 3, the existence and uniqueness results of the fractional biological population model are established. In Section 4, the $\mathbb{J}$-DM to approximate the solution of the proposed model is presented. Three examples are given in Section 4. The results obtained in Section 4 are discussed in Section 5. Concluding remarks are given in Section 6.

2. Preliminaries

This section gives some basic definitions and results on the Caputo-Fabrizio fractional derivative [8], which are considered helpful for the present study.

Definition 2.1. The Caputo-Fabrizio fractional derivative of order α of a function u is defined as

$$\mathbb{D}^{\alpha}u(t) = \frac{\mathcal{CF}(\alpha)}{1-\alpha}\int_0^t u'(\tau)\exp\left(-\frac{\alpha\,(t-\tau)}{1-\alpha}\right)d\tau, \quad 0<\alpha<1,$$

where $\mathcal{CF}(\alpha)$ is the normalization function such that $\mathcal{CF}(0)=\mathcal{CF}(1)=1$ and $u\in H^1(0,T), T>0$.

If $u\notin H^1(0,T), T>0$, then the derivative can be reformulated as

$$\mathbb{D}^{\alpha}u(t) = \frac{\alpha\mathcal{CF}(\alpha)}{1-\alpha}\int_0^t (u(t)-u(\tau))\exp\left(-\frac{\alpha\,(t-\tau)}{1-\alpha}\right)d\tau, \quad 0<\alpha<1.$$

Definition 2.2. The fractional integral of order α of a function u is defined as

(2.1)

$$\mathbb{I}^{\alpha}u(t) = \frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)}\int_0^t u(\tau)d\tau, \quad 0<\alpha<1.$$

According to the previous definition, the Caputo-Fabrizio fractional integral of a function u of order $0<\alpha\leq 1$ is an average between function u and its integral of order one.

This therefore imposes

$$\frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)} + \frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)} = 1,$$

we get an explicit formula for $\mathcal{CF}(\alpha)$, as follows

$$\mathcal{CF}(\alpha) = \frac{2}{2-\alpha}, \quad 0\leq\alpha\leq 1.$$

For this reason, we have the following definition of the fractional derivative of order $0<\alpha<1$.

Definition 2.3. The Caputo-Fabrizio fractional derivative of order α of a function u is defined as

$$\mathbb{D}^{\alpha}u(t)=\frac{1}{1-\alpha}\int_0^t u'(\tau)\exp\left(-\frac{\alpha\,(t-\tau)}{1-\alpha}\right)d\tau,\quad 0<\alpha<1,$$

and its corresponding fractional integral is defined as

$$\mathbb{I}^{\alpha}u(t)=(1-\alpha)u(t)+\alpha\int_0^t u(\tau)d\tau,\quad 0<\alpha<1.$$

Definition 2.4. The Caputo-Fabrizio fractional derivative of order $\alpha+n$ when $0<\alpha\leq 1$ and $n\geq 0$ is defined as

$$\mathbb{D}^{\alpha+n}u(t)=\mathbb{D}^{\alpha}(D^n u(t)).$$

3. Main Results

Our main objective in this section is to establish the existence and uniqueness of the solution for the fractional biological population model (1.1) subject to the initial condition (1.2), using the Banach fixed point theorem. Moreover, we propose a new numerical algorithm, which is a combined form of the $\mathbb{J}$ transform method and a new decomposition method to solve nonlinear fractional partial differential equations involving the Caputo-Fabrizio fractional derivative.

Let $B=(C[I],\|\cdot\|)$ be a Banach space of continuous mapping defined on $I=[0,T]$ with norm $\|u(x,y,t)\|=\max\limits_{t\in I}|u(x,y,t)|$.

3.1. Existence and Uniqueness.

Applying the fractional integral operator (2.1) to equation (1.1), we get

(3.1)

$$\begin{aligned}u(x,y,t)-u(x,y,0)=&\frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}\left(\left(u^2\right)_{xx}+\left(u^2\right)_{yy}+F(u)\right)\\&+\frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)}\int_0^t\left(\left(u^2\right)_{xx}+\left(u^2\right)_{yy}+F(u)\right)d\tau.\end{aligned}$$

Let

(3.2)

$$\begin{aligned}\mathcal{G}(x,y,t,u)&=\mathcal{G}(u)\\&=\left(u^2\right)_{xx}+\left(u^2\right)_{yy}+F(u).\end{aligned}$$

Theorem 3.1. *Under the following hypotheses that*

H_1) *There exists $L_F>0$, such that*

$$\|F(u_1)-F(u_2)\|\leq L_F\|u_1-u_2\|.$$

H_2) *There exists $\beta,L_x,L_y,L_F>0$, such that*

$$0<2\beta\left(L_x^2+L_y^2\right)+L_F<1.$$

Then the function $\mathcal{G}(x,y,t,u)$ satisfies the Lipschitz condition and is a contraction mapping.

Proof. Let u_1 and u_2 be two nonnegative and bounded functions. By definition of $\mathcal{G}$ and triangular inequality, we have

$$\begin{aligned}\|\mathcal{G}(x,y,t,u_1)-\mathcal{G}(x,y,t,u_2)\| &= \left\| \begin{array}{c} \left(u_1^2-u_2^2\right)_{xx}+\left(u_1^2-u_2^2\right)_{yy} \\ +F(u_1)-F(u_2) \end{array} \right\| \\ &\leq \left\|\left(u_1^2-u_2^2\right)_{xx}\right\|+\left\|\left(u_1^2-u_2^2\right)_{yy}\right\| \\ &\quad +\|F(u_1)-F(u_2)\|.\end{aligned}$$

Now, suppose that there exists constants $\beta_1,\beta_2\geq 0$ such that for all (x,y,t), $\|u_1\|\leq\beta_1$ and $\|u_2\|\leq\beta_2$. Let $\beta=\max\{\beta_1,\beta_2\}$. Then their first order derivative functions $(.)_x$ and $(.)_y$ satisfy the Lipschitz condition and there exist a numbers L_x, $L_y\geq 0$ such that

$$\begin{aligned}\|\mathcal{G}(x,y,t,u_1)-\mathcal{G}(x,y,t,u_2)\| &\leq L_x^2\left\|u_1^2-u_2^2\right\|+L_y^2\left\|u_1^2-u_2^2\right\| \\ &\quad +L_F\|u_1-u_2\| \\ &\leq \left(2\beta\left(L_x^2+L_y^2\right)+L_F\right)\|u_1-u_2\|.\end{aligned}$$

where $\left\|u_1^2-u_2^2\right\|\leq 2\beta\|u_1-u_2\|$ by since u_1 and u_2 are bounded.

Putting $L=2\beta\left(L_x^2+L_y^2\right)+L_F$, then we have

$$\|\mathcal{G}(x,y,t,u_1)-\mathcal{G}(x,y,t,u_2)\|\leq L\|u_1-u_2\|.$$

So $\mathcal{G}(x,y,t,u)$ satisfies the Lipschitz condition. Further, if $0<L<1$, then this implies a contraction.

Hence, the proof is completed. □

Theorem 3.2. *Under the following condition that*

$$\left(2\beta\left(L_x^2+L_y^2\right)+L_F\right)\left(\frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}+\frac{2\alpha T}{(2-\alpha)\mathcal{CF}(\alpha)}\right)<1, \tag{3.3}$$

then the fractional biological population model (1.1)-(1.2) admits a unique solution.

Proof. Based on equations (3.1) and (3.2), we can write

$$u(x,y,t)-u(x,y,0)=\frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}\mathcal{G}(u)+\frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)}\int_0^t\mathcal{G}(u)d\tau,$$

and the following successive formula can be obtained

$$u_n(x,y,t)=\frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}\mathcal{G}(u_{n-1})+\frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)}\int_0^t\mathcal{G}(u_{n-1})d\tau, \tag{3.4}$$

where the first term is $u_0(x,y,t)=u(x,y,0)$.

