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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 3
Pages: 227-248

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.2003241.1343



Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 3 (2024), 227-248
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.2003241.1343

Generalization of Companion of Ostrowski's Type Inequality Via Riemann-Liouville Fractional Integral and Applications in Numerical Integration, Probability Theory and Special Means

Faraz Mehmood[1*] and Akhmadjon Soleev[2]

Abstract. We apply the Riemann-Liouville fractional integral to generalize a companion of Ostrowski's type integral inequality. The present article recaptures all the results of M. W. Alomari's article and also for one more article of different authors. Applications are also deduced for numerical integration, probability theory and special means.

1. Introduction

Inequalities are one of the most powerful tools in the development of mathematics. From two decades back, scholars have researched fractional calculus because of its importance in inequalities.

We quote from [6],

"The subject of fractional calculus (that is, calculus of integrals and derivatives of an arbitrary real or complex order) was planted over 300 years ago. Since that time, fractional calculus has drawn the attention of many researchers. In recent years, the fractional calculus has played a significant role in many areas of science and engineering."

Due to the worth of fractional integral inequalities, many scholars have mentioned certain generalizations of fractional integral inequalities (see [5, 30–32]).

2020 *Mathematics Subject Classification.* 26D10, 26D15, 26D20, 26A15, 26A16, 26A42, 26A51.

Key words and phrases. Fractional Calculus, Riemann-Liouville fractional integral operator, Ostrowski's inequality, Differentiable mapping, Numerical integration, Probability density function, Special means

Received: 26 May 2023, Accepted: 15 October 2023.

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In 1938, A. M. Ostrowski, a Ukrainian mathematician, who had presented an inequality in his article [29]. Since then, this inequality has been called an Ostrowski inequality and this result was obtained by applying the Montgomery identity. Several researchers have written articles [4, 8, 9, 22, 26] about generalizations of Ostrowski's inequality in the past decades. Ostrowski's inequality has been proved to be a vast and remarkable tool for the enlargement of various fields of mathematics. Inequalities, including integral, which create bounds in the physical quantities, are of great significance in the sense that these types of inequalities are not only used in integral approximation theory, operator theory, nonlinear analysis, numerical integration, stochastic analysis, information theory, statistics and probability theory but we may also see their applications in the several branches of physics, engineering and biological sciences. We refer to the readers [14, 18–20, 23–25, 28, 33, 34] for recent contributions to the study of Ostrowski's inequality.

Here, we present an inequality from the article [11] below. Throughout the article K is a subset of $\mathbb{R}$ and K^o is the interior of the interval K.

Proposition 1.1. *Suppose $g : K \to \mathbb{R}$ is a differentiable mapping in the interval K^o such that $g' \in L[j,k]$, where $j,k \in K$ and $j < k$. If $|g'(\theta)| \leq \mathfrak{M}\ \forall\ \theta \in (j,k)$ where $\mathfrak{M} > 0$ is constant. Then*

$$\left| g(\theta) - \frac{1}{k-j}\int_j^k g(\tau)d\tau \right| \leq \mathfrak{M}(k-j)\left[\frac{1}{4} + \frac{\left(\theta - \frac{j+k}{2}\right)^2}{(k-j)^2}\right].$$

The value $\frac{1}{4}$ is the best possible constant that this can not be replaced by the smallest one.

For more results related to Ostrowski's inequality (see [1, 2, 10, 34]). Also, the reader is referred to the book [11] where several Ostrowski-type inequalities are discussed.

In [17], Guessab *et.al.* have derived a companion of Ostrowski's inequality in the below like others.

Proposition 1.2. *Let function $g : [j,k] \to \mathbb{R}$ satisfy the Lipschitz condition, that is $|g(\tau) - g(s)| \leq \mathfrak{M}|\tau - s|$. Then $\forall\ \theta \in \left[j, \frac{j+k}{2}\right]$, then*

$$\left| \frac{g(\theta) + g(j+k-\theta)}{2} - \frac{1}{k-j}\int_j^k g(\tau)d\tau \right| \leq \left[\frac{1}{8} + 2\left(\frac{\theta - \frac{3j+k}{4}}{k-j}\right)^2\right]\mathfrak{M}(k-j),$$

The value $\frac{1}{8}$ *is the best possible constant that this can not be replaced by the smallest one.*

Note that the above inequality is the best due to it gives the trapezoid type inequality for $\theta = \frac{3j+k}{4}$, i.e.,

$$\left| \frac{g\left(\frac{3j+k}{4}\right) + g\left(\frac{j+3k}{4}\right)}{2} - \frac{1}{k-j}\int_j^k g(\tau)d\tau \right| \leq \frac{\mathfrak{M}(k-j)}{8}. \tag{1.1}$$

The constant $\frac{1}{8}$ is the sharp in above the inequality.

In [13], S. S. Dragomir has derived the below companion of Ostrowski inequality.

Proposition 1.3. *Suppose function* $g : K \to \mathbb{R}$ *is an absolutely continuous in the interval* $[j,k]$*. Then*

$$\left| \frac{g(\theta)+g(j+k-\theta)}{2} - \frac{1}{k-j}\int_j^k g(\tau)d\tau \right|$$

$$\leq \begin{cases} \left[\frac{1}{8} + 2\left(\frac{\theta - \frac{3j+k}{4}}{k-j}\right)^2\right](k-j)g'_\infty, & g' \in L_\infty[j,k], \\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \times \left[\left(\frac{\theta-j}{k-j}\right)^{q+1} + \left(\frac{\frac{j+k}{2}-\theta}{k-j}\right)^{q+1}\right]^{\frac{1}{q}} & p>1, \frac{1}{p}+\frac{1}{q}=1, g' \in L_p[j,k], \\ \quad \times (k-j)^{\frac{1}{q}} g'_{[j,k],p}, & \\ \left[\frac{1}{4} + \left|\frac{\theta - \frac{3j+k}{4}}{k-j}\right|\right] g'_{[j,k],1} & \end{cases}$$

hold, $\forall\ \theta \in \left[j, \frac{j+k}{2}\right]$.

We need here to define Riemann-Liouville fractional integral (RLFI) (see[15]) in order to prove our next main result in the second section.

