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A Novel System and Iterative Schemes for Generalized Variational Inclusion Problems and its Approximation Solvability

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Abstract. The variational inclusion problems are the generalizations of the variational inequality problems. First, it was noted by Hasounni and Moudafi in 1994. Afterwards, many researchers look into the problems and try to improve the catch, and then the theory of the variational inclusion problems is enriched from time to time. There were some specific ways of solving the systems of the problems and it is called the approximation solvability of the system, which was due to R. U. Verma in 2008. Here in this study, we investigate the approximation solvability of solutions for the system of generalized nonlinear ordered variational inclusions using an iterative algorithm under some apt conditions. We employ the resolvent operator techniques in our iterative scheme as a baseline. Some exceptional cases of the problem and iterative scheme are also discussed.

1. Introduction

The theory of generalized nonlinear ordered variational inclusions (inequalities) has a great application in optimization, image processing, control theory, mathematical physics, network equilibrium problems, economics, transportation equilibrium and basic and applied sciences.

Hasounni and Moudafi pioneered variational inclusions. Their paper [11] is a useful and notable annexe of variational inequalities. In [11], they studied perturbed methods for solving the variational inclusions problem. After that, this class was generalized and extended by S. Adly [1], Ahmad and Ansari [5], and, Ding [9], and many more.

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Recently, the monotonicity adroitness was carried on and applied because of its significance in the theory of variational inequalities, variational inclusions and complementarity problems. It is a well-known fact that the projection technique is deficient when solving a variational inclusions problem. To eliminate this situation resolvent operator methods came into existence for solving various variational inclusions problems, see e.g., [2, 24, 25, 27, 29]. A problem of leisure pursuit known as the system of variational inequalities (inclusions) was pioneered in the studies. Ansari and Yao [6], Bianchi [7], Cohen and Chaplais [8] and Peng [21], deemed a system of scalar variational inequalities and the various types of problems like traffic equilibrium, the Nash equilibrium, spatial equilibrium and the general equilibrium problem can be transformed into a system of variational inequalities which Peng exhibited in his studies. Noor [17, 18, 20], Noor and Noor [19] pioneered some classes of variational inclusions (inequalities) and designed iterative schemes for approximating the solutions of variational inclusions (inequalities) in real Hilbert spaces. It is notorious that fixed point theory with applications for nonlinear mappings has been widely studied in ordered Banach spaces. Plenty of research concerned with the ordered equations and ordered variational inequalities in ordered Banach spaces has been done by Li and his co-authors, see [14, 16]. Many problems concerning ordered variational inclusions are answered by the resolvent technique linked with RME set-valued mappings [15], (α, λ)-NODM set-valued mapping [13], (γ_G, λ)-weak RRD mapping [3] and (α, λ)-weak ANODD set-valued map with strong comparison mapping A [16] and many more see, e.g., [4, 12, 22] and the citations therein. In this work, we make use of the resolvent operator approach for the approximation solvability of solutions of an implicit system of generalized nonlinear ordered variational inclusions in real ordered Hilbert spaces.

2. Some Preliminaries

Throughout the draft, we can symbolize the notion by the following characterizations.

- X stands for real ordered Hilbert spaces;
- C stands for cone with normal constant λ_{N_C};
- $\langle \cdot, \cdot \rangle$, $\|\cdot\|$ and Θ stand for inner product, norm and zero element;
- $lub\{m, n\}$ and $glb\{m, n\}$ stand for the least upper and greatest lower bounds for the set $\{m, n\}$.

In addition, suppose that $glb\{m, n\}$ and $lub\{m, n\}$ both exist.

In this section, we review some basic facts which are supplementary for the buildup of the draft.

Definition 2.1. $C \subset X(\neq \varphi)$ a nonvoid closed, convex subset is called a cone if

(a) as $m \in C$ and $\kappa > 0,\ \kappa m \in C$;
(b) if m and $-m \in C$, then $m = \Theta$.

Definition 2.2. A relation " $\leq$ " defined as $m \leq n$ iff $m - n \in C$ for $m, n \in X$ is known as partial order relation expounded by C in X, then $(X, \text{" } \leq \text{ "})$ is called a real ordered Hilbert space.

Definition 2.3 ([23])**.** For members $m, n \in X$ having relation $m \leq n$ (or $n \leq m$) are called as comparable with each other and is noted as $m \propto n$.

Definition 2.4 ([10])**.** A nonvoid subset $C \subset X$ is characterized as normal cone iff there is a constant $\lambda_{N_C} > 0$ such that $0 \leq m \leq n$ implies $\|m\| \leq \lambda_{N_C}\|n\|$, where λ_{N_C} stands for normal constant of C.