Due to Theorem 3.1, we have

$$\begin{aligned}\|u_n - u_{n-1}\| &= \left\| \frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}\mathcal{G}(u_{n-1}) + \frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)}\int_0^t \mathcal{G}(u_{n-1})d\tau \right.\\ &\quad \left. - \frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}\mathcal{G}(u_{n-2}) - \frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)}\int_0^t \mathcal{G}(u_{n-2})d\tau \right\|\\ &\leq \frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}\|\mathcal{G}(x,y,t,u_{n-1}) - \mathcal{G}(x,y,t,u_{n-2})\|\\ &\quad + \frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)}\int_0^t \|\mathcal{G}(x,y,t,u_{n-1}) - \mathcal{G}(x,y,t,u_{n-2})\|\\ &\leq \frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}\left(2\beta\left(L_x^2+L_y^2\right)+L_F\right)\|u_{n-1}-u_{n-2}\|\\ &\quad + \frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)}\left(2\beta\left(L_x^2+L_y^2\right)+L_F\right)T\|u_{n-1}-u_{n-2}\|.\end{aligned}$$

Consequently

(3.5)

$$\begin{aligned}\|u_n - u_{n-1}\| &\leq \left(2\beta\left(L_x^2+L_y^2\right)+L_F\right)\left(\frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)} + \frac{2\alpha T}{(2-\alpha)\mathcal{CF}(\alpha)}\right)\\ &\quad \times \|u_{n-1}-u_{n-2}\|,\\ \|u_n - u_{n-1}\| &\leq \left[\left(2\beta\left(L_x^2+L_y^2\right)+L_F\right)\left(\frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)} + \frac{2\alpha T}{(2-\alpha)\mathcal{CF}(\alpha)}\right)\right]^2\\ &\quad \times \|u_{n-2}-u_{n-3}\|,\\ &\vdots\\ \|u_n - u_{n-1}\| &\leq \left[\left(2\beta\left(L_x^2+L_y^2\right)+L_F\right)\left(\frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)} + \frac{2\alpha T}{(2-\alpha)\mathcal{CF}(\alpha)}\right)\right]^n\\ &\quad \times \|u_0(x,y,t)\|.\end{aligned}$$

The inequality (3.5) proves the existence and regularity of the solution of equation (3.4).

Now, we prove that the equation (3.4) converges to the solution of the fractional biological population model (1.1) subject to the initial condition (1.2).

Let

$$U_n(x,y,t) = u(x,y,t) - u_n(x,y,t).$$

Then

$$\|U_n(x,y,t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}\|\mathcal{G}(x,y,t,u) - \mathcal{G}(x,y,t,u_n)\|$$

$$+\frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)}\int_0^t \mathcal{G}(x,y,t,u)-\mathcal{G}(x,y,t,u_n).$$

Since u is a bounded function for all (x,y,t) and because $\mathcal{G}(x,y,t,u)$ satisfies the Lipschitz condition, then we can write

$$\|U_n(x,y,t)\| \leq \left[\left(2\beta\left(L_x^2+L_y^2\right)+L_F\right)\left(\frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}+\frac{2\alpha T}{(2-\alpha)\mathcal{CF}(\alpha)}\right)\right]^n \beta_1.$$

From condition (3.3), we have $\|U_n(x,y,t)\| \to 0$ when $n \to \infty$. Thus the existence of the solution of (1.1)-(1.2) is proved.

To check the uniqueness of the solution. Suppose that $u(x,y,t)$ and $\widetilde{u}(x,y,t)$ are two different solutions to the fractional biological population model (1.1)-(1.2).

By using Theorem 3.1, we obtain

$$\|u-\widetilde{u}\| \leq \left(2\beta\left(L_x^2+L_y^2\right)+L_F\right)\left(\frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}+\frac{2\alpha T}{(2-\alpha)\mathcal{CF}(\alpha)}\right)\|u-\widetilde{u}\|.$$

From condition (3.3), it follows that $\|u-\widetilde{u}\|=0$. Hence we have $u(x,y,t)=\widetilde{u}(x,y,t)$.

Hence, the proof is completed. □

3.2. **𝕁-Decomposition Method.** This subsection proposes a new numerical algorithm called 𝕁-decomposition method (𝕁-DM) to solve nonlinear fractional partial differential equation of the form

$$\mathbb{D}^{\alpha+n}u(X,t)=L\left(u(X,t)\right)+N\left(u(X,t)\right)+f(X,t), \tag{3.6}$$

subject to the initial conditions

$$\begin{aligned}\frac{\partial^k u(X,0)}{\partial t^k} &= u^{(k)}(X,0)\\ &= u_k(X), \quad k=0,1,2,\ldots,n,\end{aligned} \tag{3.7}$$

where $(X,t)\in\mathbb{R}^n\times\mathbb{R}^+$, $\mathbb{D}^{\alpha+n}u(X,t)$ is the Caputo-Fabrizio fractional derivative of the function $u(X,t)$ of order $\alpha+n$ with $0<\alpha\leq 1, n\in\mathbb{N}$, L is the linear differential operator, N represents the general nonlinear differential operator and $f(X,t)$ is the source term.

Before that, we present a new result concerning the 𝕁-transform of the Caputo-Fabrizio fractional derivative.

Definition 3.3 ([18])**.** The 𝕁-transform of the function $u:[0,\infty)\to\mathbb{R}$ of exponential order is defined over the set of functions

$$A=\left\{u(t)/\exists N,\eta_1,\eta_2>0:|u(t)|<N\exp\left(\frac{|t|}{\eta_i}\right), t\in(-1)^i\times[0,\infty)\right\},$$

by

$$\mathbb{J}\left[u(t)\right]=\mathcal{U}(s,v)$$

$$= v \int_0^\infty \exp\left(-\frac{st}{v}\right) u(t)dt,$$

for $s > 0$ and $v > 0$, provided the limit of the integral exists, where s and v are the $\mathbb{J}$-transform variables.

Some basic properties of the $\mathbb{J}$-transform are given as follows [18].

Property 3.4. Let u and w be in set A. It holds

$$\mathbb{J}\left[\lambda u(t) \pm \mu w(t)\right] = \lambda \mathbb{J}\left[u(t)\right] \pm \mu \mathbb{J}\left[w(t)\right],$$

where λ and μ are two constants.

Property 3.5. Assume that $u^{(n)}(t)$ is the $n-$th derivative of the function $u(t) \in A$. Let $\mathcal{U}(s, v)$ be the $\mathbb{J}$-transform of $u \in A$, then

$$\mathbb{J}\left[u^{(n)}(t)\right] = \frac{s^n}{v^n}\mathcal{U}(s, v) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{v^{n-(k+2)}} u^{(k)}(0).$$

Property 3.6. Let the functions $u(t), w(t) \in A$. If $\mathcal{U}(s, v)$ and $\mathcal{W}(s, v)$ are the respective $\mathbb{J}$-transforms of the functions $u(t)$ and $w(t)$. Then the convolution theorem of $\mathbb{J}$ -transform is given by

$$\mathbb{J}\left[(u * w)(t)\right] = \frac{1}{v}\mathcal{U}(s, v)\mathcal{W}(s, v),$$

where $u * w$ is the convolution of two functions $u(t)$ and $w(t)$ which is defined by

$$\begin{aligned}(u * w)(t) &= \int_0^t u(\tau)w(t - \tau)d\tau \\ &= \int_0^t u(t - \tau)w(\tau)d\tau.\end{aligned}$$

Property 3.7. Inverse $\mathbb{J}$-transform of some functions

$$\begin{aligned}&\mathbb{J}^{-1}\left[\frac{v^2}{s}\right] = 1, \\ &\mathbb{J}^{-1}\left[\frac{v^3}{s^2}\right] = t, \\ &\mathbb{J}^{-1}\left[\frac{v^{n+2}}{s^{n+1}}\right] = \frac{t^n}{n!}, \quad n = 0, 1, 2, \ldots \\ &\mathbb{J}^{-1}\left[\frac{v^2}{s - av}\right] = \exp(at).\end{aligned}$$

Theorem 3.8. *The $\mathbb{J}$-transform of the Caputo-Fabrizio fractional derivative of the function $u(t)$ of order $\alpha+n$ where $0<\alpha\leq 1$ and $n\in\mathbb{N}\cup 0$, is given by*

$$\mathbb{J}\left[\mathbb{D}^{\alpha+n}u(t)\right]=\frac{v}{s+\alpha\left(v-s\right)}\left(\frac{s^{n+1}}{v^{n+1}}\mathbb{J}\left(u(t)\right)-\sum_{k=0}^{n}\frac{s^{n-k}}{v^{n-(k+1)}}u^{(k)}(0)\right).$$