Definition 1.4. The Riemann-Liouville fractional integral operator of order $\gamma > 0$ is stated as

$$J_j^\gamma g(\theta) = \frac{1}{\Gamma(\gamma)}\int_j^\theta (\theta-\tau)^{\gamma-1} g(\tau)d\tau, \qquad J_j^0 g(\theta) = g(\theta),$$

where gamma function $\Gamma(\gamma)$ is defined as

$$\Gamma(\gamma) = \int_0^\infty \theta^{\gamma-1} e^{-\theta} d\theta.$$

In 2002, S. S. Dragomir [12] derived a few inequalities for a companion for mappings of bounded variation. In 2009 Z. Liu [21] established a few companions of the Ostrowski's type inequality for mappings whose second derivatives are absolutely continuous. In 2009, Barnett *et. al* [7] derived few companions for Ostrowski's inequality and the generalized trapezoid inequality. In 2011, M. W. Alomari [3] obtained the companion of inequality of Ostrowski (1.1) for differentiable bounded mappings and also gave the applications. Recently, authors [23] gave a companion of weighted Ostrowski's type inequality for differentiable bounded functions with application.

In the current article we will derive a companion of weighted Fractional Ostrowski's type inequality for differentiable bounded mappings and then we will give its applications.

2. Generalization of Companion of Ostrowski's Type Inequality via Riemann-Liouville Fractional Integral

Under the present section, we give our results about companion of Ostrowski's type inequality which are as follows:

Theorem 2.1. *Suppose $g : [j,k] \to \mathbb{R}$ is a differentiable mapping in the interval (j,k) and $j < k$ and $w : [j,k] \to \mathbb{R}$ is a probability density function. If $g' \in L^1[j,k]$ and $m_1 \le g'(\tau) \le M_1$, $\forall\ \tau \in [j,k]$, then*

(2.1)

$$\begin{aligned}
&\left| g(\theta) \int_j^{\frac{j+k}{2}} w(\tau)d\tau + g(j+k-\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1} \int_{\frac{j+k}{2}}^{k} w(\tau)d\tau \right. \\
&\qquad \left. - (k-\theta)^{1-\gamma}\Gamma(\gamma)J_j^{\gamma}(w(k)g(k)) + (\gamma-1)J_j^{\gamma-1}(P(\theta,k)g(k)) \right| \\
&\le (k-\theta)^{1-\gamma} \cdot \max_{\tau\in[j,k]} \left|(k-\tau)^{\gamma-1}\right| \cdot \left[\int_j^{\frac{j+\theta}{2}} \left(\int_j^{\tau} w(u)du \right) d\tau \right. \\
&\quad + \int_{\frac{j+\theta}{2}}^{\theta} \left(\int_j^{\tau} w(u)du \right) d\tau - \int_{\theta}^{\frac{j+k}{2}} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du \right) d\tau \\
&\quad + \int_{\frac{j+k}{2}}^{j+k-\theta} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du \right) d\tau - \int_{j+k-\theta}^{\frac{j-\theta+2k}{2}} \left(\int_k^{\tau} w(u)du \right) d\tau \\
&\quad \left. - \int_{\frac{j-\theta+2k}{2}}^{k} \left(\int_k^{\tau} w(u)du \right) d\tau \right] \frac{(M_1+m_1)}{2}
\end{aligned}$$

holds $\forall\ \theta \in \left[j, \frac{j+k}{2}\right]$.

Proof. For the sake of proof we state the weighted kernel as;

$$P(\theta,\tau)=(k-\theta)^{1-\gamma}\Gamma(\gamma)\begin{cases}\displaystyle\int_j^{\tau} w(u)du, & \text{if} \quad \tau\in[j,\theta],\\ \displaystyle\int_{\frac{j+k}{2}}^{\tau} w(u)du, & \text{if} \quad \tau\in(\theta,j+k-\theta],\\ \displaystyle\int_k^{\tau} w(u)du, & \text{if} \quad \tau\in(j+k-\theta,k],\end{cases}$$

$\forall\ \theta\in\left[j,\frac{j+k}{2}\right]$.

Applying RLFI operator and using integration by parts formula, obtain

(2.2)

$$\begin{aligned}J_j^{\gamma}(P(\theta,k)g(k)) &= \frac{1}{\Gamma(\gamma)}\int_j^k (k-\tau)^{\gamma-1}P(\theta,\tau)g'(\tau)d\tau\\ &= g(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau\\ &\quad + g(j+k-\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1}\int_{\frac{j+k}{2}}^k w(\tau)d\tau\\ &\quad - (k-\theta)^{1-\gamma}\Gamma(\gamma)J_j^{\gamma}(w(k)g(k))\\ &\quad + (\gamma-1)J_j^{\gamma-1}(P(\theta,k)g(k)).\end{aligned}$$

We know that

$$\int_j^k P(\theta,\tau)d\tau=0. \tag{2.3}$$

Let $C=\frac{M_1+m_1}{2}$. From (2.2) and (2.3) it follows

$$\begin{aligned}&\frac{1}{\Gamma(\gamma)}\int_j^k (k-\tau)^{\gamma-1}P(\theta,\tau)[g'(\tau)-C]d\tau\\ &\quad = g(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau\\ &\qquad + g(j+k-\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1}\int_{\frac{j+k}{2}}^k w(\tau)d\tau\\ &\qquad - (k-\theta)^{1-\gamma}\Gamma(\gamma)J_j^{\gamma}(w(k)g(k))+(\gamma-1)J_j^{\gamma-1}(P(\theta,k)g(k)).\end{aligned}$$

On the other hand, we have

$$\left|\frac{1}{\Gamma(\gamma)}\int_j^k (k-\tau)^{\gamma-1}P(\theta,\tau)[g'(\tau)-C]d\tau\right| \tag{2.4}$$
$$\leq \frac{1}{\Gamma(\gamma)}\max_{\tau\in[j,k]}\left|(k-\tau)^{\gamma-1}\right|\cdot\max_{\tau\in[j,k]}\left|g'(\tau)-C\right|\cdot\int_j^k |P(\theta,\tau)|d\tau.$$

Since

$$\max_{\tau\in[j,k]}\left|g'(\tau)-C\right|\leq\frac{M_1+m_1}{2}$$

and

(2.5)

$$\begin{aligned}\int_j^k |P(\theta,\tau)|d\tau = (k-\theta)^{1-\gamma}\Gamma(\gamma)\Bigg[&\int_j^{\frac{j+\theta}{2}}\left(\int_j^\tau w(u)du\right)d\tau\\ &+\int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^\tau w(u)du\right)d\tau-\int_\theta^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^\tau w(u)du\right)d\tau\\ &+\int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^\tau w(u)du\right)d\tau\\ &-\int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^\tau w(u)du\right)d\tau\\ &-\int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^\tau w(u)du\right)d\tau\Bigg].\end{aligned}$$

