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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 3
Pages: 213-226

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.2003041.1340



Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 3 (2024), 213-226
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.2003041.1340

On Copson Ideal Convergent Sequence Spaces

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Abstract. The present work is an investigation of some new sequence spaces $c_0^I(C)$, $c^I(C)$, $\ell_\infty^I(C)$ and $\ell_\infty(C)$ as a domain of the triangle Copson matrix via ideal convergence over an admissible ideal of $\mathbb{N}$. Also, I will define some algebraic, topological properties and inclusion relations in these spaces.

1. Introduction

The hypothesis of sequence spaces has an vital role in the diverse analysis fields. Sequence spaces have various applications in different branches of functional analysis, particularly, the theory of functions, the theory of locally convex spaces, matrix transformations, etc. The set of all real (or complex) valued sequences is symbolized by ω, which turns out to be a vector space under pointwise addition and scalar multiplication. Any vector subspace of ω is called a sequence space. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural, real and complex numbers. Throughout the paper, the notations ℓ_∞, c and c_0 represent the spaces of all bounded, convergent and convergent to zero (null sequences), respectively. After an extensive research about the usual convergence of sequences in point set topology concerning to usual metrics, the concept of Ideal convergence or I-convergence came into existence by well-known author Kostyrko et al. [20]. Ideal convergence is a gereralization of statistical convergence that was independently introduced by well-known authors Fast and Steinhaus[10, 12]. Young researchers or scholars are advised

2020 *Mathematics Subject Classification.* 46A11, 46A45, 54A20.
Key words and phrases. Copson matrix, Copson ideal convergence, Ideal convergence, Ideal, Filter.
Received: 23 May 2023, Accepted: 21 August 2023.
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to deeply analyse of the concept of usual convergence and ideal convergence, as both concepts are independent. Many sequences that are convergent but may not be I-convergent. A large number of research works have surfaced in the field of ideal convergent sequence spaces by many researchers, for further details, one may refer [3–5, 19, 22, 23, 27–29]. Further, Ideal convergence is likned with the summability theory by Šalát et al. [25, 26], which develop some new ideas from the perspective of sequence spaces. One may refer to [1, 24, 30–32] to know more about this concept. An ideal I is defined to be a family of a non-empty set X i.e., $I \subseteq 2^{\mathbf{X}}$ if $I_1, I_2 \in I$ implies that their union is in I i.e., $I_1 \cup I_2 \in I$ and $I_1 \in I, I_2 \subseteq I_1$ implies that $I_2 \in I$ whereas a filter is a family of sets $F \subseteq 2^{\mathbf{X}}$ if and only if $\emptyset \notin F$, $F_1, F_2 \in F$ implies that their intersection is in F i.e., $F_1 \cap F_2 \in F$ and $F_1 \subseteq F_2$ implies that $F_2 \in F$. If $I \neq \emptyset$ and $X \notin I$ then I is said to be non-trivial, admissible if and only if $\{\{x\} : x \in X\} \subseteq I$ and maximal if there is no ideal $J \neq I$ that contains I. For every I to be a non-trivial ideal there must be a filter $F = F(I) = \{Y : X - Y \in I\}$.

Let, $T = (t_{nk})$ be an infinite matrix of real or complex numbers t_{nk} where $n, k \in \mathbb{N}$, the sequence defined as

$$T_n(x) = \sum_{k=0}^{\infty} t_{nk} x_k, \quad \text{for each } n \in \mathbb{N} \tag{1.1}$$

which is a $T-$transform of the sequence $x = (x_k) \in \omega$ by a matrix T also assuming that the right of series (1.1) converges for each $n \in \mathbb{N}$. The mapping of convergent sequences another convergent sequence is given in Kojima-Schur Theorem [21], which is a necessary and sufficient condition as follows:

(i) $\sum_{k=1}^{\infty} |t_{nk}| \leq N$ for every $n > m$;
(ii) $\lim_{n\to\infty} t_{nk} = \beta_k$ for every fixed k;
(iii) $\sum_{k=1}^{\infty} t_{nk} = T_n \to \beta$ as $n \to \infty$.