Proof. According to definitions 2.3 and 2.4, we have

$$\begin{aligned}
&\mathbb{J}\left[\mathbb{D}^{\alpha+n}u(t)\right]\\
&\quad=\mathbb{J}\left[\mathbb{D}^{\alpha}(D^{n}u(t))\right]\\
&\quad=\frac{v}{1-\alpha}\int_{0}^{\infty}\exp\left(-\frac{st}{v}\right)\left(\int_{0}^{t}u^{(n+1)}(\tau)\exp\left(-\frac{\alpha\left(t-\tau\right)}{1-\alpha}\right)d\tau\right)dt\\
&\quad=\frac{v}{1-\alpha}\int_{0}^{\infty}\exp\left(-\frac{st}{v}\right)\left(u^{(n+1)}(t)*\exp\left(-\frac{\alpha t}{1-\alpha}\right)\right)dt\\
&\quad=\frac{1}{1-\alpha}\mathbb{J}\left[u^{(n+1)}(t)*\exp\left(-\frac{\alpha t}{1-\alpha}\right)\right].
\end{aligned}$$

From properties 3.5 and 3.6, we have

$$\begin{aligned}
&\mathbb{J}\left[\mathbb{D}^{\alpha+n}u(t)\right]\\
&\quad=\frac{1}{1-\alpha}\frac{1}{v}\mathbb{J}\left[u^{(n+1)}(t)\right]\mathbb{J}\left[\exp\left(-\frac{\alpha t}{1-\alpha}\right)\right]\\
&\quad=\frac{1}{1-\alpha}\frac{1}{v}\frac{v^{2}\left(1-\alpha\right)}{s\left(1-\alpha\right)+\alpha v}\left(\frac{s^{n+1}}{v^{n+1}}\mathbb{J}\left(u(t)\right)-\sum_{k=0}^{n}\frac{s^{n-k}}{v^{n-(k+1)}}u^{(k)}(0)\right)\\
&\quad=\frac{v}{s+\alpha\left(v-s\right)}\left(\frac{s^{n+1}}{v^{n+1}}\mathbb{J}\left(u(t)\right)-\sum_{k=0}^{n}\frac{s^{n-k}}{v^{n-(k+1)}}u^{(k)}(0)\right).
\end{aligned}$$

Hence, the proof is completed. □

Now, to present our proposed algorithm, we first need to present the following results.

Theorem 3.9. *Let*

$$u(X,t)=\sum_{i=0}^{\infty}u_{i}(X,t).$$

For the parameter λ, we define

$$u_{\lambda}(X,t)=\sum_{i=0}^{\infty}\lambda^{i}u_{i}(X,t).$$

Then

1) *The linear operator $L(u_\lambda)$ satisfies the following property*

$$L(u_\lambda) = L\left(\sum_{i=0}^{\infty} \lambda^i u_i(X,t)\right) = \sum_{i=0}^{\infty} \lambda^i L(u_i(X,t)).$$

2) *The nonlinear operator $N(u_\lambda)$ satisfies the following property*

$$N(u_\lambda) = N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right) = \sum_{n=0}^{\infty} \left[\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}\right] \lambda^n.$$

Proof. 1) Clear.

2) Using the Maclaurin expansion of $N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)$ with respect to λ, we have

(3.8)

$$\begin{aligned} N(u_\lambda) &= N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right) \\ &= \left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0} + \left[\frac{d}{d\lambda}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}\right] \lambda \\ &\quad + \left[\frac{1}{2!}\frac{d}{d\lambda}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}\right] \lambda^2 \\ &\quad + \left[\frac{1}{3!}\frac{d}{d\lambda}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}\right] \lambda^3 + \cdots \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}\right] \lambda^n \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i u_i\right) + N\left(\sum_{i=n+1}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}\right] \lambda^n \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}\right] \lambda^n. \end{aligned}$$

Hence, the proof is completed. □

Definition 3.10. The polynomials $K_n(u_0, u_1, \ldots, u_n)$ can be defined as

$$K_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}. \tag{3.9}$$

Remark 3.11. Let $K_n = K_n(u_0, u_1, \ldots, u_n)$ be as in Definition 3.10. Then by using Theorem 3.9, the nonlinear operator $N(u_\lambda)$ can be expressed in terms of K_n

$$N(u_\lambda) = \sum_{n=0}^{\infty}\lambda^n K_n. \tag{3.10}$$

Theorem 3.12. *According to $\mathbb{J}$-DM, the solution of equations (1.1) and (1.2) is presented in the form*

(3.11)

$$\begin{aligned} u(X,t) &= \sum_{i=0}^{\infty} u_i(X,t) \\ &= \sum_{k=0}^{n} u_k(X)\frac{t^k}{k!} \\ &\quad + \sum_{i=1}^{\infty}\mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))\,v^n}{s^{n+1}}\mathbb{J}\left[L\left(u_{i-1}(X,t)\right) + K_{i-1} + f(X,t)\right]\right), \end{aligned}$$

where $\mathbb{J}[.]$ is the $\mathbb{J}$-transform and K_i are polynomials of $u_0, u_1, \ldots, u_n$ defined by (3.9).

Proof. Let the solution $u(X,t)$ of equations (3.6)-(3.7) be as follows

$$u(X,t) = \sum_{i=0}^{\infty} u_i(X,t). \tag{3.12}$$

For solving the equations (3.6)-(3.7), we consider

(3.13)

$$\mathbb{D}^{\alpha+n}u_\lambda(X,t) = \lambda\left[L\left(u_\lambda(X,t)\right) + N\left(u_\lambda(X,t)\right) + f(X,t)\right], \quad \lambda \in [0,1],$$

subject to the initial conditions

$$\begin{aligned} \frac{\partial^k u_\lambda(X,0)}{\partial\zeta^k} &= u_\lambda^{(k)}(X,0) \\ &= \phi_k(X), \quad k = 0,1,2,\ldots,n. \end{aligned} \tag{3.14}$$

Now, we assume that equations (3.13)-(3.14) have a solution given by

$$u_\lambda(X,t) = \sum_{i=0}^{\infty}\lambda^i u_i(X,t). \tag{3.15}$$

An application of $\mathbb{J}$-transform on both sides of (3.13), gives

$$\mathbb{J}\left[\mathbb{D}^{\alpha+n}u_\lambda(X,t)\right] = \lambda\mathbb{J}\left[L\left(u_\lambda(X,t)\right) + N\left(u_\lambda(X,t)\right) + f(X,t)\right]. \tag{3.16}$$

Moreover, Theorem 3.8 implies

(3.17)

$$\begin{aligned}\mathbb{J}\left(u_\lambda(X,t)\right) = &\sum_{k=0}^{n} \frac{v^{k+2}}{s^{k+1}} u^{(k)}(X,0) \\ &+ \frac{\left(s+\alpha\left(v-s\right)\right)v^n}{s^{n+1}}\lambda\mathbb{J}\left[\begin{array}{c} L\left(u_\lambda(X,t)\right) + N\left(u_\lambda(X,t)\right) \\ +f(X,t) \end{array}\right].\end{aligned}$$

Operating the inverse $\mathbb{J}$-transform on both sides of equation (3.17), we get

(3.18)

$$\begin{aligned}u_\lambda(X,t) = &\sum_{k=0}^{n} \phi_k(X)\frac{t^k}{k!} \\ &+ \lambda\left[\mathbb{J}^{-1}\left(\frac{\left(s+\alpha\left(v-s\right)\right)v^n}{s^{n+1}}\mathbb{J}\left[\begin{array}{c} L\left(u_\lambda(X,t)\right) + N\left(u_\lambda(X,t)\right) \\ +f(X,t) \end{array}\right]\right)\right].\end{aligned}$$

Substituting (3.15) into (3.18), we obtain

(3.19)

$$\begin{aligned}\sum_{i=0}^{\infty}\lambda^i u_i(X,t) = &\sum_{k=0}^{n} \phi_k(X)\frac{t^k}{k!} \\ &+ \lambda\left[\mathbb{J}^{-1}\left(\frac{\left(s+\alpha\left(v-s\right)\right)v^n}{s^{n+1}}\mathbb{J}\left[L\left(\sum_{i=0}^{\infty}\lambda^i u_i(X,t)\right)\right.\right.\right. \\ &\left.\left.\left.+N\left(\sum_{i=0}^{\infty}\lambda^i u_i(X,t)\right) + f(X,t)\right]\right)\right].\end{aligned}$$