Now from (2.4) to (2.5), it follows that

$$\begin{aligned}&\Bigg|g(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau+g(j+k-\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1}\int_{\frac{j+k}{2}}^k w(\tau)d\tau\\ &\quad-(k-\theta)^{1-\gamma}\Gamma(\gamma)J_j^\gamma(w(k)g(k))+(\gamma-1)J_j^{\gamma-1}(P(\theta,k)g(k))\Bigg|\\ &\leq(k-\theta)^{1-\gamma}\cdot\max_{\tau\in[j,k]}\left|(k-\tau)^{\gamma-1}\right|\cdot\Bigg[\int_j^{\frac{j+\theta}{2}}\left(\int_j^\tau w(u)du\right)d\tau\\ &\quad+\int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^\tau w(u)du\right)d\tau-\int_\theta^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^\tau w(u)du\right)d\tau\\ &\quad+\int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^\tau w(u)du\right)d\tau-\int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^\tau w(u)du\right)d\tau\end{aligned}$$

$$- \int_{\frac{j-\theta+2k}{2}}^{k} \left(\int_{k}^{\tau} w(u)du \right) d\tau \Bigg] \frac{(M_1 + m_1)}{2},$$

$\forall\ \theta \in \left[j, \frac{j+k}{2} \right]$. □

Remark 2.2. If put $\gamma = 1$ and $w = \frac{1}{k-j}$ in Theorem 2.1, then we recapture the result of Theorem 4 of [3].

Remark 2.3. If put $\gamma = 1$ in Theorem 2.1, then we recapture the result of Theorem 2.1 of [23].

Corollary 2.4. *In the inequality* (2.1), *select*

(*i*) $\theta = \frac{3j+k}{4}$, *obtain*

(2.6)

$$\begin{aligned}
&\Bigg| g\left(\frac{3j+k}{4}\right) \int_{j}^{\frac{j+k}{2}} w(\tau)d\tau + 3^{1-\gamma} g\left(\frac{j+3k}{4}\right) \int_{\frac{j+k}{2}}^{k} w(\tau)d\tau \\
&\quad - \left(\frac{3}{4}(k-j)\right)^{1-\gamma} \Gamma(\gamma) J_j^{\gamma}(w(k)g(k)) \\
&\quad + (\gamma-1) J_j^{\gamma-1}\left(P\left(\frac{3j+k}{4}, k\right) g(k)\right) \Bigg| \\
&\leq \left(\frac{3}{4}(k-j)\right)^{1-\gamma} \cdot \max_{\tau\in[j,k]} \left|(k-\tau)^{\gamma-1}\right| \cdot \Bigg[\int_{j}^{\frac{7j+k}{8}} \left(\int_{j}^{\tau} w(u)du \right) d\tau \\
&\quad + \int_{\frac{7j+k}{8}}^{\frac{3j+k}{4}} \left(\int_{j}^{\tau} w(u)du \right) d\tau - \int_{\frac{3j+k}{4}}^{\frac{j+k}{2}} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du \right) d\tau \\
&\quad + \int_{\frac{j+k}{2}}^{\frac{j+3k}{4}} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du \right) d\tau - \int_{\frac{j+3k}{4}}^{\frac{j+7k}{8}} \left(\int_{k}^{\tau} w(u)du \right) d\tau \\
&\quad - \int_{\frac{j+7k}{8}}^{k} \left(\int_{k}^{\tau} w(u)du \right) d\tau \Bigg] \frac{(M_1 + m_1)}{2},
\end{aligned}$$

(*ii*) $\theta = \frac{2j+k}{3}$, *obtain*

$$\begin{aligned}
&\Bigg| g\left(\frac{2j+k}{3}\right) \int_{j}^{\frac{j+k}{2}} w(\tau)d\tau + 2^{1-\gamma} g\left(\frac{j+2k}{3}\right) \int_{\frac{j+k}{2}}^{k} w(\tau)d\tau \\
&\quad - \left(\frac{2}{3}(k-j)\right)^{1-\gamma} \Gamma(\gamma) J_j^{\gamma}(w(k)g(k)) \\
&\quad + (\gamma-1) J_j^{\gamma-1}\left(P\left(\frac{2j+k}{3}, k\right) g(k)\right) \Bigg|
\end{aligned}$$

$$\leq \left(\frac{2}{3}(k-j)\right)^{1-\gamma} \cdot \max_{\tau\in[j,k]} \left|(k-\tau)^{\gamma-1}\right| \cdot \left[\int_j^{\frac{5j+k}{6}} \left(\int_j^{\tau} w(u)du\right) d\tau \right.$$
$$+ \int_{\frac{5j+k}{6}}^{\frac{2j+k}{3}} \left(\int_j^{\tau} w(u)du\right) d\tau - \int_{\frac{2j+k}{3}}^{\frac{j+k}{2}} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right) d\tau$$
$$+ \int_{\frac{j+k}{2}}^{\frac{j+2k}{3}} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right) d\tau - \int_{\frac{j+2k}{3}}^{\frac{j+5k}{6}} \left(\int_k^{\tau} w(u)du\right) d\tau$$
$$\left. - \int_{\frac{j+5k}{6}}^{k} \left(\int_k^{\tau} w(u)du\right) d\tau\right] \frac{(M_1+m_1)}{2},$$

(iii) $\theta = \frac{j+k}{2}$, *obtain*

$$\left| g(\frac{j+k}{2}) \int_j^k w(\tau)d\tau - \left(\frac{k-j}{2}\right)^{1-\gamma} \Gamma(\gamma) J_j^{\gamma}(w(k)g(k)) \right.$$
$$\left. + (\gamma-1) J_j^{\gamma-1} \left(P\left(\frac{j+k}{2}, k\right) g(k)\right) \right|$$
$$\leq \left(\frac{k-j}{2}\right)^{1-\gamma} \cdot \max_{\tau\in[j,k]} \left|(k-\tau)^{\gamma-1}\right| \cdot \left[\int_j^{\frac{3j+k}{4}} \left(\int_j^{\tau} w(u)du\right) d\tau \right.$$
$$+ \int_{\frac{3j+k}{4}}^{\frac{j+k}{2}} \left(\int_j^{\tau} w(u)du\right) d\tau - \int_{\frac{j+k}{2}}^{\frac{j+3k}{4}} \left(\int_k^{\tau} w(u)du\right) d\tau$$
$$\left. - \int_{\frac{j+3k}{4}}^{k} \left(\int_k^{\tau} w(u)du\right) d\tau\right] \frac{(M_1+m_1)}{2}.$$

In the following we present special cases of above corollary.