Definition 2.5 ([3])**.** For some elements $m,\ n \in X$. The relations $\vee$ and $\oplus$ are defined as follows:

$$m \vee n = lub\{m, n\}$$
$$m \oplus n = (m - n) \vee (n - m).$$

Definition 2.6 ([3])**.** A single-valued map $A : X \to X$ is said to be comparison map, if for each $m, n \in X$, $m \propto n$ then $A(m) \propto A(n),\ m \propto A(m)$ and $n \propto A(n)$.

Definition 2.7 ([3])**.** A single-valued map $A : X \to X$ is termed as β-ordered compression, if it is comparison map and

$$A(m) \oplus A(n) \leq \beta(m \oplus n), \quad \text{for } 0 < \beta < 1.$$

Lemma 2.8 ([10])**.** *Let X be a real ordered Hilbert space having normal cone C with normal constant λ_{N_C}. For every $m, n \in X$, the following relation hold:*

(a) $\|\Theta \oplus \Theta\| = \|\Theta\| = \Theta$;
(b) $\|m \wedge n\| \leq \|m\| \wedge \|n\| \leq \|m\| + \|n\|$;
(c) $\|m \oplus n\| \leq \|m - n\| \leq \lambda_{N_C}\|m \oplus n\|$;
(d) *if $m \propto n$, then $\|m \oplus n\| = \|m - n\|$.*

Lemma 2.9 ([13, 15])**.** *Let X be a real ordered Hilbert space with a cone C having normal constant λ_{N_C} and " $\leq$ " be a relation defined in Definition 2.2. Then, we have the relations below:*

(a) $m \oplus n = n \oplus m,\ m \oplus m = \Theta$;
(b) $\Theta \leq m \oplus \Theta$;
(c) *if κ is real, then $(\kappa m) \oplus (\kappa n) = |\kappa|(m \oplus n)$;*
(d) *if m, n and l are be comparable to each others, then $(m \oplus n) \leq (m \oplus l) + (l \oplus n)$;*

(e) *suppose* $(m+n)\vee(s+t)$ *exist and if* $m\propto s,t$ *and* $n\propto s,t$, *then* $(m+n)\oplus(s+t)\leq(m\oplus s+n\oplus t)\wedge(m\oplus t+n\oplus s)$;
(f) *if* m,n,r,w *all are comparable with each other, then* $(m\wedge n)\oplus(r\wedge w)\leq((m\oplus r)\vee(n\oplus w))\wedge((m\oplus w)\vee(n\oplus r))$;
(g) *if* $m\leq n$ *and* $s\leq t$, *then* $m+s\leq n+t$;
(h) *if* $m\propto\Theta$, *then* $-m\oplus\Theta\leq m\leq m\oplus\Theta$;
(i) *if* $m\propto n$, *then* $(m\oplus\Theta)\oplus(n\oplus\Theta)\leq(m\oplus n)\oplus\Theta=m\oplus n$;
(j) $(m\oplus\Theta)-(n\oplus\Theta)\leq(m-n)\oplus\Theta$;
(k) *if* $\Theta\leq m$ *and* $m\neq\Theta$ *and* $\alpha>0$, *then* $\Theta\leq\alpha m$ *and* $\alpha m\neq\Theta$, *for all* $m,n,r,s,t,l\in X$ *and* $\alpha,\kappa\in\mathbb{R}$.

Definition 2.10 ([13])**.** Let $G:X\to X$ to be a single-valued map.

(a) G is said to be a δ-order non-extended map if there is a positive constant $\delta>0$ such that

$$\delta(m\oplus n)\leq G(m)\oplus G(n)\quad\text{for all}\quad m,n\in X;$$

(b) G is said to be a strongly comparison map if it is comparison map and $G(m)\propto G(n)$ iff $m\propto n$, for all $m,n\in X$.

Definition 2.11 ([16])**.** For a set-valued map $\mathfrak{M}:X\to 2^X$ and a single-valued map $G:X\to X$.

(a) $\mathfrak{M}$ is called as a weak-comparison map, if for $t_m\in\mathfrak{M}(m), m\propto t_m$ and if $m\propto n$, then $\exists\ t_m\in\mathfrak{M}(m)$ and $t_n\in\mathfrak{M}(n)$ such that $t_m\propto t_n$, for all $m,n\in X$;
(b) $\mathfrak{M}$ is called as an κ-weak-non-ordinary difference map linked with G, if it is a weak comparison map and for each $m,n\in X$, $\exists\ \kappa>0$ and $t_m\in\mathfrak{M}(G(m))$ and $t_n\in\mathfrak{M}(G(n))$ such that

$$(t_m\oplus t_n)\oplus\kappa(G(m)\oplus G(n))=\Theta;$$

(c) $\mathfrak{M}$ is called as ξ-order different weak-comparison map linked with G if $\exists\ \xi>0$, for all $m,n\in X$ and there exist $t_m\in\mathfrak{M}(G(m)), t_n\in\mathfrak{M}(G(n))$ such that

$$\xi(t_m-t_n)\propto m-n;$$

(d) If $\mathfrak{M}$ is a weak comparison map having the properties of being both as κ-weak-non-ordinary difference map and ξ-order different weak-comparison map, respectively linked with G and $(G+\xi\mathfrak{M})(X)=X$, for $\kappa,\xi>0$. Then $\mathfrak{M}$ is called as an ordered (κ_G,ξ)-weak-ANODM map.