Using Theorem 3.9, then equation (3.19) becomes

(3.20)

$$\begin{aligned}\sum_{i=0}^{\infty}\lambda^i u_i(X,t) = &\sum_{k=0}^{n} \phi_k(X)\frac{t^k}{k!} + \lambda\left[\mathbb{J}^{-1}\left(\frac{\left(s+\alpha\left(v-s\right)\right)v^n}{s^{n+1}}\mathbb{J}\left[f(X,t)\right]\right)\right] \\ &+ \lambda\left[\mathbb{J}^{-1}\left(\frac{\left(s+\alpha\left(v-s\right)\right)v^n}{s^{n+1}}\mathbb{J}\left[\sum_{i=0}^{\infty}L\left(\lambda^i u_i(X,t)\right)\right]\right)\right] \\ &+ \lambda\left[\mathbb{J}^{-1}\left(\frac{\left(s+\alpha\left(v-s\right)\right)v^n}{s^{n+1}}\right.\right.\end{aligned}$$

$$\times\mathbb{J}\left[\sum_{n=0}^{\infty}\left[\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}\right]\lambda^n\right]\Bigg)\Bigg].$$

Using Definition 3.10, then equation (3.20) becomes

(3.21)
$$\begin{aligned}\sum_{i=0}^{\infty}\lambda^i u_i(X,t) = \sum_{k=0}^{n}\phi_k(X)\frac{t^k}{k!} &+ \lambda\left[\mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))\,v^n}{s^{n+1}}\mathbb{J}\left[f(X,t)\right]\right)\right]\\ &+ \lambda\left[\mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))\,v^n}{s^{n+1}}\mathbb{J}\left[\sum_{k=0}^{\infty}L\left(\lambda^i u_i(X,t)\right)\right]\right)\right]\\ &+ \lambda\left[\mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))\,v^n}{s^{n+1}}\mathbb{J}\left[\sum_{n=0}^{\infty}\lambda^n K_n\right]\right)\right].\end{aligned}$$

By equating the terms in equation (3.21) with identical powers of λ, we obtain the following components

(3.22)
$$\begin{aligned}u_0(X,t) &= \sum_{k=0}^{n}\phi_k(X)\frac{t^k}{k!},\\ u_1(X,t) &= \mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))\,v^n}{s^{n+1}}\mathbb{J}\left[f(X,t)+L\left(u_0(X,t)\right)+K_0\right]\right),\\ &\vdots\\ u_i(X,t) &= \mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))\,v^n}{s^{n+1}}\mathbb{J}\left[L\left(u_{i-1}(X,t)\right)+K_{i-1}\right]\right), i=2,3,4,\ldots\end{aligned}$$

Substituting the components of (3.22) into equation (3.15) gives the solution of equations (3.13)-(3.14).

Now, from equations (3.12) and (3.15), we obtain

(3.23)
$$\begin{aligned}u(X,t) &= \lim_{\lambda\to 1}u_\lambda(X,t)\\ &= u_0(X,t)+u_1(X,t)+\sum_{i=2}^{\infty}u_i(X,t).\end{aligned}$$

Using the first equation of (3.22), we see that

(3.24)
$$\frac{\partial^k u(X,0)}{\partial t^k} = \lim_{\lambda\to 1}\frac{\partial^k u_\lambda(X,0)}{\partial t^k},$$

which implies that $u_k(X)=\phi_k(X)$.

Substituting equations (3.22) into equation (3.23), we get

(3.25)

$$u(X,t) = \sum_{k=0}^{n} u_k(X)\frac{t^k}{k!} + \sum_{i=1}^{\infty} \mathbb{J}^{-1}\left(\frac{(s+\alpha\,(v-s))\,v^n}{s^{n+1}}\mathbb{J}\left(L\left(u_{i-1}(X,t)\right) + K_{i-1} + f(X,t)\right]\right).$$

Hence, the proof is completed. □

3.3. **Convergence Analysis.** This subsection discusses the convergence of the solution.

Theorem 3.13. *Under the following conditions that*

C_1) *There exists* $\eta_1 > 0$, *such that*

$$|L(u_1) - L(u_2)| \leq \eta_1 |u_1 - u_2|.$$

C_2) *There exists* $\eta_2 > 0$, *such that*

$$|N(u_1) - N(u_2)| \leq \eta_2 |u_1 - u_2|. \tag{3.26}$$

C_3) *There exists* $\delta = (\eta_1 + \eta_2)\left((1-\alpha)M_1T + \alpha M_2T\right)$, *such that*

$$0 < \delta < 1. \tag{3.27}$$

Then the series solution obtained by $\mathbb{J}$*-DM of equation (3.6), is convergent.*

Proof. Let $S_i = \sum_{j=0}^{i} u_j(X,t)$ be the i^{th} partial sum.

First, we prove that $\{S_i\}_{i\geq 0}$ is a Cauchy sequence in Banach space B. Taking into consideration a new formulation of polynomials, we get

$$N(S_i) = K_i + \sum_{j=0}^{i-1} K_j. \tag{3.28}$$

Now,

(3.29)

$$\begin{aligned}\|S_i - S_q\| = \max_{t\in I}|S_i - S_q| &= \max_{t\in I}\left|\sum_{j=q+1}^{i} u_j(X,t)\right| \\ &\leq \max_{t\in I}\left|\begin{array}{l}\mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))v^n}{s^{n+1}}\mathbb{J}\left[\sum_{j=q+1}^{i} L\left(u_{j-1}(X,t)\right)\right]\right) \\ +\mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))v^n}{s^{n+1}}\mathbb{J}\left[\sum_{j=q+1}^{i} K_{j-1}(X,t)\right]\right)\end{array}\right|\end{aligned}$$

$$
\begin{aligned}
&= \max_{t\in I} \left| \begin{array}{c} \mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))v^n}{s^{n+1}}\mathbb{J}\left[\sum_{j=q}^{i-1} L\left(u_j(X,t)\right)\right]\right) \\ +\mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))v^n}{s^{n+1}}\mathbb{J}\left[\sum_{j=q}^{i-1} K_j(X,t)\right]\right) \end{array} \right| \\
&\leq \max_{t\in I} \left| \begin{array}{c} \mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))v^n}{s^{n+1}}\mathbb{J}\left[L\left(S_{i-1}\right)-L\left(S_{q-1}\right)\right]\right) \\ +\mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))v^n}{s^{n+1}}\mathbb{J}\left[N\left(S_{i-1}\right)-N\left(S_{q-1}\right)\right]\right) \end{array} \right| \\
&\leq \max_{t\in I} \left[\begin{array}{c} \mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))v^n}{s^{n+1}}\mathbb{J}\left|L\left(S_{i-1}\right)-L\left(S_{q-1}\right)\right|\right) \\ +\mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))v^n}{s^{n+1}}\mathbb{J}\left|N\left(S_{i-1}\right)-N\left(S_{q-1}\right)\right|\right) \end{array} \right].
\end{aligned}
$$

According to the conditions C_1 and C_2, we have

(3.30)

$$
\begin{aligned}
\|S_i - S_q\| &\leq \max_{t\in I} \left[\begin{array}{c} \eta_1\mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))v^n}{s^{n+1}}\mathbb{J}\left|S_{i-1}-S_{q-1}\right|\right) \\ +\eta_2\mathbb{J}^{-1}\left(\frac{(s+\alpha(v-s))v^n}{s^{n+1}}\mathbb{J}\left|S_{i-1}-S_{q-1}\right|\right) \end{array} \right] \\
&\leq (\eta_1+\eta_2)\max_{t\in I}\mathbb{J}^{-1}\left(\frac{\left(s+\alpha\left(v-s\right)\right)v^n}{s^{n+1}}\mathbb{J}\left|S_{i-1}-S_{q-1}\right|\right) \\
&= (\eta_1+\eta_2)\max_{t\in I}\mathbb{J}^{-1}\left(\left((1-\alpha)\frac{v^n}{s^n}+\alpha\frac{v^{n+1}}{s^{n+1}}\right)\mathbb{J}\left|S_{i-1}-S_{q-1}\right|\right) \\
&= (\eta_1+\eta_2)\mathbb{J}^{-1}\left(\left((1-\alpha)\frac{v^n}{s^n}+\alpha\frac{v^{n+1}}{s^{n+1}}\right)\frac{v^2}{s}\left\|S_{i-1}-S_{q-1}\right\|\right) \\
&= (\eta_1+\eta_2)\left((1-\alpha)\frac{t^n}{n!}+\alpha\frac{t^{n+1}}{(n+1)!}\right)\left\|S_{i-1}-S_{q-1}\right\|.
\end{aligned}
$$

Using convolution theorem, we have

(3.31)

$$
\|S_i - S_q\| \leq \int_0^t (\eta_1+\eta_2)\left((1-\alpha)\frac{(t-\tau)^n}{n!}+\alpha\frac{(t-\tau)^{n+1}}{(n+1)!}\right)\|S_{i-1}-S_{q-1}\|\,d\tau.
$$

Using mean value theorem of integral calculus [9], we get

$$
\|S_i - S_q\| \leq (\eta_1+\eta_2)\left((1-\alpha)M_1T+\alpha M_2T\right)\|S_{i-1}-S_{q-1}\|, \tag{3.32}
$$

where $M_1 = \max_{t\in I}(\zeta-\tau)^n$ and $M_2 = \max_{t\in I}(\zeta-\tau)^{n+1}$.