Special Case 1: By putting $w = \frac{1}{k-j}$ and $\gamma = 1$ in (ii) of Corollary 2.4, then we get

$$\left| \frac{g\left(\frac{2j+k}{3}\right) + g\left(\frac{j+2k}{3}\right)}{2} - \frac{1}{k-j}\int_j^k g(\tau)d\tau \right| \leq \frac{5(k-j)}{72}(M_1+m_1).$$

Special Case 2: By putting $w = \frac{1}{k-j}$ and $\gamma = 1$ in (iii) of Corollary 2.4, then we get

$$\left| g(\frac{j+k}{2}) - \frac{1}{k-j}\int_j^k g(\tau)d\tau \right| \leq \frac{(k-j)}{4}(M_1+m_1).$$

Remark 2.5. (i) First by putting $\gamma = 1$ and $w = \frac{1}{k-j}$ in Theorem 2.1 and then put $\theta = j$ in obtained inequality, we recapture the Corollary 2 of [3].

(ii) By putting $\gamma = 1$ and $w = \frac{1}{k-j}$ in (i) of Corollary 2.4, we recapture the Corollary 1 of [3].

Remark 2.6. (i) First by putting $\gamma = 1$ in Theorem 2.1 and then put $\theta = j$ in obtained inequality, we recapture the Corollary 2.3(iv) of [23].
(ii) By putting $\gamma = 1$ in (i) of Corollary 2.4, we recapture the Corollary 2.3(i) of [23].
(iii) By putting $\gamma = 1$ in (ii) of Corollary 2.4, we recapture the Corollary 2.3(ii) of [23].
(iv) By putting $\gamma = 1$ in (iii) of Corollary 2.4, we recapture the Corollary 2.3(iii) of [23].

Ostrowski's type inequality can be defined in the form of following corollary.

Corollary 2.7. *Let the suppositions of Theorem* 2.1 *be valid. Further, if g is symmetric about the θ-axis, i.e., $g(j+k-\theta) = g(\theta)$, then*

$$\begin{aligned}
&\left| g(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau + g(\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1}\int_{\frac{j+k}{2}}^{k} w(\tau)d\tau \right. \\
&\qquad \left. - (k-\theta)^{1-\gamma}\Gamma(\gamma)J_j^{\gamma}(w(k)g(k)) + (\gamma-1)J_j^{\gamma-1}(P(\theta,k)g(k)) \right| \\
&\leq (k-\theta)^{1-\gamma}\cdot \max_{\tau\in[j,k]}\left|(k-\tau)^{\gamma-1}\right| \cdot \left[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\tau} w(u)du\right)d\tau \right. \\
&\quad + \int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\tau} w(u)du\right)d\tau - \int_{\theta}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau \\
&\quad + \int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau - \int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^{\tau} w(u)du\right)d\tau \\
&\quad \left. - \int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^{\tau} w(u)du\right)d\tau\right]\frac{(M_1+m_1)}{2}
\end{aligned}$$

holds.

Remark 2.8. First by putting $\gamma = 1$ and $w = \frac{1}{k-j}$ in Corollary 2.7 and then put $\theta = j$ in obtained inequality, we recapture the Corollary 3 of [3].

Remark 2.9. First by putting $\gamma = 1$ in Corollary 2.7 and then put $\theta = j$ in the obtained inequality, we recapture the Corollary 2.5 of [23].

3. Application to Numerical Integration

Let $K_n : j = \theta_0 < \theta_1 < \cdots < \theta_n = k$ be division of interval $[j,k]$ and $h_i = \theta_{i+1} - \theta_i$, $(i = 0, 1, 2, \ldots, n-1)$.

Consider the quadrature formula

$$Q_n(K_n, g) := \sum_{i=0}^{n-1} \left[g\left(\frac{3\theta_i + \theta_{i+1}}{4}\right) \int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\tau)d\tau + 3^{1-\gamma} g\left(\frac{\theta_i + 3\theta_{i+1}}{4}\right) \right.$$
$$\left. \times \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\tau)d\tau + (\gamma-1) J_{\theta_i}^{\gamma-1}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4}, \theta_{i+1}\right) g(\theta_{i+1})\right) \right].$$

Here, we prove following result.

Theorem 3.1. *Suppose $g : K \to \mathbb{R}$ is a differentiable mapping in the interval K^o and $w : [j,k] \to \mathbb{R}$ is a probability density function, where $j, k \in K$ with $j < k$. If $g' \in L^1[j,k]$ and $m_1 \le g'(\theta) \le M_1$, for all $\theta \in [j,k]$, then following holds*

$$\Gamma(\gamma) \sum_{i=0}^{n-1} \left(\frac{3}{4} h_i\right)^{1-\gamma} J_{\theta_i}^{\gamma}(w(\theta_{i+1}) g(\theta_{i+1})) = Q_n(K_n, g) + R_n(K_n, g),$$

where $Q_n(K_n, g)$ is stated as above and the following remainder $R_n(K_n, g)$ satisfies the estimates

$$|R_n(K_n, g)|$$
$$\le \frac{(M_1 + m_1)}{2} \sum_{i=0}^{n-1} \left[\left(\frac{3}{4} h_i\right)^{1-\gamma} \max_{\tau \in [\theta_i, \theta_{i+1}]} |(\theta_{i+1} - \tau)^{\gamma-1}| \right.$$
$$\times \left[\int_{\theta_i}^{\frac{7\theta_i+\theta_{i+1}}{8}} \left(\int_{\theta_i}^{\tau} w(u)du \right) d\tau + \int_{\frac{7\theta_i+\theta_{i+1}}{8}}^{\frac{3\theta_i+\theta_{i+1}}{4}} \left(\int_{\theta_i}^{\tau} w(u)du \right) d\tau \right.$$
$$- \int_{\frac{3\theta_i+\theta_{i+1}}{4}}^{\frac{\theta_i+\theta_{i+1}}{2}} \left(\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\tau} w(u)du \right) d\tau + \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\frac{\theta_i+3\theta_{i+1}}{4}} \left(\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\tau} w(u)du \right) d\tau$$
$$\left. \left. - \int_{\frac{\theta_i+3\theta_{i+1}}{4}}^{\frac{\theta_i+7\theta_{i+1}}{8}} \left(\int_{\theta_{i+1}}^{\tau} w(u)du \right) d\tau - \int_{\frac{\theta_i+7\theta_{i+1}}{8}}^{\theta_{i+1}} \left(\int_{\theta_{i+1}}^{\tau} w(u)du \right) d\tau \right] \right].$$