Also, if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all $n \in \mathbb{N}$, then $T = t_{nk}$ is known to be a trangle matrix which has a unique inverse T^{-1} for $|T| \neq 0$ and T^{-1} is a triangle matrix. Moreover, let the matrix domain

$$\lambda_T := \{x = (x_k) \in \omega : Tx \in \lambda\}$$

for every sequence $x = (x_k) \in \lambda$ which is also a sequense space. λ_T is a BK-space normed by $\|x\|_{\lambda_T} = \|Tx\|_\lambda$ for $x \in \lambda_T$ [13] only if λ is a BK-space and T is triangle matrix. A number of research papers have been published on this idea and its generalization [7, 8, 15–18] which

motivated us to investigate some of the new sequence spaces by using a well known Copson matirx.

Recalling, the Copson matrix of order n, $C^n = \left(c^n_{j,k}\right)$ is an infinite matrix defined as,

$$c^n_{j,k} = \begin{cases} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}}, & \text{if } j \leq k, \\ 0, & \text{otherwise,} \end{cases}$$

where all subscripts in $\mathbb{N}$. The Copson matrix has been used in the analysis of sequence spaces, and also, over ℓ_p-space, it is contemplated as a bounded linear operator. According to the Helinger-Toeplitz theorem, the ℓ_p-norm is defined as $\|C^n\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)}$. Recently, by using the Copson square matrix of order n, Roopaei [11] has investigated $c_0\left(C^n\right)$, $c(C^n)$ and $\ell_p(C^n)$ as sequence spaces with all sequences whose Copson-transformation of the sequence $x = (x_k)$ are in c_0, c, ℓ_p which are also sequence spaces respectively i.e.,

$$\lambda(C^n) = \left\{ x = (x_j) \in \omega : \lim_{j\to\infty} \left(\sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} \right) x_k \in \lambda \right\}$$

for $\lambda \in \{c_0, c, \ell_p\}$. Throughout the paper c_0^I, c^I and ℓ_∞^I, will be representing the sequence spaces of all sequences whhich are null, convergent and bounded via an ideal I.

In this paper, by usnig the Copson matrix of order n and via an ideal convergence, we investigated $c_0^I\left(C\right)$, $c^I\left(C\right)$, $\ell_\infty^I\left(C\right)$ and $\ell_\infty\left(C\right)$ as sequences spaces with all sequences whose $C-$transformation of $x = (x_k)$ are in c_0^I, c^I, ℓ_∞^I and ℓ_∞, respectively. We define the sequence $C^n\left(x\right)$ as the C-tranformation of a sequence $x = (x_k)$ as follows:

$$C^n x = \lim_{j\to\infty} \left(\sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} \right) x_k.$$

In order to define main results, we recall some useful definitions and lemmas related to this investigation.

Definition 1.1 ([12])**.** If $E = \{s \in E : s \leq n\} \subset \mathbb{N}$, then the natural density of the set E is defined as

$$d(E) = \lim_{n\to\infty} \frac{1}{n}|E| \text{ exists}$$

where, $|E|$ is the cardinality of pre-defined set E.

Definition 1.2 ([10]). A sequence $x = (x_k)$ is statistically converges to a number $\zeta \in \mathbb{R}$ if, for every $\varepsilon > 0$ however small, $d(\{k \in \mathbb{N} : |x_k - \zeta| \geq \varepsilon\}) = 0$. and represented as $st \lim x_k = \zeta$. In case $\zeta = 0$, then the sequence $x = (x_k)$ is said to be st-null.

Definition 1.3 ([25]). A sequence $x = (x_k)$ is said to be I-Cauchy if, for every $\varepsilon > 0$ however small, $\exists$ a number $m = m(\varepsilon)$ such that the set $\{k \in \mathbb{N} : |x_k - x_m| \geq \varepsilon\}$ belongs to an ideal I.