Let $i = q+1$, then

$$
\begin{aligned}
\|S_{q+1} - S_q\| &\leq \delta\|S_q - S_{q-1}\| \\
&\leq \delta^2\|S_{q-1} - S_{q-2}\| \\
&\vdots
\end{aligned}
$$

$$\leq \delta^q \|S_1 - S_0\|,$$

where $\delta = (\eta_1 + \eta_2)((1-\alpha)M_1T + \alpha M_2T)$.

By using the triangle inequality, we have

(3.33)

$$\begin{aligned}
\|S_i - S_q\| &= \|S_{q+1} - S_q + S_{q+2} - S_{q+1} + \cdots + S_i - S_{i-1}\| \\
&\leq \|S_{q+1} - S_q\| + \|S_{q+2} - S_{q+1}\| + \cdots + \|S_i - S_{i-1}\| \\
&\leq \delta^q \|S_1 - S_0\| + \delta^{q+1} \|S_1 - S_0\| + \cdots + \delta^{i-1} \|S_1 - S_0\| \\
&= \delta^q \left(1 + \delta + \cdots + \delta^{i-q-1}\right) \|S_1 - S_0\| \\
&\leq \delta^q \left(\frac{1-\delta^{i-q}}{1-\delta}\right) \|u_1\|.
\end{aligned}$$

Since $0 < \delta < 1$, we have $1 - \delta^{i-q} < 1$, then

$$\|S_i - S_q\| \leq \frac{\delta^q}{1-\delta} \|u_1\|. \tag{3.34}$$

Also u_1 is bounded, therefore $\|S_i - S_q\| = 0$ as $q \to +\infty$. So $\{S_i\}_{i\geq 0}$ is a Cauchy sequence in the Banach space B, thus the series $\sum_{i=0}^{+\infty} u_i(X,t)$ is convergent.

Hence, the proof is completed. □

3.4. **Error Estimate.** This subsection gives a result related to the maximum absolute truncation error for the $\mathbb{J}$-DM series solution.

Theorem 3.14. *The maximum absolute truncation error of the series solution of equation (3.6) can be obtained by*

$$\max_{t\in I} \left| u(X,t) - \sum_{j=1}^{m} u_j(X,t) \right| \leq \frac{\delta^{m+1}}{1-\delta} \max_{t\in I} |u_1|. \tag{3.35}$$

Proof. From Theorem 3.13, we have

$$\|S_n - S_m\| \leq \frac{\delta^{m+1}}{1-\delta} \|u_1\|. \tag{3.36}$$

As $n \to +\infty$, $S_n \to u(X,t)$, thus

$$\begin{aligned}
\max_{t\in I} \left| u(X,t) - \sum_{j=1}^{m} u_j(X,t) \right| &= \max_{t\in I} |S_n - S_m| \\
&= \|S_n - S_m\| \\
&\leq \frac{\delta^{m+1}}{1-\delta} \|u_1\|
\end{aligned} \tag{3.37}$$

$$= \frac{\delta^{m+1}}{1-\delta} \max_{t\in I} |u_1| .$$

Hence, the proof is completed. □

4. Examples

This section shows the validity and efficiency of J-DM, by demonstrating three examples of fractional biological population models involving the Caputo-Fabrizio fractional derivative.

Example 4.1. Consider a Caputo-Fabrizio fractional biological population model with Malthusian law of the following form

$$\mathbb{D}^{\alpha} u = \left(u^2\right)_{xx} + \left(u^2\right)_{yy} + u, \quad 0 < \alpha \leq 1, \tag{4.1}$$

subject to the initial condition

$$\begin{aligned} u(x, y, 0) &= u_0(x, y) \\ &= \sqrt{\sin(x)\sinh(y)}. \end{aligned} \tag{4.2}$$

The exact solution of equations (4.1)-(4.2) when $\alpha = 1$, is [21]

$$u(x, y, t) = \left(\sqrt{\sin x \sinh y}\right) \exp(t).$$

For solving equations (4.1)-(4.2), we compare (4.1)-(4.2) with equations (3.6)-(3.7), we obtain

$$\mathbb{D}^{\alpha} u = N(u) + u,$$

where $N(u) = \left(u^2\right)_{xx} + \left(u^2\right)_{yy}$.

We assume equations (4.1)-(4.2) has a solution given by

$$u(x, y, t) = \sum_{i=0}^{\infty} u_i(x, y, t). \tag{4.3}$$

To obtain the series solution of equation (4.1), we consider the following equation

$$\mathbb{D}^{\alpha} u_{\lambda} = \lambda \left[N(u_{\lambda}) + u_{\lambda}\right], \quad \lambda \in [0, 1], \tag{4.4}$$

subject to the initial condition

$$u_{\lambda}(x, y, 0) = \phi_0(x, y), \tag{4.5}$$

and we assume that equations (4.4)-(4.5) has a solution of the following form

$$u_{\lambda}(x, y, t) = \sum_{i=0}^{\infty} \lambda^i u_i(x, y, t). \tag{4.6}$$

An application of $\mathbb{J}$-transform on both sides of (4.4), Theorem 3.8 and the initial condition (4.5), gives

$$\mathbb{J}[u_\lambda] = \frac{v^2}{s}\phi_0 + \lambda\frac{s+\alpha(v-s)}{s}\mathbb{J}[N(u_\lambda)+u_\lambda]. \tag{4.7}$$

Taking the inverse $\mathbb{J}$-transform on both sides of equation (4.7), we get

$$u_\lambda = \phi_0 + \lambda\left[\mathbb{J}^{-1}\left(\frac{s+\alpha(v-s)}{s}\mathbb{J}[N(u_\lambda)+u_\lambda]\right)\right]. \tag{4.8}$$

Using Remark 3.11 and equation (4.6) into equation (4.8), we obtain

$$\sum_{i=0}^{\infty}\lambda^i u_i = \phi_0 + \lambda\left[\mathbb{J}^{-1}\left(\frac{s+\alpha(v-s)}{s}\mathbb{J}\left[\sum_{i=0}^{\infty}\lambda^i K_i + \sum_{i=0}^{\infty}\lambda^i u_i\right]\right)\right], \tag{4.9}$$

where K_i are polynomials of $u_0, u_1, \ldots, u_n$, and it can be calculated by formula (3.9).

For the convenience of the reader, the first few components of the polynomials K_i are given by

$$\begin{aligned} K_0 &= \left(u_0^2\right)_{xx} + \left(u_0^2\right)_{yy}, \\ K_1 &= 2\left(u_0u_1\right)_{xx} + 2\left(u_0u_1\right)_{yy}, \\ K_2 &= \left(2u_0u_2+u_1^2\right)_{xx} + \left(2u_0u_2+u_1^2\right)_{yy}. \end{aligned} \tag{4.10}$$

We equate the terms in equation (4.9) with identical powers of λ, we obtain the following components

$$\begin{aligned} u_0 &= \phi_0, \\ u_1 &= \mathbb{J}^{-1}\left(\frac{s+\alpha(v-s)}{s}\mathbb{J}[K_0+u_0]\right), \\ u_i &= \mathbb{J}^{-1}\left(\frac{s+\alpha(v-s)}{s}\mathbb{J}[K_{i-1}+u_{i-1}]\right), \quad i=2,3,4,\ldots \end{aligned} \tag{4.11}$$

Using equations (4.3) and (4.6), we can write

$$\begin{aligned} u(x,y,t) &= \lim_{\lambda\to 1} u_\lambda(x,y,t) \\ &= u_0(x,y,t) + u_1(x,y,t) + \sum_{i=2}^{\infty} u_i(x,y,t). \end{aligned} \tag{4.12}$$

Using equation (4.12), we have $u(x,y,0) = \lim_{\lambda\to 1} u_\lambda(x,t,0)$, it gives that $u_0 = \phi_0$.