Proof. Applying inequality (2.6) on the intervals $[\theta_i, \theta_{i+1}]$, we get

(3.1)

$$R_i(K_i, g) = \Gamma(\gamma) \left(\frac{3}{4} h_i\right)^{1-\gamma} J_{\theta_i}^{\gamma}(w(\theta_{i+1}) g(\theta_{i+1}))$$
$$- \left[g\left(\frac{3\theta_i + \theta_{i+1}}{4}\right) \int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\tau)d\tau + 3^{1-\gamma} g\left(\frac{\theta_i + 3\theta_{i+1}}{4}\right) \right.$$

$$\times \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\tau)d\tau + (\gamma-1)J_{\theta_i}^{\gamma-1}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4},\theta_{i+1}\right)g(\theta_{i+1})\right)\Bigg].$$

Summing (3.1) over i from 0 to $n-1$, then

$$\begin{aligned}R_n(K_n,g) &= \Gamma(\gamma)\sum_{i=0}^{n-1}\left(\frac{3}{4}h_i\right)^{1-\gamma} J_{\theta_i}^{\gamma}(w(\theta_{i+1})g(\theta_{i+1}))\\ &\quad -\sum_{i=0}^{n-1}\Bigg[g\left(\frac{3\theta_i+\theta_{i+1}}{4}\right)\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\tau)d\tau + 3^{1-\gamma}g\left(\frac{\theta_i+3\theta_{i+1}}{4}\right)\\ &\quad \times \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\tau)d\tau + (\gamma-1)J_{\theta_i}^{\gamma-1}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4},\theta_{i+1}\right)g(\theta_{i+1})\right)\Bigg],\end{aligned}$$

which follows the form of (2.6), i.e.

$$\begin{aligned}|R_n(K_n,g)| &= \Bigg|\Gamma(\gamma)\sum_{i=0}^{n-1}\left(\frac{3}{4}h_i\right)^{1-\gamma} J_{\theta_i}^{\gamma}(w(\theta_{i+1})g(\theta_{i+1}))\\ &\quad -\sum_{i=0}^{n-1}\Bigg[g\left(\frac{3\theta_i+\theta_{i+1}}{4}\right)\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\tau)d\tau\\ &\quad + 3^{1-\gamma}g\left(\frac{\theta_i+3\theta_{i+1}}{4}\right)\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\tau)d\tau\\ &\quad + (\gamma-1)J_{\theta_i}^{\gamma-1}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4},\theta_{i+1}\right)g(\theta_{i+1})\right)\Bigg]\Bigg|\\ &\le \frac{(M_1+m_1)}{2}\sum_{i=0}^{n-1}\Bigg[\left(\frac{3}{4}h_i\right)^{1-\gamma}\max_{\tau\in[\theta_i,\theta_{i+1}]}|(\theta_{i+1}-\tau)^{\gamma-1}|\\ &\quad \times\Bigg[\int_{\theta_i}^{\frac{7\theta_i+\theta_{i+1}}{8}}\left(\int_{\theta_i}^{\tau} w(u)du\right)d\tau\\ &\quad + \int_{\frac{7\theta_i+\theta_{i+1}}{8}}^{\frac{3\theta_i+\theta_{i+1}}{4}}\left(\int_{\theta_i}^{\tau} w(u)du\right)d\tau\\ &\quad - \int_{\frac{3\theta_i+\theta_{i+1}}{4}}^{\frac{\theta_i+\theta_{i+1}}{2}}\left(\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\tau} w(u)du\right)d\tau\\ &\quad + \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\frac{\theta_i+3\theta_{i+1}}{4}}\left(\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\tau} w(u)du\right)d\tau\\ &\quad - \int_{\frac{\theta_i+3\theta_{i+1}}{4}}^{\frac{\theta_i+7\theta_{i+1}}{8}}\left(\int_{\theta_{i+1}}^{\tau} w(u)du\right)d\tau\end{aligned}$$

$$- \int_{\frac{\theta_i+7\theta_{i+1}}{8}}^{\theta_{i+1}} \left(\int_{\theta_{i+1}}^{\tau} w(u)du \right) d\tau \Bigg] \Bigg].$$

This completes the required proof. □

Remark 3.2. If put $\gamma = 1$ and $w = \frac{1}{k-j}$ in Theorem 3.1, we can get the result of Theorem 5 of [3].

Remark 3.3. By putting $\gamma = 1$ in Theorem 3.1, we recapture the result of Theorem 3.1 of [23].

4. Applications to Probability Theory

Throughout this section we consider $w : [j, k] \to [0, 1]$. Suppose Y is a random variable taking values in the finite interval $[j, k]$ with probability density function $g : [j, k] \to [0, 1]$ and with cumulative distribution function $G : [j, k] \to [0, 1]$ is defined by

$$\begin{aligned} G(\theta) &= P(Y \leq \theta) \\ &= \Gamma(\gamma) J_j^{\gamma}(w(\theta)g(\theta)) \\ &= \int_j^{\theta} (\theta - \tau)^{\gamma-1} w(\tau)g(\tau)d\tau; \quad j \leq \theta \leq \frac{j+k}{2}, \end{aligned}$$

and

$$E(Y) = \int_j^k \tau g(\tau)d\tau \ , \ E_w(Y) = \int_j^k \tau w(\tau)g(\tau)d\tau$$

$$E_{wf}(Y) = \Gamma(\gamma) J_j^{\gamma}(kw(k)g(k)) = \int_j^k \tau(k-\tau)^{\gamma-1} w(\tau)g(\tau)d\tau$$

$$E_{wf1}(Y) = \Gamma(\gamma) J_j^{\gamma-1}(kw(k)g(k)) = \int_j^k \tau(k-\tau)^{\gamma-2} w(\tau)g(\tau)d\tau,$$

$$E_{wf2}(Y) = \Gamma(\gamma) J_j^{\gamma}(kw'(k)g(k)) = \int_j^k \tau(k-\tau)^{\gamma-1} w'(\tau)g(\tau)d\tau,$$

$$E_{wf3}(Y) = \Gamma(\gamma) J_j^{\gamma}(kw(k)g'(k)) = \int_j^k \tau(k-\tau)^{\gamma-1} w(\tau)g'(\tau)d\tau,$$

are the expectation, weighted expectation and weighted fractional expectation of random variable 'Y' in interval $[j, k]$. Then we can write the following theorem as:

Theorem 4.1. *Suppose $g : [j, k] \to \mathbb{R}$ is differentiable mapping in interval (j, k) and $j < k$. If $g' \in L^1[j, k]$ and $m_1 \leq g'(\tau) \leq M_1$, for all*

$\tau \in [j,k]$. *Further, suppose that the function* w *is differentiable. Then*

(4.1)

$$\begin{aligned}
&\left| G(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau + G(j+k-\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1}\int_{\frac{j+k}{2}}^{k} w(\tau)d\tau \right.\\
&\quad -(k-\theta)^{1-\gamma}\left((\gamma-1)E_{wf1}(Y)-E_{wf2}(Y)-E_{wf3}(Y)\right)\\
&\quad \left.+(\gamma-1)J_j^{\gamma-1}(P(\theta,k)G(k))\right|\\
&\leq (k-\theta)^{1-\gamma}\cdot\max_{\tau\in[j,k]}\left|(k-\tau)^{\gamma-1}\right|\cdot\left[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\tau} w(u)du\right)d\tau\right.\\
&\quad +\int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\tau} w(u)du\right)d\tau-\int_{\theta}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau\\
&\quad +\int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau-\int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^{\tau} w(u)du\right)d\tau\\
&\quad \left.-\int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^{\tau} w(u)du\right)d\tau\right]\frac{(M_1+m_1)}{2}
\end{aligned}$$

holds $\forall\ \theta \in \left[j, \frac{j+k}{2}\right]$.

Proof. Select $g = G$ and by applying the below identity in Theorem 2.1, we obtain (4.1).

$$\begin{aligned}
\Gamma(\gamma)J_j^{\gamma}(w(k)g(k)) &= \int_j^k (k-\tau)^{\gamma-1}w(\tau)g(\tau)d\tau\\
&= (\gamma-1)E_{wf1}(Y)-E_{wf2}(Y)-E_{wf3}(Y).
\end{aligned}$$

Since $G(j)=0$ and $G(k)=1$. □

Corollary 4.2. *Select* $\gamma = 1$ *in Theorem* 4.1. *Then, we get the following*

$$\begin{aligned}
&\left| G(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau + G(j+k-\theta)\int_{\frac{j+k}{2}}^{k} w(\tau)d\tau + E_w(Y)\right.\\
&\quad \left.+\int_j^k \tau w'(\tau)G(\tau)d\tau - kw(k)\right|\\
&\leq \left[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\tau} w(u)du\right)d\tau+\int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\tau} w(u)du\right)d\tau\right.
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\theta}^{\frac{j+k}{2}} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du \right) d\tau + \int_{\frac{j+k}{2}}^{j+k-\theta} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du \right) d\tau \\
&- \int_{j+k-\theta}^{\frac{j-\theta+2k}{2}} \left(\int_{k}^{\tau} w(u)du \right) d\tau \\
&- \int_{\frac{j-\theta+2k}{2}}^{k} \left(\int_{k}^{\tau} w(u)du \right) d\tau \Bigg] \frac{(M_1+m_1)}{2}
\end{aligned}
$$

holds $\forall\ \theta \in \left[j, \frac{j+k}{2}\right]$, *where* $E_w(Y)$ *is the weighted expectation of* Y.

Remark 4.3. If we put $w = \frac{1}{k-j}$ in Corollary 4.2 and taking the expectation $E(Y) = \int_j^k \tau G(\tau)d\tau = k - \int_j^k G(\tau)d\tau$, we recapture Theorem 6 of [3].

Corollary 4.4. *Select* $\theta = \frac{3j+k}{4}$ *in Theorem* 4.1, *we get*

$$
\begin{aligned}
&\Bigg| G\left(\frac{3j+k}{4}\right) \int_j^{\frac{j+k}{2}} w(\tau)d\tau + G\left(\frac{j+3k}{4}\right) \left(\frac{3}{4}(k-j)\right)^{1-\gamma} \left(\frac{k-j}{4}\right)^{\gamma-1} \int_{\frac{j+k}{2}}^{k} w(\tau)d\tau \\
&\quad - \left(\frac{3}{4}(k-j)\right)^{1-\gamma} ((\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y)) \\
&\quad + (\gamma-1)J_j^{\gamma-1}(P(\frac{3j+k}{4}, k)G(k)) \Bigg| \\
&\leq \left(\frac{3}{4}(k-j)\right)^{1-\gamma} \cdot \max_{\tau\in[j,k]} \left|(k-\tau)^{\gamma-1}\right| \\
&\quad \cdot \Bigg[\int_j^{\frac{7j+k}{8}} \left(\int_j^{\tau} w(u)du \right) d\tau + \int_{\frac{7j+k}{8}}^{\frac{3j+k}{4}} \left(\int_j^{\tau} w(u)du \right) d\tau \\
&\quad - \int_{\frac{3j+k}{4}}^{\frac{j+k}{2}} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du \right) d\tau + \int_{\frac{j+k}{2}}^{\frac{j+3k}{4}} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du \right) d\tau \\
&\quad - \int_{\frac{j+3k}{4}}^{\frac{j+7k}{8}} \left(\int_k^{\tau} w(u)du \right) d\tau - \int_{\frac{j+7k}{8}}^{k} \left(\int_k^{\tau} w(u)du \right) d\tau \Bigg] \frac{(M_1+m_1)}{2}.
\end{aligned}
$$

Remark 4.5. By putting $\gamma = 1$ and $w = \frac{1}{k-j}$ in Corollary 4.4, we recapture Corollary 4 of [3].