Definition 1.4 ([20]). A sequence $x = (x_k)$ is said to be I-convergent to a number $\zeta \in \mathbb{R}$ if, for every $\varepsilon > 0$ however small, the set $\{k \in \mathbb{N} : |x_k - \zeta| \geq \varepsilon\}$ belongs to an ideal I and represented as $I \lim x_k = \zeta$. In case $\zeta = 0$, then (x_k) is said to be I-null.

Definition 1.5 ([27]). A sequence $x = (x_k)$ is said to be I-bounded if there exists a positive real number $M > 0$ however large, such that, the set $\{k \in \mathbb{N} : |x_k| > M\}$ belongs to an ideal I.

Definition 1.6 ([25]). Let there exists two sequences $x = (x_k)$ and $y = (y_k)$. We say that $x_k = y_k$ for almost all k relative to I if the set $\{k \in \mathbb{N} : x_k \neq y_k\}$ belongs to an ideal I.

Definition 1.7 ([25]). A sequence space S is said to be normal or solid, if the Cauchy product $(\alpha_k x_k)$ belongs to S, whenever $(x_k) \in S$ and for any sequence of scalars (α_k) with the condition $|\alpha_k| < 1$, for every $k \in \mathbb{N}$.

Definition 1.8 ([25]). Let $S = \{s_i \in \mathbb{N} : s_1 < s_2 < \cdots\} \subseteq \mathbb{N}$ and K be a sequence space. A S-step space of K is a sequence space

$$\lambda_S^K = \{(x_{s_i}) \in \omega : (x_s) \in K\}.$$

A canonical pre-image of a sequence $(x_{s_i}) \in \lambda_S^K$ is a sequence $(y_s) \in \omega$ defined as follows:

$$y_s = \begin{cases} x_s, & \text{if } s \in S, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical pre-image of a step space λ_S^K is a set of canonical pre-images of all elements in λ_S^K, i.e., y is in canonical pre-image of λ_S^K iff y is canonical pre-image of some element $x \in \lambda_S^K$.

Definition 1.9 ([25]). A sequence space S is said to be monotone, if it contains the canonical pre-images of its step space.

Lemma 1.10 ([25]). *Every solid space $\implies$ monotone space.*

Lemma 1.11 ([26]). *Let $K_1 \in \mathcal{F}(I)$ and $K_2 \subseteq \mathbb{N}$. If $K_2 \notin I$, then $K_1 \cap K_2 \notin I$.*

2. Main results

In this section, we investigated some new sequence spaces $c_0^I(C)$, $c^I(C)$, $\ell_\infty^I(C)$ and $\ell_\infty(C)$ defined by a Copson transformation $C^n(x)$ of a sequence $x=(x_k)$ over an admissible ideal I of subsets of $\mathbb{N}$ and study some algebraic, topological properties and prove some inclusion relations on these spaces.

$$c_0^I(C)=\{x=(x_k)\in\omega:\{n\in\mathbb{N}:|C^n(x)|\geq\varepsilon\}\in I\}$$

$$c^I(C)=\{x=(x_k)\in\omega:\{n\in\mathbb{N}:|C^n(x)-\zeta|\geq\varepsilon,\ \text{for some } \zeta\in\mathbb{R}\}\in I\}$$

$$\ell_\infty^I(C)=\{x=(x_k)\in\omega:\exists M>0 \text{ s.t } \{n\in\mathbb{N}:|C^n(x)|\geq M\}\in I\}$$

$$\ell_\infty(C)=\{x=(x_k)\in\omega:\sup_n|C^n(x)|<\infty\}.$$

Also,

$$m_0^I(C):=c_0^I(C)\cap\ell_\infty(C)\quad\text{and}\quad m^I(C):=c^I(C)\cap\ell_\infty(C).$$

Sequence spaces $c_0^I(C)$, $c^I(C)$, $\ell_\infty^I(C)$, $m^I(C)$ and $m_0^I(C)$ can be redefined as follows:

$$c_0^I(C)=\left(c_0^I\right)_C,\qquad c^I=\left(c^I\right)_C,\qquad \ell_\infty^I(C)=\left(\ell_\infty^I\right)_C,$$

$$m^I(C)=(m^I)_C,\quad m_0^I(C)=\left(m_0^I\right)_C.$$

Definition 2.1. A sequence $x=(x_k)$ is said to be Copson I-Cauchy if for each $\varepsilon>0$, however small, there exists a positive integer $m_{(\varepsilon)}\in\mathbb{N}$ such that

$$\{n\in\mathbb{N}:|C^n(x)-C^m(x)|\geq\varepsilon\}$$

belongs to I, where $I\subseteq\mathbb{N}$ be an admissible ideal.