The other components of the solution can be easily found using the relations (4.10) and (4.11), so we have

$u_0(x,y,t) = \sqrt{\sin(x)\sinh(y)}$,

$$u_1(x,y,t) = \sqrt{\sin(x)\sinh(y)}\left((1-\alpha)+\alpha t\right),$$
$$u_2(x,y,t) = \sqrt{\sin(x)\sinh(y)}\left((1-\alpha)^2+2(1-\alpha)\alpha t+\alpha^2\frac{t^2}{2!}\right),$$
$$u_3(x,y,t) = \sqrt{\sin(x)\sinh(y)}\left((1-\alpha)^3+3(1-\alpha)^2\alpha t+3\alpha^2(1-\alpha)\frac{t^2}{2!}+\alpha^3\frac{t^3}{3!}\right),$$
$$\vdots$$

and so on.

Using the values of $u_0(x,y,t), u_1(x,y,t), u_2(x,y,t), \ldots$, thus the approximate solution of equations (4.1)-(4.2) is presented as follows

(4.13)
$$u(x,y,t) = \sqrt{\sin(x)\sinh(y)}\Bigg(-\left(\alpha^3-4\alpha^2+6\alpha-4\right)+\left(3\alpha^3-8\alpha^2+6\alpha\right)t - \left(3\alpha^3-4\alpha^2\right)\frac{t^2}{2!}+\alpha^3\frac{t^3}{3!}+\cdots\Bigg).$$

Putting $\alpha = 1$ in equation (4.13), we obtain

$$u(x,y,t) = \sqrt{\sin(x)\sinh(y)}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right),$$

which converge very rapidly to the exact solution

$$u(x,y,t) = \sqrt{\sin(x)\sinh(y)}\exp(t).$$

The result obtained is the same as that obtained by CRDTM [1] and MGTFSM [14].

Example 4.2. Consider a Caputo-Fabrizio fractional biological population model with Verhulst law of the following form

$$\mathbb{D}^\alpha u = \left(u^2\right)_{xx}+\left(u^2\right)_{yy}+u(1-ru), \quad 0<\alpha\leq 1, \tag{4.14}$$

subject to the initial condition

$$u(x,y,0) = u_0(x,y) = \exp\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right). \tag{4.15}$$

The exact solution of equations (4.14)-(4.15) when $\alpha = 1$, is [14]

$$u(x,y,t) = \exp\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)+t\right).$$

For solving equations (4.14)-(4.15), we compare (4.14)-(4.15) with equations (3.6)-(3.7), we obtain

$$\mathbb{D}^\alpha u = N(u)+u,$$

where $N(u) = \left(u^2\right)_{xx} + \left(u^2\right)_{yy} - ru^2$.

We assume equations (4.14)-(4.15) has a solution given by

$$u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t). \tag{4.16}$$

To obtain the series solution of equation (4.14), we consider the following equation

$$\mathbb{D}^{\alpha} u_\lambda = \lambda \left[N(u_\lambda) + u_\lambda\right], \quad \lambda \in [0,1], \tag{4.17}$$

subject to the initial condition

$$u_\lambda(x,y,0) = \phi_0(x,y), \tag{4.18}$$

and we assume that equations (4.17)-(4.18) has a solution of the following form

$$u_\lambda(x,y,t) = \sum_{i=0}^{\infty} \lambda^i u_i(x,y,t). \tag{4.19}$$

An application of $\mathbb{J}$-transform on both sides of (4.17), Theorem 3.8 and the initial condition (4.18), gives

$$\mathbb{J}\left[u_\lambda\right] = \frac{v^2}{s}\phi_0 + \lambda \frac{s + \alpha\,(v-s)}{s} \mathbb{J}\left[N(u_\lambda) + u_\lambda\right]. \tag{4.20}$$

Taking the inverse $\mathbb{J}$-transform on both sides of equation (4.20), we get

$$u_\lambda = \phi_0 + \lambda \left[\mathbb{J}^{-1}\left(\frac{s + \alpha\,(v-s)}{s} \mathbb{J}\left[N(u_\lambda) + u_\lambda\right]\right)\right]. \tag{4.21}$$

Using Remark 3.11 and equation (4.19) into equation (4.21), we obtain

$$\sum_{i=0}^{\infty} \lambda^i u_i = \phi_0 + \lambda \left[\mathbb{J}^{-1}\left(\frac{s + \alpha\,(v-s)}{s} \mathbb{J}\left[\sum_{i=0}^{\infty} \lambda^i K_i + \sum_{i=0}^{\infty} \lambda^i u_i\right]\right)\right], \tag{4.22}$$

where K_i are polynomials of $u_0, u_1, \ldots, u_n$, and it can be calculated by formula (3.9).

For the convenience of the reader, the first few components of the polynomials K_i are given by

$$\begin{aligned} K_0 &= \left(u_0^2\right)_{xx} + \left(u_0^2\right)_{yy} - ru_0^2, \\ K_1 &= 2\left(u_0 u_1\right)_{xx} + 2\left(u_0 u_1\right)_{yy} - 2ru_0 u_1, \\ K_2 &= \left(2u_0 u_2 + u_1^2\right)_{xx} + \left(2u_0 u_2 + u_1^2\right)_{yy} - r\left(2u_0 u_2 + u_1^2\right). \end{aligned} \tag{4.23}$$

We equate the terms in equation (4.22) with identical powers of λ, we obtain the following components

$$
\begin{aligned}
(4.24)\qquad & u_0 = \phi_0, \\
& u_1 = \mathbb{J}^{-1}\left(\frac{s+\alpha\,(v-s)}{s}\mathbb{J}\left[K_0+u_0\right]\right), \\
& u_i = \mathbb{J}^{-1}\left(\frac{s+\alpha\,(v-s)}{s}\mathbb{J}\left[K_{i-1}+u_{i-1}\right]\right), \quad i=2,3,4,\ldots
\end{aligned}
$$

Using equations (4.16) and (4.19), we can write

$$
\begin{aligned}
(4.25)\qquad u(x,y,t) &= \lim_{\lambda\to 1} u_\lambda(x,y,t) \\
&= u_0(x,y,t)+u_1(x,y,t)+\sum_{i=2}^{\infty} u_i(x,y,t).
\end{aligned}
$$

Using equation (4.25), we have $u(x,y,0)=\lim\limits_{\lambda\to 1}u_\lambda(x,t,0)$, it gives that $u_0=\phi_0$.

The other components of the solution can be easily found using the relations (4.23) and (4.24), so we have

$$
\begin{aligned}
u_0(x,y,t) &= \exp\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right), \\
u_1(x,y,t) &= \exp\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)\left((1-\alpha)+\alpha t\right), \\
u_2(x,y,t) &= \exp\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)\left((1-\alpha)^2+2\,(1-\alpha)\,\alpha t+\alpha^2\frac{t^2}{2!}\right), \\
u_3(x,y,t) &= \exp\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right) \\
&\quad\times\left((1-\alpha)^3+3\,(1-\alpha)^2\,\alpha t+3\alpha^2\,(1-\alpha)\,\frac{t^2}{2!}+\alpha^3\frac{t^3}{3!}\right), \\
&\ \ \vdots
\end{aligned}
$$

and so on.

Using the values of $u_0(x,y,t), u_1(x,y,t), u_2(x,y,t),\ldots$, thus the approximate solution of equations (4.14)-(4.15) is presented as follows

(4.26)

$$
\begin{aligned}
u(x,y,t) = \exp\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)\Bigg(&-\left(\alpha^3-4\alpha^2+6\alpha-4\right) \\
&+\left(3\alpha^3-8\alpha^2+6\alpha\right)t-\left(3\alpha^3-4\alpha^2\right)\frac{t^2}{2!}+\alpha^3\frac{t^3}{3!}+\cdots\Bigg).
\end{aligned}
$$

Putting $\alpha = 1$ in equation (4.26), we obtain

$$u(x,y,t) = \exp\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right),$$

which converge very rapidly to the exact solution

$$\begin{aligned} u(x,y,t) &= \exp\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)\exp(t) \\ &= \exp\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y) + t\right). \end{aligned}$$

The result obtained is the same as that obtained by CRDTM [1] and MGTFSM [14].

Example 4.3. Consider a Caputo-Fabrizio fractional biological population model with porous media of the following form

$$\mathbb{D}^{\alpha}u = \left(u^2\right)_{xx} + \left(u^2\right)_{yy} - u\left(1 + \frac{8}{9}u\right), \quad 0 < \alpha \leq 1, \tag{4.27}$$

subject to the initial condition

$$\begin{aligned} u(x,y,0) &= u_0(x,y) \\ &= \exp\left(\frac{x+y}{3}\right). \end{aligned} \tag{4.28}$$

The exact solution of equations (4.27)-(4.28) when $\alpha = 1$, is [27]

$$u(x,y,t) = \exp\left(\frac{x+y}{3} - t\right).$$

For solving equations (4.27)-(4.28), we compare (4.27)-(4.28) with equations (3.6)-(3.7), we obtain

$$\mathbb{D}^{\alpha}u = N(u) - u,$$

where $N(u) = \left(u^2\right)_{xx} + \left(u^2\right)_{yy} - \frac{8}{9}u^2$.