Definition 2.12 ([16])**.** A map $R^{\mathfrak{M}}_{G,\lambda}:X\to X$ linked with a single-valued δ-ordered non-extended map and a set-valued κ-non-ordinary difference map, $G:X\to X$ and $\mathfrak{M}:X\to 2^X$, respectively is called as resolvent operator and is defined as

$$R^{\mathfrak{M}}_{G,\xi}(m)=(G+\xi\mathfrak{M})^{-1}(m),\quad\text{for all } m\in X, \tag{2.1}$$

where $\delta, \kappa, \xi > 0$ are constants.

Lemma 2.13 ([16]). *The resolvent operator $R^{\mathfrak{M}}_{G,\lambda}$ defined in Definition 2.12 is single-valued for $\kappa\xi \neq 1$, where $\kappa, \xi > 0$, i.e., $R^{\mathfrak{M}}_{G,\xi} : X \to X$ of $\mathfrak{M}$ lives.*

Lemma 2.14 ([16]). *Let $R^{\mathfrak{M}}_{G,\xi} : X \to X$ be an operator linked with a (κ_G, ξ)-weak-ANODD set-valued map $\mathfrak{M} : X \to 2^X$ and a strongly comparison single-valued mapping $G : X \to X$, respectively. Then $R^{\mathfrak{M}}_{G,\xi}$ is a comparison map.*

Lemma 2.15 ([16]). *An ordered (κ_G, ξ)-weak-ANODD map $\mathfrak{M} : X \to 2^X$ and a δ-ordered non-extended map $G : X \to X$ linked with $R^{\mathfrak{M}}_{G,\xi}$ for $\kappa_G > \frac{1}{\xi}$, then the following inequality hold:*

$$R^{\mathfrak{M}}_{G,\xi}(m) \oplus R^{\mathfrak{M}}_{G,\xi}(n) \leq \frac{1}{\delta(\kappa_G\xi - 1)}(m \oplus n), \quad \text{for all } m, n \in X. \tag{2.2}$$

3. Formulation of the Problem

Let C be a normal cone having a normal constant λ_{N_C} in a real ordered Hilbert space X. Let $\mathfrak{N} : X \times X \to X$, $\mathfrak{M} : X \to CB(X)$ and $G, f, g : X \to X$ are the single-valued bifunction, set-valued and single-valued maps, respectively. Now we design the underneath catch:

Acquire $(a, b) \in X \times X$ such that

$$\begin{cases} 0 \in G(f(a)) - G(g(b)) + [\xi_1\mathfrak{N}(g(b), f(a)) + \lambda_1\mathfrak{M}(f(a))], \\ 0 \in G(g(b)) - G(f(a)) + [\xi_2\mathfrak{N}(f(a), g(b)) + \lambda_2\mathfrak{M}(g(b))]. \end{cases} \tag{3.1}$$

This catch is stated as the generalized system of nonlinear variational inclusions (in short GSNVI).

4. Special Cases

(1). If we choose $f,\ g = I$ (Identity maps on X), $\xi_1 = \lambda_1$ and $\xi_2 = \lambda_2$ then the problem GSNVI (3.1) commutes to the catch of obtaining an element $(a, b) \in X \times X$ such that

$$\begin{cases} 0 \in G(a) - G(b) + \lambda_1[\mathfrak{N}(b, a) + \mathfrak{M}(a)], \\ 0 \in G(b) - G(a) + \lambda_2[\mathfrak{N}(a, b) + \mathfrak{M}(b)]. \end{cases} \tag{4.1}$$

This problem was initiated by Wang and Wu [27] and is stated a system of nonlinear variational inclusions problem.

(2). If we choose $f,\ g = I$ (Identity maps on X), $a = b$ and $\xi_1 = \lambda_1 = \xi_2 = \lambda_2$ then the problem GSNVI (3.1) commutes to the catch of obtaining $a \in X$ such that

$$0 \in \mathfrak{N}(a, a) + \mathfrak{M}(a). \tag{4.2}$$

This catch was initiated by Verma [26] and is called as nonlinear variational inclusions problem.

(3). If we choose $f = g = I$ (Identity maps on X), $a = b$ and $\mathfrak{N} = 0$, then the problem GSNVI (3.1) reverted to the underneath zero point problem of obtaining $q \in X$ such that

$$0 \in \mathfrak{M}(q). \tag{4.3}$$

Which was encountered by Li [13] and coined as a class of general nonlinear ordered variational inclusions.