Example 2.2. Define a class of finite subsets of $\mathbb{N}$ i.e $I^f=\{N\subseteq\mathbb{N}:N \text{ is finite}\}$ is an admissible ideal in $\mathbb{N}$ and $c^{I^f}(C)=C_c$.

Example 2.3. Let, S_c will denote the the space of all Copson statistically convergent sequences i.e.,

$$S_c:=\left\{x=(x_k):d\left(\{n\in\mathbb{N}:|C^n(x)-\zeta|\geq\varepsilon\}\right)=0,\ \text{for any real } \zeta\right\}.$$

We define $I^d=\{N\subseteq\mathbb{N}:d(N)=0\}$ a non trivial ideal that imples that $c^{I^d}(C)=S_c$, where $d(N)$ represents natural density of the set N.

This follows from the following example:

Example 2.4. Every usual Copson convergent sequence converges Copson statistically but the converse may not be true. To prove this result we consider a sequence $x=(x_k)$ defined as follows:

$$C^n(x)=\begin{cases}1, & \text{if } n \text{ is a square,}\\ 0, & \text{otherwise.}\end{cases}$$

That is $C^n(x) = \{1, 0, 0, 1, 0, 0, 0, 0, 1, 0, \dots\}$ and taking the limit $\zeta = 0$. Then we have the inclusion

$$\{n \in \mathbb{N} : |C^n(x) - \zeta| \geq \varepsilon\} \subset \{1, 4, 9, 16, \dots, m^2, (m+1)^2 \dots\}$$

Since, Natural density of the set on right side of above equation is zero i.e., the set of squares of natural numbers, so as a result we get,

$$d(\{n \in \mathbb{N} : |C^n(x) - \zeta| \geq \varepsilon\}) = 0.$$

This implies that, the sequence Copson statistically convergent $(x_k) \in S_c$, but the sequence is not usual Copson convergent $(x_k) \notin C_c$.

Theorem 2.5. *The spaces $c^I(C)$, $c_0^I(C)$, $\ell_\infty^I(C)$, $m_0^I(C)$ and $m^I(C)$ are linear spaces over the real numbers $\mathbb{R}$.*

Proof. Let $x = (x_k)$, $y = (y_k) \in c^I(C)$ be two arbitrary sequences and α_1, α_2 are scalars. Now, since $x, y \in c^I(C)$, then for given $\varepsilon > 0$, there exist $\zeta_1, \zeta_2 \in \mathbb{R}$, such that

$$\left\{n \in \mathbb{N} : |C^n(x) - \zeta_1| \geq \frac{\varepsilon}{2}\right\} \in I$$

and

$$\left\{n \in \mathbb{N} : |C^n(y) - \zeta_2| \geq \frac{\varepsilon}{2}\right\} \in I.$$

Now, let

$$A_1 = \left\{n \in \mathbb{N} : |C^n(x) - \zeta_1| < \frac{\varepsilon}{2|\alpha_1|}\right\} \in \mathcal{F}(I)$$
$$A_2 = \left\{n \in \mathbb{N} : |C^n(y) - \zeta_2| < \frac{\varepsilon}{2|\alpha_2|}\right\} \in \mathcal{F}(I)$$

be such that $A_1^c, A_2^c \in I$. Then

$$\begin{aligned} A_3 &= \{n \in \mathbb{N} : |C^n(\alpha_1 x + \alpha_2 y) - (\alpha_1\zeta_1 + \alpha_2\zeta_2)| < \varepsilon\} \\ &\supseteq \left\{n \in \mathbb{N} : |C^n(x) - \zeta_1| < \frac{\varepsilon}{2|\alpha_1|}\right\} \\ &\quad \cap \left\{n \in \mathbb{N} : |C^n(y) - \zeta_2| < \frac{\varepsilon}{2|\alpha_2|}\right\}. \end{aligned}$$