We assume equations (4.27)-(4.28) has a solution given by

$$u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t). \tag{4.29}$$

To obtain the series solution of equation (4.27), we consider the following equation

$$\mathbb{D}^{\alpha}u_{\lambda} = \lambda\left[N(u_{\lambda}) - u_{\lambda}\right], \quad \lambda \in [0,1], \tag{4.30}$$

subject the initial condition

$$u_{\lambda}(x,y,0) = \phi_0(x,y), \tag{4.31}$$

and we assume that equations (4.30)-(4.31) has a solution of the following form

$$u_\lambda(x,y,t) = \sum_{i=0}^{\infty} \lambda^i u_i(x,y,t). \tag{4.32}$$

An application of $\mathbb{J}$-transform on both sides of (4.30), Theorem 3.8 and the initial condition (4.31), gives

$$\mathbb{J}[u_\lambda] = \frac{v^2}{s}\phi_0 + \lambda \frac{s+\alpha(v-s)}{s}\mathbb{J}[N(u_\lambda) - u_\lambda]. \tag{4.33}$$

Taking the inverse $\mathbb{J}$-transform on both sides of equation (4.33), we get

$$u_\lambda = \phi_0 + \lambda\left[\mathbb{J}^{-1}\left(\frac{s+\alpha(v-s)}{s}\mathbb{J}[N(u_\lambda) - u_\lambda]\right)\right]. \tag{4.34}$$

Using Remark 3.11 and equation (4.32) into equation (4.34), we obtain

$$\sum_{i=0}^{\infty} \lambda^i u_i = \phi_0 + \lambda\left[\mathbb{J}^{-1}\left(\frac{s+\alpha(v-s)}{s}\mathbb{J}\left[\sum_{i=0}^{\infty}\lambda^i K_i - \sum_{i=0}^{\infty}\lambda^i u_i\right]\right)\right], \tag{4.35}$$

where K_i are polynomials of $u_0, u_1, \ldots, u_n$, and it can be calculated by formula (3.9).

For the convenience of the reader, the first few components of the polynomials K_i are given by

$$\begin{aligned} K_0 &= \left(u_0^2\right)_{xx} + \left(u_0^2\right)_{yy} - \frac{8}{9}u_0^2, \\ K_1 &= 2\left(u_0u_1\right)_{xx} + 2\left(u_0u_1\right)_{yy} - \frac{16}{9}u_0u_1, \\ K_2 &= \left(2u_0u_2 + u_1^2\right)_{xx} + \left(2u_0u_2 + u_1^2\right)_{yy} - \frac{8}{9}\left(2u_0u_2 + u_1^2\right). \end{aligned} \tag{4.36}$$

We equate the terms in equation (4.35) with identical powers of λ, we obtain the following components

$$\begin{aligned} u_0 &= \phi_0, \\ u_1 &= \mathbb{J}^{-1}\left(\frac{s+\alpha(v-s)}{s}\mathbb{J}[K_0 - u_0]\right), \\ u_i &= \mathbb{J}^{-1}\left(\frac{s+\alpha(v-s)}{s}\mathbb{J}[K_{i-1} - u_{i-1}]\right), \quad i = 2,3,4,\ldots \end{aligned} \tag{4.37}$$

Using equations (4.29) and (4.32), we can write

$$u(x,y,t) = \lim_{\lambda\to 1} u_\lambda(x,y,t) \tag{4.38}$$

$$= u_0(x,y,t) + u_1(x,y,t) + \sum_{i=2}^{\infty} u_i(x,y,t).$$

Using equation (4.38), we have $u(x,y,0) = \lim_{\lambda \to 1} u_\lambda(x,t,0)$, it gives that $u_0 = \phi_0$.

The other components of the solution can be easily found using the relations (4.36) and (4.37), so we have

$$\begin{aligned}
u_0(x,y,t) &= \exp\left(\frac{x+y}{3}\right), \\
u_1(x,y,t) &= -\exp\left(\frac{x+y}{3}\right)(1-\alpha) + \alpha t, \\
u_2(x,y,t) &= \exp\left(\frac{x+y}{3}\right)\left((1-\alpha)^2 + 2(1-\alpha)\alpha t + \alpha^2 \frac{t^2}{2!}\right), \\
u_3(x,y,t) &= -\exp\left(\frac{x+y}{3}\right) \\
&\quad \times \left((1-\alpha)^3 + 3(1-\alpha)^2 \alpha t + 3\alpha^2 (1-\alpha)\frac{t^2}{2!} + \alpha^3 \frac{t^3}{3!}\right), \\
&\vdots
\end{aligned}$$

and so on.

Using the values of $u_0(x,y,t), u_1(x,y,t), u_2(x,y,t), \ldots$, thus the approximate solution of equations (4.27)-(4.28) is presented as follows

(4.39)

$$\begin{aligned}
u(x,y,t) = \exp\left(\frac{x+y}{3}\right)\Bigg(&\left(\alpha^3 - 2\alpha^2 + 2\alpha\right) - \left(3\alpha^3 - 4\alpha^2 + 2\alpha\right) t \\
&+ \left(3\alpha^3 - 2\alpha^2\right)\frac{t^2}{2!} - \alpha^3 \frac{t^3}{3!} + \cdots \Bigg).
\end{aligned}$$

Putting $\alpha = 1$ in equation (4.39), we obtain

$$u(x,y,t) = \exp\left(\frac{x+y}{3}\right)\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right),$$

which converge very rapidly to the exact solution

$$\begin{aligned}
u(x,y,t) &= \exp\left(\frac{x+y}{3}\right)\exp(-t) \\
&= \exp\left(\frac{x+y}{3} - t\right).
\end{aligned}$$

The result obtained is the same as that obtained by CRDTM [1] and MGTFSM [14].

5. Results and Discussion

In this section, we discuss the relationship and comparison between the exact and the $\mathbb{J}$-DM results for the three fractional biological population model cases using 2D-Plots, 3D-Plots and tables.

Figures 1, 3 and 5 represent the surface plots of the exact and approximate solutions obtained by $\mathbb{J}$-DM for different fractional orders α when $t = 1.5$ for Examples 4.1 and 4.2 and $t = 0.5$ for Example 4.3. Figures 2, 4 and 6 represent the two-dimensional behavior of the exact and approximate solutions obtained using the present method for different fractional orders α when $x = y = 1$ for Examples 4.1 and 4.3 and $x = y = r = 1$ for Example 4.2. From these figures, we observe that the exact and approximate solutions at $\alpha = 1$ for Examples 4.1–4.3 are in excellent agreement with each other. Tables 1, 3 and 5 show the numerical values of the exact and approximate solutions obtained by $\mathbb{J}$-DM for different fractional orders of α when $x = y = 0.1$ for Examples 4.1 and 4.3 and $x = y = 0.1$ and $r = 1$ for Example 4.2. It is clear from these tables that the accuracy increases with the increase in the order of approximation. Tables 2, 4 and 6 show the absolute error comparison of the $\mathbb{J}$-DM with CRDTM [1] and MGTFSM [14] at $\alpha = 1$. From these tables, it is evident that our method converges just as quickly as the other methods.