5. Algorithm for the GSNVI

This draft section is devoted to the fixed point articulation and iterative schemes for the catch GSNVI (3.1).

Lemma 5.1. *Let $G, f, g : X \to X$ be the single-valued maps such that f and g are comparison maps and G is strongly comparison map and λ_A-ordered compression map with respect to both f and g, respectively. Suppose $\mathfrak{M} : X \to CB(X)$ is (κ_G, ξ)-weak-ANODD set-valued map. Then $(a^*, b^*) \in X \times X$ is the result of the catch GSNVI (3.1) if and only if it agrees*

$$f(a^*) = R^{\mathfrak{M}}_{G,\lambda_1}[G(g(b)) - \xi_1\mathfrak{N}(g(b), f(a))],$$
$$g(b^*) = R^{\mathfrak{M}}_{G,\lambda_2}[G(f(a)) - \xi_2\mathfrak{N}(f(a), g(b))].$$

Proof. Proof of this lemma immediately concluded from the definition of the resolvent operators $R^{\mathfrak{M}}_{G,\lambda_1}$ and $R^{\mathfrak{M}}_{G,\lambda_2}$, respectively. □

5.1. Iterative Algorithm.

- Main Algorithm: For starting points $a_0, b_0 \in X$, determine the sequences $\{a_n\}$ and $\{b_n\}$ by the following iterative scheme:

$$\begin{cases} f(a_{n+1}) = (1 - \varrho_n)f(a_n) + \varrho_n R^{\mathfrak{M}}_{G,\lambda_1}[G(g(b_n)) - \xi_1\mathfrak{N}(g(b_n), f(a_n))], \\ g(b_n) = R^{\mathfrak{M}}_{G,\lambda_2}[G(f(a_n)) - \xi_2\mathfrak{N}(f(a_n), g(b_n))]. \end{cases} \tag{5.1}$$

- **Case 1:** If $f = g = I$ (Identity maps on X) in algorithm (5.1), then we obtain the following:
  For starting points $a_0, b_0 \in X$, determine the sequences a_n and b_n by the iterative scheme:

$$\begin{cases} a_{n+1} = (1 - \varrho_n)a_n + \varrho_n R^{\mathfrak{M}}_{G,\lambda_1}[G(b_n) - \xi_1\mathfrak{N}(b_n, a_n)], \\ b_n = R^{\mathfrak{M}}_{G,\lambda_2}[G(a_n) - \xi_2\mathfrak{N}(a_n, b_n)]. \end{cases} \tag{5.2}$$

- **Case 2:** If $a_n = b_n \forall\ n$, $f = g = I$ (Identity maps on X) and $\lambda_1 = \lambda_2 = \xi_1 = \xi_2 = \lambda$ (say) in algorithm (5.1), then we obtain the following:

For starting point $a_0 \in X$, determine the sequence $\{a_n\}$ by the iterative scheme:

$$a_{n+1} = (1-\varrho_n)a_n + \varrho_n R^{\mathfrak{M}}_{G,\lambda}[G(a_n) - \lambda\mathfrak{N}(a_n, a_n)]. \tag{5.3}$$

- **Case 3:** If $f = g = G = I$, (Identity maps on X), $\mathfrak{N} = 0$, $\varrho_n = \varrho$ and $a_n = b_n \ \forall\ n$, in scheme (5.3), then we obtain the following:
  For starting point $a_0 \in X$, determine the sequence $\{a_n\}$ by the iterative scheme:

$$a_{n+1} = (1-\varrho)a_n + \varrho R_{\mathfrak{M},\lambda}(a_n). \tag{5.4}$$

Lemma 5.2 ([28]). *Let $\{\beta_n\}_{n=0}^{\infty}$ be a nonnegative real sequence and let δ_n be a real sequence in [0, 1] such that $\sum_{i=1}^{\infty}\delta_i = \infty$, $\sigma_n = O(\delta_n)$ and agrees with*

$$\beta_{n+1} \leq (1-\delta_n)\beta_n + \sigma_n.$$

Then $\lim_{n\to\infty}\beta_n = 0$.

6. Main Result

This section of the draft is devoted to our main result in which we prove a convergence result for generalized system of nonlinear variational inclusions problem (3.1) (in short GSNVI).