As a result, we get here that the right side of the above equation belongs to the filter $\mathcal{F}(I)$ associated with I over the set of natural numbers, which implies that its complement set always belongs to Ideal I, and hence, we get $(\alpha_1 x + \alpha_2 y) \in c^I(C)$. $\implies$ $c^I(C)$ is linear space over $\mathbb{R}$.

The proof for the remaining spaces $c_0^I(C)$, $\ell_\infty^I(C)$, $m_0^I(C)$ and $m^I(C)$ can be proven by following the similar way as above. □

Theorem 2.6. *Spaces $\lambda(C)$ are normed spaces with respect to the sup-norm as follows:*

$$\|x\|_{\lambda(C)} = \sup_n |C^n(x)|, \quad \text{where } \lambda \in \left\{c^I, c_0^I, \ell_\infty^I, \ell_\infty\right\}. \tag{2.1}$$

Theorem 2.7. *A sequence $x = (x_k)$ is said to be Copson I-convergent if and only if for every $\varepsilon > 0$, $\exists\ m = m(\varepsilon) \in \mathbb{N}$, such that*

$$\{n \in \mathbb{N} : |C^n(x) - C^m(x)| < \varepsilon\} \in \mathcal{F}(I). \tag{2.2}$$

Proof. Let, the sequence $x = (x_k)$ is Copson I-convergent to some number $\zeta \in \mathbb{R}$, then for a given $\varepsilon > 0$ however small, we have

$$A_\varepsilon = \left\{n \in \mathbb{N} : |C^n(x) - \zeta| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(I).$$

Fix an integer $m = m(\varepsilon) \in A_\varepsilon$. Then we have

$$\begin{aligned} |C^n(x) - C^m(x)| &\leq |C^n(x) - \zeta| + |\zeta - C^m(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

for all $n \in A_\varepsilon$. Hence (2.2) holds.

Conversely, suppose that (2.2) holds for all $\varepsilon > 0$. Then

$$B_\varepsilon = \{n \in \mathbb{N} : C^n(x) \in [C^n(x) - \varepsilon, C^n(x) + \varepsilon]\} \in \mathcal{F}(I), \text{ for all } \varepsilon > 0.$$

Let $J_\varepsilon = [C^n(x) - \varepsilon, C^n(x) + \varepsilon]$. Fixing $\varepsilon > 0$, we have $B_\varepsilon \in \mathcal{F}(I)$ and $B_{\frac{\varepsilon}{2}} \in \mathcal{F}(I)$. Hence $B_\varepsilon \cap B_{\frac{\varepsilon}{2}} \in \mathcal{F}(I)$ provided

$$J = J_\varepsilon \cap J_{\frac{\varepsilon}{2}} \neq \emptyset,$$

which implies that,

$$\{n \in \mathbb{N} : C^n(x) \in J\} \in \mathcal{F}(I)$$

and hence

$$\operatorname{diam}(J) \leq \frac{1}{2} \operatorname{diam}(J_\epsilon),$$

where, $\operatorname{diam}(J)$ represents length of interval J and by induction we get sequence of closed intervals as follows: $J_\epsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$ s.t

$$\operatorname{diam}(I_n) \leq \frac{1}{2} \operatorname{diam}(I_{n-1}), \quad \text{for } n = (2, 3, \dots).$$

As a result we get,

$$\{n \in \mathbb{N} : C^n(x) \in I_n\} \in \mathcal{F}(I).$$

This implies that $\exists\ \zeta \in \bigcap_{n\in\mathbb{N}} I_n$ and it is a routine work to verify that $\zeta = I\lim C^n(x)$ showing that $x = (x_k)$ is Copson I-convergent. Provided that result follows. □