Corollary 4.6. *If* G *is symmetric about the* $\theta-$*axis in the Theorem* 4.1, *i.e.,* $G(j+k-\theta) = G(\theta)$, *then*

$$
\begin{aligned}
&\Bigg| G(\theta) \int_j^{\frac{j+k}{2}} w(\tau)d\tau + G(\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1} \int_{\frac{j+k}{2}}^{k} w(\tau)d\tau \\
&\qquad - (k-\theta)^{1-\gamma} ((\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y))
\end{aligned}
$$

$$
\begin{aligned}
&+(\gamma-1)J_j^{\gamma-1}(P(\theta,k)G(k))\Bigg| \\
&\leq (k-\theta)^{1-\gamma}\cdot\max_{\tau\in[j,k]}\left|(k-\tau)^{\gamma-1}\right|\cdot\Bigg[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\tau}w(u)du\right)d\tau \\
&+\int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\tau}w(u)du\right)d\tau-\int_{\theta}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\tau}w(u)du\right)d\tau \\
&+\int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^{\tau}w(u)du\right)d\tau-\int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^{\tau}w(u)du\right)d\tau \\
&-\int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^{\tau}w(u)du\right)d\tau\Bigg]\frac{(M_1+m_1)}{2}
\end{aligned}
$$

holds $\forall\ \theta\in\left[j,\frac{j+k}{2}\right]$.

Remark 4.7. If put $\gamma=1$ and $w=\frac{1}{k-j}$ in Corollary 4.6, we recapture Corollary 5 of [3].

We would present some special means before the applications to special means and these means will apply in the 5th section.

Special Means: These means can be found in [35].

(a) The Arithmetic Mean

$$A=A(j,k)=\frac{j+k}{2};\quad j,k\geq 0.$$

(b) The Geometric Mean

$$G=G(j,k)=\sqrt{jk};\quad j,k\geq 0.$$

(c) The Harmonic Mean

$$H=H(j,k)=\frac{2}{\frac{1}{j}+\frac{1}{k}};\quad j,k>0.$$

(d) The Logarithmic Mean

$$L=L(j,k)=\begin{cases}j, & \text{if}\quad j=k\\ \dfrac{k-j}{\ln k-\ln j}, & \text{if}\quad j\neq k;\end{cases}\qquad j,k>0.$$

(e) Identric Mean

$$I = I(j,k) = \begin{cases} j, & \text{if} \quad j = k \\ \ln\left(\dfrac{\left(\frac{k^k}{j^j}\right)^{\frac{1}{k-j}}}{e}\right), & \text{if} \quad j \neq k; \end{cases} \qquad j,k > 0.$$

(f) $p-$Logarithmic Mean

$$L_p = L_p(j,k) = \begin{cases} j, & \text{if} \quad j = k \\ \left(\dfrac{k^{p+1} - j^{p+1}}{(p+1)(k-j)}\right)^{\frac{1}{p}}, & \text{if} \quad j \neq k, \end{cases}$$

where $p \in \mathbb{R}\backslash\{-1,0\}, j,k > 0$. It is known that L_p is monotonically over $p \in \mathbb{R}$, $L_0 = I$ and $L_{-1} = L$.

5. Application to Special Means

Example 5.1. Consider

$$\begin{aligned} &\gamma = 1, \\ &g(\theta) = \theta^p,\ p \in \mathbb{R}\backslash\{-1,0\},\ \text{then for } j < k, \\ &\text{then } \frac{1}{(k-j)}\int_j^k g(\tau)d\tau = L_p^p(j,k), \\ &\frac{g(j)+g(k)}{2} = A(j^p,k^p), \\ &\text{and } \frac{j+k}{2} = A, \end{aligned}$$

where $\theta \in \left[j, \frac{j+k}{2}\right]$.
Therefore, (2.1) becomes

$$\begin{aligned} \left|\frac{\theta^p + (2A-\theta)^p}{2} - L_p^p(j,k)\right| \leq \Bigg[&\int_j^{\frac{j+\theta}{2}} \left(\int_j^\tau w(u)du\right)d\tau + \int_{\frac{j+\theta}{2}}^{\theta} \left(\int_j^\tau w(u)du\right)d\tau \\ &- \int_\theta^A \left(\int_A^\tau w(u)du\right)d\tau + \int_A^{2A-\theta} \left(\int_A^\tau w(u)du\right)d\tau \\ &- \int_{2A-\theta}^{A+\frac{k-\theta}{2}} \left(\int_k^\tau w(u)du\right)d\tau \\ &- \int_{A+\frac{k-\theta}{2}}^{k} \left(\int_k^\tau w(u)du\right)d\tau \Bigg] \frac{(M_1+m_1)}{2}. \end{aligned}$$

If we choose $\theta = j$ in (2.1), we get

$$\left|A(j^p,k^p) - L_p^p(j,k)\right|$$

$$\leq \left[\int_A^k \left(\int_A^\tau w(u)du\right) d\tau - \int_j^A \left(\int_A^\tau w(u)du\right) d\tau\right] \frac{(M_1+m_1)}{2}.$$

Example 5.2. Consider

$$\begin{aligned}
&\gamma = 1,\\
&g(\theta) = \frac{1}{\theta}, \quad \theta \neq 0\\
&\text{then } \frac{1}{k-j}\int_j^k g(\tau)d\tau = L^{-1}(j,k),\\
&\frac{g(j)+g(k)}{2} = \frac{A}{G^2},\\
&\text{and } \frac{j+k}{2} = A,
\end{aligned}$$

where $\theta \in \left[j, \frac{j+k}{2}\right] \subset (0,\infty)$.
Therefore, (2.1) becomes

$$\begin{aligned}
\left|\frac{A}{\theta(j+k-\theta)} - L^{-1}(j,k)\right| \leq \Bigg[&\int_j^{\frac{j+\theta}{2}} \left(\int_j^\tau w(u)du\right) d\tau + \int_{\frac{j+\theta}{2}}^{\theta} \left(\int_j^\tau w(u)du\right) d\tau\\
&- \int_\theta^A \left(\int_A^\tau w(u)du\right) d\tau + \int_A^{2A-\theta} \left(\int_A^\tau w(u)du\right) d\tau\\
&- \int_{2A-\theta}^{A+\frac{k-\theta}{2}} \left(\int_k^\tau w(u)du\right) d\tau\\
&- \int_{A+\frac{k-\theta}{2}}^{k} \left(\int_k^\tau w(u)du\right) d\tau\Bigg] \frac{(M_1+m_1)}{2}.
\end{aligned}$$

If we choose $\theta = j$ in (2.1), we get

$$\left|\frac{A}{G^2} - L^{-1}(j,k)\right| \leq \left[\int_A^k \left(\int_A^\tau w(u)du\right) d\tau - \int_j^A \left(\int_A^\tau w(u)du\right) d\tau\right] \frac{(M_1+m_1)}{2}.$$