Theorem 6.1. *Suppose X be a real ordered Hilbert space with a normal cone C having normal constant λ_{N_C}. Let $G, f, g : X \to X$ are the single-valued maps such that G is strongly comparison and λ_A-ordered compression map with respect to both f and g, f be λ_f-ordered compression and γ_f-ordered non-extended map and g be λ_g-ordered compression and γ_g-ordered non-extended map, respectively. Let $\mathfrak{N} : X \times X \to X$ be a single-valued bifunction such that $\mathfrak{N}$ is λ_{N_1}-ordered compression in first spot and λ_{N_2}-ordered compression in second spot with respect to both f and g, respectively. Presume that $\mathfrak{M} : X \to CB(X)$ is (κ_G, ξ)-weak-ANODD set-valued map.*

In addition, if $G \propto f$, $G \propto g$, $f \propto \Theta$, $g \propto \Theta$, $f \propto g$, $\lambda_1, \lambda_2 > 0$ and $\xi_1, \xi_2 > 0$, respectively. Presume that the underneath conditions are fulfils:

(i) $\varrho_n \subset (0, 1]$, $\sum \varrho_n$ *diverges;*

(ii) $\Omega_2 + \Omega_3\Omega_4 < \Omega_1$, *where*

$$\Omega_1 = \left(\frac{\lambda_{N_C}\lambda_f}{\gamma_f}\right),$$

$$\Omega_2 = \left(\frac{L_1\xi_1\lambda_f\lambda_{N_C}\lambda_{N_2}}{\gamma_f}\right),$$

$$\Omega_3 = \left(\frac{\lambda_{N_C} L_1 \lambda_A \lambda_g + \lambda_{N_C} L_1 \xi_1 \lambda_{N_1} \lambda_g}{\gamma_f}\right),$$
$$\Omega_4 = \left(\frac{L_2 \lambda_{N_C} (\lambda_A \lambda_f + \xi_2 \lambda_{N_1} \lambda_f)}{(\gamma_g - L_2 \xi_2 \lambda_{N_2} \lambda_g)}\right),$$

(iii) $\lambda_{N_2} \lambda_g L_2 \xi_2 < \gamma_g$, *where* $L_1 = \frac{1}{\delta(\kappa_G \lambda_1 - 1)}$ *and* $L_2 = \frac{1}{\delta(\kappa_G \lambda_2 - 1)}$.

Then, $\{a_n\}$ *and* $\{b_n\}$ *strongly converges to* a^* *and* b^**, respectively.*

Proof. Allow $(a^*, b^*) \in X \times X$ be the answer of catch (3.1), we have

$$f(a^*) = (1 - \varrho_n) f(a^*) + \varrho_n R^{\mathfrak{M}}_{G,\lambda_1} \left[G(g(b^*)) - \xi_1 \mathfrak{N}(g(b^*), f(a^*))\right]; \tag{6.1}$$

$$g(b^*) = R^{\mathfrak{M}}_{G,\lambda_2} \left[G(f(a^*)) - \xi_2 \mathfrak{N}(f(a^*), g(b^*))\right]. \tag{6.2}$$

Using the γ-ordered non-extended mapping f, we have

$$\gamma_f(a_{n+1} \oplus a_n) \leq \left[f(a_{n+1}) \oplus f(a_n)\right]. \tag{6.3}$$

By the iterative algorithm (5.1), Lemma 2.9, Lemma 2.15 and (6.1), we have

$$\begin{aligned}
\Theta &\leq f(a_{n+1}) \oplus f(a^*) \\
&\leq \Big[(1 - \varrho_n) f(a_n) + \varrho_n R^{\mathfrak{M}}_{G,\lambda_1} \left[G(g(b_n)) - \xi_1 \mathfrak{N}(g(b_n), f(a_n))\right]\Big] \\
&\quad \oplus \Big[(1 - \varrho_n) f(a^*) + \varrho_n R^{\mathfrak{M}}_{G,\lambda_1} \left[G(g(b^*)) - \xi_1 \mathfrak{N}(g(b^*), f(a^*))\right]\Big] \\
&\leq (1 - \varrho_n) \left(f(a_n) \oplus f(a^*)\right) \\
&\quad + \varrho_n \Big(R^{\mathfrak{M}}_{G,\lambda_1} \left[G(g(b_n)) - \xi_1 \mathfrak{N}(g(b_n), f(a_n))\right] \\
&\quad \oplus R^{\mathfrak{M}}_{G,\lambda_1} \left[G(g(b^*)) - \xi_1 \mathfrak{N}(g(b^*), f(a^*))\right]\Big) \\
&\leq (1 - \varrho_n) \lambda_f (a_n \oplus a^*) + \varrho_n L_1 \Big(\left[G(g(b_n)) \oplus G(g(a^*))\right] \\
&\quad + \left[\xi_1 \mathfrak{N}(g(b_n), f(a_n)) \oplus \xi_1 \mathfrak{N}(g(b^*), f(a^*))\right]\Big)
\end{aligned}$$