Theorem 2.8. *The inclusions* $\ell_\infty^I(C) \supset c^I(C) \supset c_0^I(C)$ *hold and are strict.*

Proof. The inclusion $c^I(C) \supset c_0^I(C)$ is obviously true. We will show only its strictness for this we take a sequence $x = (x_k)$ such that $C^n(x) = 2$ which implies that $C^n(x)$ belongs to c^I but not belongs to c_0^I.
Moreover, let $x = (x_k) \in c^I(C)$ then there exists a real number ζ such that $I \lim C^n(x) = \zeta$ i.e.,

$$\{n \in \mathbb{N} : |C^n(x) - \zeta| \geq \varepsilon\} \in I.$$

We have

$$|C^n(x)| = |C^n(x) - \zeta + \zeta| \leq |C^n(x) - \zeta| + |\zeta|.$$

From the above result we can say that the sequence (x_k) must be an element of $\ell_\infty^I(C)$.

Also to show strictness of inclusion i.e., $\ell_\infty^I(C) \supset c^I(C)$ we consider an example as follows

Example 2.9. Define a sequence $x = (x_k)$ such that

$$C^n(x) = \begin{cases} 1, & \text{if } n \text{ is odd non-square,} \\ 0, & \text{if } n \text{ is even non-square,} \\ \sqrt{n}, & \text{if } n \text{ is square,} \end{cases}$$

which implies the sequence $C^n(x)$ belongs to ℓ_∞^I, but $C^n(x)$ not belongs to c^I provided the sequence $x \in \ell_\infty^I(C) \setminus c^I(C)$.
As a result, we get that the inclusion $\ell_\infty^I(C) \supset c^I(C) \supset c_0^I(C)$ follows strictly.

This follows from the following example.

Example 2.10. Every Copson bounded sequence is Copson I-bounded but the converse may not be true.To prove this we consider an example by defining a sequence $x = (x_k)$ such that

$$C^n(x) = \begin{cases} \frac{n^2}{n+1}, & \text{for prime n,} \\ 0, & \text{otherwise,} \end{cases}$$

which proves that $C^n(x)$ is not bounded sequence but the set $\{n \in \mathbb{N} : |C^n(x)| \geq 1\}$ belongs to ideal. Hence the sequence (x_k) is Copson I-bounded. □

Theorem 2.11. *The sequence spaces:*

(i) $c^I(C)$ *and* $\ell_\infty(C)$*, do not contain each other but overlap only.*
(ii) $c_0^I(C)$ *and* $\ell_\infty(C)$*, do not contain each other but overlap only.*

Proof. (i) We shall prove the spaces $c^I(C)$ and $\ell_\infty(C)$ are not disjoint spaces for this we consider a sequence $x = (x_k)$ s.t $C^n(x) = \frac{1}{n}$ for n belongs to $\mathbb{N}$ then $x \in c^I(C)$ and $x \in \ell_\infty(C)$ both. Moreover,we define a sequence $x = (x_k)$ such that

$$C^n(x) = \begin{cases} \sqrt{n}, & n \text{ is a square,} \\ 0, & \text{otherwise,} \end{cases}$$

so that, $x \in c^I(C)$ but $x \notin \ell_\infty(C)$. Moreover, we again define a sequence $x = (x_k)$ s.t

$$C^n(x) = \begin{cases} n, & n \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

then as a result we get, $(x) \in \ell_\infty(C)$ but $x \notin c^I(C)$.
The proof of the part (ii) is as similar as that of part (i) hence can be omitted. □

Theorem 2.12. *The spaces defined as:*

$$m_0^I(C) := c_0^I(C) \cap \ell_\infty(C) \quad \text{and} \quad m^I(C) := c^I(C) \cap \ell_\infty(C).$$

are closed in $\ell_\infty(C)$ as a subspace.