Example 5.3. Consider

$$\begin{aligned}
&\gamma = 1,\\
&g(\theta) = \ln\theta, \quad \theta \in (0,\infty)\\
&\text{then } \frac{1}{k-j}\int_j^k g(\tau)d\tau = \ln(I(j,k)),\\
&\frac{g(j)+g(k)}{2} = \ln G,\\
&\text{and } \frac{j+k}{2} = A,
\end{aligned}$$

where $\theta \in \left[j, \frac{j+k}{2}\right] \subset (0, \infty)$.
Therefore, (2.1) becomes

$$\begin{aligned}\left|\ln\left[\frac{[\theta(2A-\theta)]^{\frac{1}{2}}}{I(j,k)}\right]\right| \leq \Bigg[&\int_j^{\frac{j+\theta}{2}} \left(\int_j^{\tau} w(u)du\right) d\tau \\ &+ \int_{\frac{j+\theta}{2}}^{\theta} \left(\int_j^{\tau} w(u)du\right) d\tau - \int_{\theta}^{A} \left(\int_A^{\tau} w(u)du\right) d\tau \\ &+ \int_A^{2A-\theta} \left(\int_A^{\tau} w(u)du\right) d\tau - \int_{2A-\theta}^{A+\frac{k-\theta}{2}} \left(\int_k^{\tau} w(u)du\right) d\tau \\ &- \int_{A+\frac{k-\theta}{2}}^{k} \left(\int_k^{\tau} w(u)du\right) d\tau\Bigg] \frac{(M_1+m_1)}{2}.\end{aligned}$$

If we choose $\theta = j$ in (2.1), we get

$$\left|\ln\left[\frac{G}{I(j,k)}\right]\right| \leq \left[\int_A^k \left(\int_A^{\tau} w(u)du\right) d\tau - \int_j^A \left(\int_A^{\tau} w(u)du\right) d\tau\right] \frac{(M_1+m_1)}{2}.$$

Example 5.4. Consider

$$\begin{aligned}&\gamma = 1, \\ &g(\theta) = e^{\theta}, \quad \theta \in (-\infty, \infty) \\ &\text{then } \frac{1}{k-j}\int_j^k g(\tau)d\tau = \frac{e^k - e^j}{k-j}, \\ &\frac{g(j)+g(k)}{2} = A(e^j, e^k), \text{ and } \frac{j+k}{2} = A,\end{aligned}$$

where $\theta \in \left[j, \frac{j+k}{2}\right]$.
Therefore, (2.1) becomes

$$\begin{aligned}&\left|\frac{e^{\theta}+e^{(2A-\theta)}}{2} - \frac{e^k - e^j}{k-j}\right| \\ &\leq \Bigg[\int_j^{\frac{j+\theta}{2}} \left(\int_j^{\tau} w(u)du\right) d\tau + \int_{\frac{j+\theta}{2}}^{\theta} \left(\int_j^{\tau} w(u)du\right) d\tau \\ &\quad - \int_{\theta}^{A} \left(\int_A^{\tau} w(u)du\right) d\tau + \int_A^{2A-\theta} \left(\int_A^{\tau} w(u)du\right) d\tau \\ &\quad - \int_{2A-\theta}^{A+\frac{k-\theta}{2}} \left(\int_k^{\tau} w(u)du\right) d\tau - \int_{A+\frac{k-\theta}{2}}^{k} \left(\int_k^{\tau} w(u)du\right) d\tau\Bigg] \frac{(M_1+m_1)}{2}.\end{aligned}$$

If we choose $\theta = j$ in (2.1), we get

$$\left|A(e^j, e^k) - \frac{e^k - e^j}{k-j}\right| \leq \left[\int_A^k \left(\int_A^\tau w(u)du\right) d\tau - \int_j^A \left(\int_A^\tau w(u)du\right) d\tau\right] \frac{(M_1 + m_1)}{2}.$$

Example 5.5. Consider

$$\begin{aligned}
&\gamma = 1,\\
&g(\theta) = \tan\theta, \quad \theta \neq \frac{\pi}{2} \pm n\pi\\
&\text{then } \frac{1}{k-j}\int_j^k g(\tau)d\tau = \ln\left[\frac{\sec k}{\sec j}\right]^{k-j},\\
&\frac{g(j)+g(k)}{2} = A(\tan j, \tan k), \text{ and } \frac{j+k}{2} = A,
\end{aligned}$$

where $\theta \in \left[j, \frac{j+k}{2}\right]$.
Therefore, (2.1) becomes

$$\begin{aligned}
&\left|\frac{\tan\theta + \tan(2A-\theta)}{2} - \ln\left[\frac{\sec k}{\sec j}\right]^{k-j}\right|\\
&\leq \left[\int_j^{\frac{j+\theta}{2}} \left(\int_j^\tau w(u)du\right) d\tau + \int_{\frac{j+\theta}{2}}^\theta \left(\int_j^\tau w(u)du\right) d\tau\right.\\
&\quad - \int_\theta^A \left(\int_A^\tau w(u)du\right) d\tau + \int_A^{2A-\theta} \left(\int_A^\tau w(u)du\right) d\tau\\
&\quad \left. - \int_{2A-\theta}^{A+\frac{k-\theta}{2}} \left(\int_k^\tau w(u)du\right) d\tau - \int_{A+\frac{k-\theta}{2}}^k \left(\int_k^\tau w(u)du\right) d\tau\right] \frac{(M_1+m_1)}{2}.
\end{aligned}$$

If we choose $\theta = j$ in (2.1), we get

$$\begin{aligned}
&\left|A(\tan j, \tan k) - \ln\left[\frac{\sec k}{\sec j}\right]^{k-j}\right|\\
&\leq \left[\int_A^k \left(\int_A^\tau w(u)du\right) d\tau - \int_j^A \left(\int_A^\tau w(u)du\right) d\tau\right] \frac{(M_1+m_1)}{2}.
\end{aligned}$$

6. Conclusion

In this article, we aimed to generalize the results of [3] and [23]. We have obtained generalization of companion of Ostrowski's type integral inequality by applying the Riemann-Liouville fractional integral. Using suitable substitutions we recaptured all the results of M. W. Alomari's article [3] and gave some special cases and we have also recaptured all the results of one more article [23] by different authors. Further, we

have deduced applications to numerical integration, probability theory and special means.

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