Since f is λ_f-ordered compression, g is λ_g-ordered compression, A is λ_A-ordered compression mappings and N is λ_{N_1}-ordered compression in first argument and λ_{N_2}-ordered compression mapping in second argument with respect to both f and g, respectively. We have

$$\begin{aligned}
f(a_{n+1}) \oplus f(a^*) &\leq (1 - \varrho_n) \lambda_f (a_n \oplus a^*) + \varrho_n L_1 \Big(\lambda_A \lambda_g (b_n \oplus b^*) \\
&\quad + \xi_1 \Big(\mathfrak{N}(g(b_n), f(a_n)) \oplus \mathfrak{N}(g(b^*), f(a^*)) \\
&\quad \oplus \mathfrak{N}(g(b^*), f(a_n)) \oplus \mathfrak{N}(g(b^*), f(a_n))\Big)\Big) \\
&\leq (1 - \varrho_n) \lambda_f (a_n \oplus a^*) + \varrho_n L_1 \lambda_A \lambda_g (b_n \oplus b^*)
\end{aligned}$$

$$
\begin{aligned}
&+ \xi_1 \varrho_n L_1 \Big(\lambda_{N_1}(g(b_n) \oplus g(b^*)) \\
&\oplus \lambda_{N_2}(f(a_n) \oplus f(a^*)) \Big) \\
&\leq (1-\varrho_n)\lambda_f(a_n \oplus a^*) + \varrho_n L_1 \lambda_A \lambda_g(b_n \oplus b^*) \\
&+ \xi_1 \varrho_n L_1 \left(\lambda_{N_1}\lambda_g(b_n \oplus b^*) \oplus \lambda_{N_2}\lambda_f(a_n \oplus a^*)\right)
\end{aligned}
$$

From (6.3), we have

$$
\begin{aligned}
(a_{n+1} \oplus a_n) \leq \left(\frac{1}{\gamma_f}\right) \Big[&(1-\varrho_n)\lambda_f(a_n \oplus a^*) + \varrho_n L_1 \lambda_A \lambda_g(b_n \oplus b^*) \\
&+ \xi_1 \varrho_n L_1 \left[\lambda_{N_1}\lambda_g(b_n \oplus b^*) \oplus \lambda_{N_2}\lambda_f(a_n \oplus a^*)\right] \Big].
\end{aligned} \tag{6.4}
$$

Using Definition 2.4 and (6.4), we have

(6.5)

$$
\begin{aligned}
\|a_{n+1} \oplus a^*\| &\leq \left(\frac{\lambda_{N_C}}{\gamma_f}\right) \Big\| \Big[(1-\varrho_n)\lambda_f(a_n \oplus a^*) + \varrho_n L_1 \lambda_A \lambda_g(b_n \oplus b^*) \\
&\quad + \xi_1 \varrho_n L_1 \left[\lambda_{N_1}\lambda_g(b_n \oplus b^*) \oplus \lambda_{N_2}\lambda_f(a_n \oplus a^*)\right] \Big] \Big\| \\
&\leq \left(\frac{\lambda_{N_C}\lambda_f}{\gamma_f}\right)(1-\varrho_n)\|(a_n \oplus a^*)\| \\
&\quad + \left(\frac{\lambda_{N_C}\varrho_n L_1 \lambda_A \lambda_g}{\gamma_f}\right)\|(b_n \oplus b^*)\| \\
&\quad + \left(\frac{\lambda_{N_C}\xi_1 \varrho_n L_1}{\gamma_f}\right)\|[\lambda_{N_1}\lambda_g(b_n \oplus b^*) \oplus \lambda_{N_2}\lambda_f(a_n \oplus a^*)]\| \\
&\leq \left(\frac{\lambda_{N_C}\lambda_f}{\gamma_f}\right)(1-\varrho_n)\|(a_n \oplus a^*)\| \\
&\quad + \left(\frac{\lambda_{N_C}\varrho_n L_1 \lambda_A \lambda_g}{\gamma_f}\right)\|(b_n \oplus b^*)\| \\
&\quad + \left(\frac{\lambda_{N_C}\xi_1 \varrho_n L_1 \lambda_{N_1}\lambda_g}{\gamma_f}\right)\|b_n \oplus b^*\| \\
&\quad + \left(\frac{\lambda_{N_C}\xi_1 \varrho_n L_1 \lambda_{N_2}\lambda_f}{\gamma_f}\right)\|a_n \oplus a^*\| \\
&\leq [(1-\varrho_n)\Omega_1 + \varrho_n \Omega_2]\,\|a_n \oplus a^*\| + \varrho_n \Omega_3 \|b_n \oplus b^*\|,
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega_1 &= \left(\frac{\lambda_{N_C}\lambda_f}{\gamma_f}\right), \\
\Omega_2 &= \left(\frac{L_1 \xi_1 \lambda_f \lambda_{N_C} \lambda_{N_2}}{\gamma_f}\right),
\end{aligned}
$$

$$\Omega_3 = \left(\frac{\lambda_{N_C} L_1 \lambda_A \lambda_g + \lambda_{N_C} L_1 \xi_1 \lambda_{N_1} \lambda_g}{\gamma_f} \right).$$