The plots and tables, show that the proposed method is more accurate and very effective in analysing the considered fractional biological population model.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	u_{exact}
0.01	0.14301	0.12602	0.11221	0.10102	0.10102
0.03	0.14567	0.12849	0.11447	0.10305	0.10305
0.05	0.14836	0.13101	0.11677	0.10513	0.10513
0.07	0.15109	0.13357	0.11911	0.10725	0.10725
0.09	0.15386	0.13616	0.12150	0.10942	0.10942

Table 1. The numerical values of the $\mathbb{J}$-DM-approximate and exact solutions of Example 4.1 for different values of α when $x = y = 0.1$

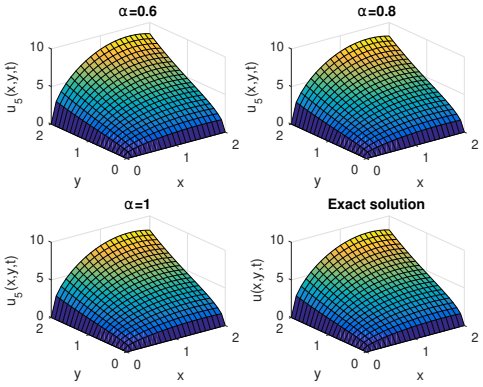


FIGURE 1. The 3D plots of the 5^{th}-order approximate and exact solutions for Example 4.1 for different values of α when $t = 1.5$

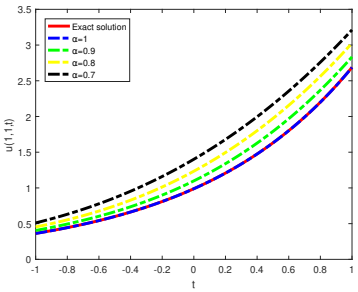


FIGURE 2. The 2D plots of the 5^{th}-order approximate and exact solutions for Example 4.1 for different values of α when $x = y = 1$

	CRDTM	MGTFSM	$\mathbb{J}$-DM
t	$\lvert u_{exact} - u_{CRDTM}\rvert$	$\lvert u_{exact} - u_{MGTFSM}\rvert$	$\lvert u_{exact} - u_{\mathbb{J}-DM}\rvert$
0.01	4.1750×10^{-11}	4.1750×10^{-11}	4.1750×10^{-11}
0.03	3.3953×10^{-9}	3.3953×10^{-9}	3.3953×10^{-9}
0.05	2.6304×10^{-8}	2.6304×10^{-8}	2.6304×10^{-8}
0.07	1.0146×10^{-7}	1.0146×10^{-7}	1.0146×10^{-7}
0.09	2.7837×10^{-7}	2.7837×10^{-7}	2.7837×10^{-7}

TABLE 2. Comparison of the $\mathbb{J}$-DM with CRDTM [1] and MGTFSM [14] in terms of absolute error for Example 4.1 at $\alpha = 1$

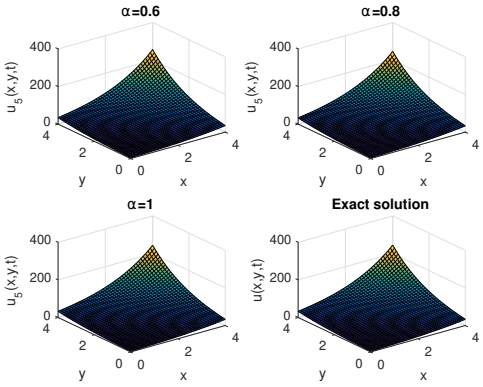


FIGURE 3. The 3D plots of the 5^{th}-order approximate and exact solutions for Example 4.2 for different values of α when $r = 1$ and $t = 1.5$

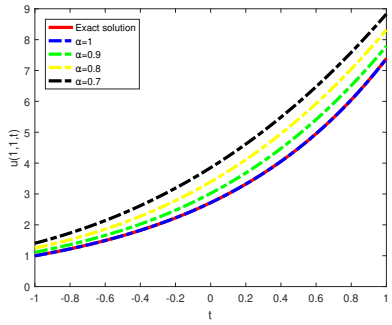


FIGURE 4. The 2D plots of the 5^{th}-order approximate and exact solutions for Example 4.2 for different values of α when $x = y = r = 1$

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	u_{exact}
0.01	1.5349	1.3525	1.2043	1.0841	1.0841
0.03	1.5634	1.3791	1.2286	1.1060	1.1060
0.05	1.5923	1.4061	1.2532	1.1283	1.1283
0.07	1.6216	1.4335	1.2784	1.1511	1.1511
0.09	1.6514	1.4614	1.3040	1.1743	1.1743

TABLE 3. The numerical values of the $\mathbb{J}$-DM-approximate and exact solutions of Example 4.2 for different values of α when $r = 1$ and $x = y = 0.1$

	CRDTM	MGTFSM	$\mathbb{J}$-DM
t	$\lvert u_{exact} - u_{CRDTM}\rvert$	$\lvert u_{exact} - u_{MGTFSM}\rvert$	$\lvert u_{exact} - u_{\mathbb{J}-DM}\rvert$
0.01	4.4809×10^{-10}	4.4809×10^{-10}	4.4809×10^{-10}
0.03	3.6441×10^{-8}	3.6441×10^{-8}	3.6441×10^{-8}
0.05	2.8232×10^{-7}	2.8232×10^{-7}	2.8232×10^{-7}
0.07	1.0889×10^{-6}	1.0889×10^{-6}	1.0889×10^{-6}
0.09	2.9877×10^{-6}	2.9877×10^{-6}	2.9877×10^{-6}

TABLE 4. Comparison of the $\mathbb{J}$-DM with CRDTM [1] and MGTFSM [14] in terms of absolute error for Example 4.2 at $\alpha = 1$ and $r = 1$

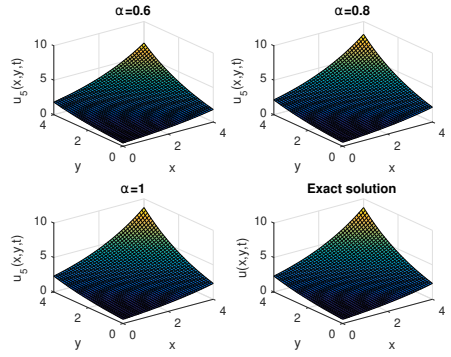


FIGURE 5. The 3D plots of the 5^{th}-order approximate and exact solutions for Example 4.3 for different values of α when $t = 0.5$

6. CONCLUSIONS

In the present paper, we have studied the existence, uniqueness and convergence of solutions of the fractional biological population model

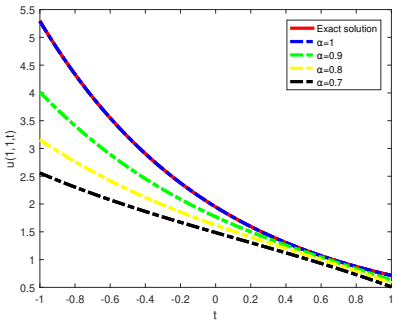


FIGURE 6. The 2D plots of the 5^{th}-order approximate and exact solutions for Example 4.3 for different values of α when $x = y = 1$

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	u_{exact}
0.01	0.81059	0.88321	0.96371	1.0583	1.0583
0.03	0.80058	0.87101	0.94798	1.0373	1.0373
0.05	0.79059	0.85890	0.93248	1.0168	1.0168
0.07	0.78061	0.84690	0.91721	0.99667	0.99667
0.09	0.77065	0.83499	0.90216	0.97694	0.97694

TABLE 5. The numerical values of the $\mathbb{J}$-DM-approximate and exact solutions of Example 4.3 for different values of α when $x = y = 0.1$

	CRDTM	MGTFSM	$\mathbb{J}$-DM
t	$\lvert u_{exact} - u_{CRDTM}\rvert$	$\lvert u_{exact} - u_{MGTFSM}\rvert$	$\lvert u_{exact} - u_{\mathbb{J}-DM}\rvert$
0.01	4.4450×10^{-10}	4.4450×10^{-10}	4.4450×10^{-10}
0.03	3.5861×10^{-8}	3.5861×10^{-8}	3.5861×10^{-8}
0.05	2.7561×10^{-7}	2.7561×10^{-7}	2.7561×10^{-7}
0.07	1.0546×10^{-6}	1.0546×10^{-6}	1.0546×10^{-6}
0.09	2.8704×10^{-6}	2.8704×10^{-6}	2.8704×10^{-6}

TABLE 6. Comparison of the $\mathbb{J}$-DM with CRDTM [1] and MGTFSM [14] in terms of absolute error for Example 4.3 at $\alpha = 1$

given by the Caputo-Fabrizio sense using the Banach fixed point theorem. Moreover, we proposed a new hybrid method, called $\mathbb{J}$-decomposition method ($\mathbb{J}$-DM), which is successfully implemented to solve the proposed model. For illustration purposes, we considered three different examples

to demonstrate our results. Approximate solutions are compared with exact and existing solutions by other methods. The study demonstrates that $\mathbb{J}$-DM requires less computational work than existing methods while providing reliable results. The present method offers significant advantages in simplicity, applicability, computational efficiency and accuracy for finding approximate analytical solutions to many nonlinear fractional problems. Hence, it would also be interesting to study various other problems that use the Caputo-Fabrizio fractional derivative for future work.

Acknowledgment. The author wish to thank the SCMA Editors and anonymous reviewers for their helpful suggestions, which improved the final version of this paper.

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