Proof. We consider a Cauchy sequence $\left(x_k^{(i)}\right)$ in $m^I(C) \subset \ell_\infty(C)$. Then $\left(x_k^{(i)}\right)$ converges to a point in $\ell_\infty(C)$ and $\lim_{i\to\infty} C^{n(i)}(x) = C^n(x)$. Let $I - \lim C^{n(i)}(x) = \zeta_i$ for every $i \in \mathbb{N}$. Then we only need to show that

(a) Sequence (ζ_i) converges to ζ;
(b) The limit, $I - \lim C^n(x) = \zeta$ exists.

(a) As $\left(x_k^{(i)}\right)$ is a Cauchy sequence then for each $\varepsilon > 0$ however small, there always exists a positive integer $m \in \mathbb{N}$ such that

$$\left|C^{n(i)}(x) - C^{n(j)}(x)\right| < \frac{\varepsilon}{3}, \quad \text{for all } i, j \geq m. \tag{2.3}$$

Now, consider two sets A_i and A_j in an ideal I defined as:

$$A_i = \left\{n \in \mathbb{N} : \left|C^{n(i)}(x) - \zeta_i\right| \geq \frac{\varepsilon}{3}\right\} \tag{2.4}$$

and

$$A_j = \left\{n \in \mathbb{N} : \left|C^{n(j)}(x) - \zeta_j\right| \geq \frac{\varepsilon}{3}\right\}. \tag{2.5}$$

Moreover, let $i, j \geq m$ and $n \notin A_i \cap A_j$ then we get,

$$\begin{aligned} |\zeta_i - \zeta_j| &\leq \left|C^{n(i)}(x) - \zeta_i\right| + \left|C^{n(j)}(x) - \zeta_j\right| + \left|C^{n(i)}(x) - C^{n(j)}(x)\right| \\ &< \varepsilon \end{aligned}$$

by (2.3), (2.4),and(2.5).
Thus (ζ_i) is a Cauchy sequence and thus convergens to $\zeta \in \mathbb{R}$
i.e., $\lim_{i\to\infty} \zeta_i = \zeta$.

(b) Let $\delta > 0$ however small, be given, then we have a positive integer n_0 such that

$$|\zeta_i - \zeta| < \frac{\delta}{3}, \quad \text{for every } i > n_0, \tag{2.6}$$

which implies that $\left(x_k^{(i)}\right) \to x_k$ as $i \to \infty$. Thus

$$\left|C^{n(i)}(x) - C^n(x)\right| < \frac{\delta}{3}, \quad \text{for every } i > n_0. \tag{2.7}$$

Since $\left(C^{n(j)}(x)\right)$ is I-convergent to a real no. ζ_j then $\exists\ \mathcal{E} \in I$ such that for every $n \notin \mathcal{E}$ so we get,

$$\left|C^{n(j)}(x) - \zeta_j\right| < \frac{\delta}{3}. \tag{2.8}$$

Moreover, let $j > n_0$ then $\forall\ n \notin \mathcal{E}$, we get by (2.6), (2.7) and (2.8) s.t

$$|C^n(x) - \zeta| \leq \left|C^n(x) - C^{n(j)}(x)\right| + \left|C^{n(j)}(x) - \zeta_j\right| + |\zeta_j - \zeta| < \delta.$$

Therefore, (x_k) is Copson I-convergent to a real no. ζ. Thus $m^I(C)$ is closed in $\ell_\infty(C)$ as a subspace. The other part can be established following similar technique. □

Theorem 2.13. *Sequence spaces $c^I(C), c_0^I(C)$ and $\ell_\infty^I(C)$ are BK-spaces with respect to the sup-norm as follows:*

$$\|x\|_{\lambda(C)} = \sup_n |C^n(x)|, \quad \textit{where } \lambda \in \left\{c^I, c_0^I, \ell_\infty^I, \ell_\infty\right\}. \tag{2.9}$$

Proof. It is known that the spaces c^I, c_0^I and ℓ_∞^I are BK-spaces. Moreover,(1.2) satisfies and the Copson matrix is a triangle matrix. Now, by considering all these three facts and also by Theorem of Wilansky [2], we have concluded that spaces are BK-spaces and hence the proof is completed. □

In the view of Theorem 2.12 and since the inclusions $m^I(C) \subset \ell_\infty(C)$ and $m_0^I(C) \subset \ell_\infty(C)$ are strict, we formulate the following result without proof.