Using the non-extendedness of g, we have

$$\begin{aligned}
(b_n \oplus b^*) &\leq \frac{1}{\gamma_g}(g(b_{n+1}) \oplus g(b^*)) \\
&\leq \frac{1}{\gamma_g}\Big[R^{\mathfrak{M}}_{G,\lambda_2}\left[G(f(a_n)) - \xi_2 \mathfrak{N}(f(a_n), g(b_n))\right] \\
&\quad \oplus R^{\mathfrak{M}}_{G,\lambda_2}\left[G(f(a^*)) - \xi_2 \mathfrak{N}(f(a^*), g(b^*))\right] \Big] \\
&\leq \frac{1}{\gamma_g}\Big[L_2 \Big[\{G(f(a_n)) \oplus G(f(a^*))\} \\
&\quad + \xi_2 \left[\mathfrak{N}(f(a_n), g(b_n)) \oplus \mathfrak{N}(f(a^*), g(b^*))\right] \Big] \Big] \\
&\leq \frac{1}{\gamma_g}\Big[L_2 \Big[\lambda_A (f(a_n) \oplus f(a^*)) \\
&\quad + \xi_2 \left[\lambda_{N_1}(f(a_n) \oplus f(a^*)) \oplus \lambda_{N_2}(g(b_n) \oplus g(b^*))\right] \Big] \Big] \\
&\leq \frac{1}{\gamma_g}\Big[L_2 \Big[\lambda_A \lambda_f (a_n \oplus a^*) \\
&\quad + \xi_2 \left[\lambda_{N_1} \lambda_f (a_n \oplus a^*) \oplus \lambda_{N_2} \lambda_g (b_n \oplus b^*)\right] \Big] \Big] \\
&\leq \frac{1}{\gamma_g}\left[L_2 \left[\lambda_A \lambda_f + \xi_2 \lambda_{N_1} \lambda_f\right](a_n \oplus a^*) + \left[\xi_2 \lambda_{N_2} \lambda_g\right](b_n \oplus b^*) \right].
\end{aligned}$$

Now, we have

$$\begin{aligned}
\left[\frac{\gamma_g - L_2 \xi_2 \lambda_{N_2} \lambda_g}{\gamma_g} \right](b_n \oplus b^*) &\leq \frac{L_2}{\gamma_g}\left[\lambda_A \lambda_f + \xi_2 \lambda_{N_1} \lambda_f\right](a_n \oplus a^*) \\
(b_n \oplus b^*) &\leq \frac{L_2(\lambda_A \lambda_f + \xi_2 \lambda_{N_1} \lambda_f)}{(\gamma_g - L_2 \xi_2 \lambda_{N_2} \lambda_g)}(a_n \oplus a^*).
\end{aligned}$$

Using Definition 2.4 on the above expression, we obtain

$$\|b_n \oplus b^*\| \leq \frac{L_2 \lambda_{N_C}(\lambda_A \lambda_f + \xi_2 \lambda_{N_1} \lambda_f)}{(\gamma_g - L_2 \xi_2 \lambda_{N_2} \lambda_g)} \|a_n \oplus a^*\|,$$

that is,

$$\|b_n \oplus b^*\| \leq \Omega_4 \|a_n \oplus a^*\|, \tag{6.6}$$

where

$$\Omega_4 = \frac{L_2 \lambda_{N_C}(\lambda_A \lambda_f + \xi_2 \lambda_{N_1} \lambda_f)}{(\gamma_g - L_2 \xi_2 \lambda_{N_2} \lambda_g)}.$$

Using (6.6) into (6.5), we obtain

$$\|a_{n+1} \oplus a^*\| \leq \left[(1 - \varrho_n)\Omega_1 + \varrho_n \Omega_2\right] \|a_n \oplus a^*\| + \varrho_n \Omega_3 \Omega_4 \|b_n \oplus b^*\|. \tag{6.7}$$

With the help of Lemma 2.8, (6.7) can be transformed as

(6.8)

$$\begin{aligned}\|a_{n+1} \oplus a^*\| &= \|a_{n+1} - a^*\| \\ &\leq [(1-\varrho_n)\Omega_1 + \varrho_n\Omega_2]\,\|a_n - a^*\| + \alpha_n\Omega_3\Omega_4\|b_n - b^*\| \\ &\leq [(1-\varrho_n)\Omega_1 + \varrho_n\Omega_2]\,\|a_n - a^*\| + \varrho_n\Omega_3\Omega_4\|a_n - a^*\| \\ &\leq [(1-\varrho_n)\Omega_1 + \varrho_n(\Omega_2 + \Omega_3\Omega_4)]\,\|a_n - a^*\| \\ &\leq [1 - \varrho_n(1 - (1 - \Omega_1 + \Omega_2 + \Omega_3\Omega_4))]\,\|a_n - a^*\|,\end{aligned}$$

where

$$\begin{aligned}\Omega_1 &= \left(\frac{\lambda_{N_C}\lambda_f}{\gamma_f}\right), \\ \Omega_2 &= \left(\frac{L_1\xi_1\lambda_f\lambda_{N_C}\lambda_{N_2}}{\gamma_f}\right), \\ \Omega_3 &= \left(\frac{\lambda_{N_C}L_1\lambda_A\lambda_g + \lambda_{N_C}L_1\xi_1\lambda_{N_1}\lambda_g}{\gamma_f}\right) \\ \Omega_4 &= \frac{L_2\lambda_{N_C}(\lambda_A\lambda_f + \xi_2\lambda_{N_1}\lambda_f)}{(\gamma_g - L_2\xi_2\lambda_{N_2}\lambda_g)}.\end{aligned}$$