Theorem 2.14. *Spaces $m^I(C)$ and $m_0^I(C)$ are nowhere dense in $\ell_\infty(C)$ as a subset.*

Theorem 2.15. *Spaces $c_0^I(C)$ and $m_0^I(C)$ are monotone and solid respectively.*

Proof. First we shall prove the result only for $c_0^I(C)$.

Let $x = (x_k) \in c_0^I(C)$. For $\varepsilon > 0$ however small, we have

$$\{n \in \mathbb{N} : |C^n(x)| \geq \varepsilon\} \in I. \tag{2.10}$$

Let $a = (a_k)$ be a scalar sequence satisfies $|a| \leq 1 \; \forall \; k \in \mathbb{N}$, then we have

$$|C^n(ax)| = |aC^n(x)| \leq |a|\,|C^n(x)| \leq |C^n(x)|, \quad \forall n \in \mathbb{N}. \tag{2.11}$$

From the above (2.10), (2.11) equations, we conclude that:

$$\{n \in \mathbb{N} : |C^n(ax)| \geq \varepsilon\} \subseteq \{n \in \mathbb{N} : |C^n(x)| \geq \varepsilon\} \in I$$

which implies that te set,

$$\{n \in \mathbb{N} : |C^n(\alpha x)| \geq \varepsilon\} \text{ belongs to ideal } I.$$

Therefore as a result we get the sequence $(ax_k) \in c_0^I(C)$.
$\Longrightarrow$ Space $c_0^I(C)$ is a solid space.
Also, as we know that every solid space is monotone(1.10) which implies that the space $c_0^I(H)$ is monotone space. □

Corollary 2.16. *If the ideal I is neither maximal nor $I = I_f$, then the sequence spaces $c^I(C)$ and $m^I(C)$ are neither solid nor monotone.*

Proof. We shall prove the above corollary by introducing an example as follows:

Example 2.17. Let $I = I_f$ and let $S = \{n \in \mathbb{N} : n \text{ is an odd integer}\}$. Consider the S-step space S_K of K as:

$$S_K = \{(x_k) \in \omega : (x_k) \in S\}.$$

Now, defining a sequence $(y_k) \in S_K$ s.t

$$C^n(y) = \begin{cases} C^n(x), & \text{if } n \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we consider a sequence (x_k) defined as $C^n(x) = 3 \; \forall \; n \in \mathbb{N}$ then the sequence $(x_k) \in E(C)$, but its S-step space preimage does not belongs to $E(C)$, where $E = c^I$ and m^I.

In this way, as a result we find $E(C)$ are not monotone and by following the lemma(1.10) spaces $E(C)$ are not solid. □

Theorem 2.18. *For a sequence $x = (x_k)$ and a non-trivial admissible ideal I in $\mathbb{N}$ if there exists a sequence $y = (y_k) \in c^I(C)$ such that $C^n(x) = C^n(y)$ for almost all n relative to I, then $x \in c^I(C)$.*

Proof. Suppose that $C^n(x) = C^n(y)$ for almost all n relative to I i.e.,

$$\{n \in \mathbb{N} : C^n(x) \neq C^n(y)\} \in I.$$

Consider the sequence (y_k) is Copson I-convergent to ζ then for every $\varepsilon > 0$ however small, we have the following set belongs to ideal I i.e.,

$$\{n \in \mathbb{N} : |C^n(y) - \zeta| \geq \varepsilon\} \in I.$$

Since, we have considered I as an admissible ideal of set of natural numbers so we have the result by following the below inclusion:

$$\{n \in \mathbb{N} : |C^n(x) - \zeta| \geq \varepsilon\} \subseteq \{n \in \mathbb{N} : C^n(x) \neq C^n(y)\} \cup \{n \in \mathbb{N} : |C^n(y) - \zeta| \geq \varepsilon\}. \qquad \square$$

Acknowledgment. The authors thank the anonymous referee for his/her comments that helped us improve this article.

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