Using conditions (i)-(iii) and Lemma 5.2 on (6.8), one can find the desired result. □

With the help of Theorem 6.1, one can immediately conclude the underneath results.

7. Applications

Corollary 7.1. *Allow X to be a real ordered Hilbert space with a normal cone C having normal constant λ_{N_C}. Let $G : X \to X$ be a single-valued map with a virtue of strong comparison and λ_A-ordered compression map. Presume that $\mathfrak{N} : X \times X \to X$ be a single-valued bifunction such that $\mathfrak{N}$ is λ_{N_1}-ordered compression in first slot and λ_{N_2}-ordered compression in second slot, respectively. Let $\mathfrak{M} : X \to CB(X)$ be a set-valued (κ_G, ξ)-weak-ANODD mapping.*

In addition, if $G \propto \Theta$, $\lambda_1, \lambda_2 > 0$ and $\xi_1, \xi_2 > 0$, respectively. Suppose the underneath relations are lives:

(i) $\varrho_n \subset (0, 1]$, $\sum \varrho_n$ *diverges;*
(ii) $\Omega_1\Omega_2\Omega_3 < 1$, *where*

$$\begin{aligned}\Omega_1 &= (L_2\lambda_A + L_2\xi_2\lambda_{N_1}), \\ \Omega_2 &= L_1(\lambda_A + \xi_1\lambda_{N_1}), \\ \Omega_3 &= \left(\frac{L_1\xi_1\lambda_{N_2}}{1 - L_2\xi_2\lambda_{N_2}}\right),\end{aligned}$$

(iii) $\lambda_{N_2} L_2 \xi_2 < 1$, *where* $L_1 = \frac{1}{\gamma(\kappa_G \lambda_1 - 1)}$ *and* $L_2 = \frac{1}{\gamma(\kappa_G \lambda_2 - 1)}$.

Then, $\{a_n\}$ *and* $\{b_n\}$ *generated from (5.2) converges strongly to* a^* *and* b^* *of the problem (4.1), respectively.*

Corollary 7.2. *Allow* X *to be a real ordered Hilbert space with a normal cone* C *having normal constant* λ_{N_C}. *Suppose* $G : X \to X$ *be a single-valued map with a virtue of strong comparison and* λ_G*-ordered compression map. Presume that* $\mathfrak{N} : X \times X \to X$ *be a single-valued bifunction such that* N *is* λ_{N_1}*-ordered compression in first slot and* λ_{N_2}*-ordered compression in second slot, respectively. Allow* $\mathfrak{M} : X \to CB(X)$ *to be a set-valued* (κ_G, ξ)*-weak-ANODD mapping.*

In addition, if $G \propto \Theta$ *and* $\lambda > 0$. *Suppose the underneath relations are lives:*

(i) $\varrho_n \subset (0, 1]$, $\sum \varrho_n$ *diverges;*
(ii) $L(\lambda_A + \lambda(\lambda_{N_1} + \lambda_{N_2})) < 1$, *where* $L = \frac{1}{\gamma(\kappa_G \xi - 1)}$;
(iii) $\kappa_G > \frac{1}{\xi}$.

Then, $\{a_n\}$ *and* $\{b_n\}$ *generated from (5.3) strongly converges to* a^* *and* b^* *of the problem (4.2), respectively.*

Corollary 7.3. *Allow* X *to be a real ordered Hilbert space with a normal cone* C *having normal constant* λ_{N_C}. *Allow* $\mathfrak{M} : X \to CB(X)$ *to be a set-valued* (κ, ξ)*-NODM map, then a sequence* $\{a_n\}$ *generated by algorithm (5.4), strongly converges to an answer* a^* *of the catch (4.3), for* $\kappa > \frac{2}{\xi} > 0$.

8. Conclusion

In this paper, we have discussed the generalized system of ordered variational inclusion problems and investigated the approximation solvability of solutions for the system of generalized nonlinear ordered variational inclusions using an iterative algorithm merged with the resolvent operator techniques. The scope of this paper is clearly to enrich the mathematical literature on the theory of variational inclusion and community working in this field of study. This paper opens up a new dimension to study the ordered real Hilbert spaces, ordered Banach spaces, in the setting of variational inequalities and inclusions.

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