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Some New Notions of Bases for the Range of Operators in Hilbert Spaces

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Abstract. This paper is devoted to introduce a new concept of bases for the range of the operator K. Actually, we consider controlled K-orthonormal and controlled K-Riesz bases which are a generalization of ordinary bases in Hilbert spaces. In the sequel, we give some characterizations of these sequences. Next, we construct some new controlled bases with operator theory tools. Finally, some dual characterizations are given.

1. Introduction

The notion of Frames, which are used as a powerful substitute to Hilbert space bases, dates back to Gabor [19] and Duffin and Schaeffer [17]. These sequences, are redundant systems that provide non-unique representations of vectors in Hilbert spaces. They are also very important for applications, e.g. in physics [2, 14], signal processing [6, 10, 11] and acoustics [7, 26]. There are several generalizations of the concept of frames; for instance, fusion frames, g-frames, weaving frames, etc.

Găvruţa [20] proposed the concept of K-frames during the study of the atomic decomposition of bounded linear operators for subspaces, were first introduced by Feichtinger and Werther in [18]. K-frames, which are more general than ordinary frames, are limited to the range of a bounded linear operator in Hilbert spaces. In fact, they replace the lower bound condition $A\|f\|^2$ of classical frames by new lower bound condition $A\|K^*f\|^2$.

Controlled frames for spherical wavelets were first introduced in [12] and the related theory for general frames was reformulated in [5]. Since

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then, controlled frames have been widely studied in [30, 31]. The main advantage of these frames lies in the fact that they keep all the properties of standard frames but additionally they give a generalized way to check the frame condition while offering a numerical advantage in the sense of preconditioning.

A Riesz basis for a separable Hilbert space $\mathcal{H}$ is a sequence of the form $\{Ue_i\}_{i\in\mathbb{I}}$ where $\{e_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}$ and U is a bounded bijective operator from $\mathcal{H}$ onto $\mathcal{H}$. Riesz bases were introduced by Bari [8, 9], and many properties and equivalent characterizations were already determined in [9]. As is well known Riesz basis is not only a basis but also a special frame. The research of frame and Riesz basis plays important role in theoretical research of wavelet analysis [21]. There are some generalizations of Riesz bases such as g-Riesz bases [33], K-Riesz bases [22] and Riesz fusion bases [3].

The main purpose of this paper is to introduce the concepts of controlled K-orthonormal and controlled K-Riesz bases in Hilbert spaces. Next, we give some characterizations of these sequences. Moreover, we propose several methods to construct controlled orthonormal and controlled Riesz bases. Finally, we mention some duality results.

2. Notation and Definitions

In this section, we collect the basic notation and some preliminary results. Throughout the paper, $\mathcal{H}$ is a separable Hilbert space and $\mathbb{I}$ is an at most countable index set. Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded and linear operators on $\mathcal{H}$. For $U \in \mathcal{B}(\mathcal{H})$, the notations U^*, $R(U)$ and $N(U)$ denote respectively the adjoint operator, the range and the null space of U. We define $\mathcal{GL}(\mathcal{H})$ as the set of all bounded and linear operators on $\mathcal{H}$ with a bounded inverse. One can easily check that if $U, V \in \mathcal{GL}(\mathcal{H})$, then U^* and UV are also in $\mathcal{GL}(\mathcal{H})$. A bounded operator T is called positive, if $\langle Tf, f\rangle > 0$, for all $0 \neq f \in \mathcal{H}$. Every positive operator is obviously self-adjoint. If $U \in \mathcal{B}(\mathcal{H})$ is positive, then there exists a unique positive operator V such that $V^2 = U$. This will be denoted by $V = U^{\frac{1}{2}}$. Also, the operator V commutes with every operator that commutes with U. The set of positive operators in $\mathcal{GL}(\mathcal{H})$ will be denoted by $\mathcal{GL}^+(\mathcal{H})$. Note that, by [5, Proposition 2.4], $U \in \mathcal{GL}^+(\mathcal{H})$ if and only if U is positive and $U^{\frac{1}{2}} \in \mathcal{GL}(\mathcal{H})$.

For non-invertible operators, various types of generalized inverses exist in the literature. One of the most important of these generalized inverses is called the pseudo-inverse operator. In the next lemma, we collect some properties of this operator.

Lemma 2.1 ([13])**.** *Suppose that U is a bounded and invertible operator on $\mathcal{B}(\mathcal{H})$ with closed range $R(U)$. Then, there exists a unique operator*

$U^\dagger \in \mathcal{B}(\mathcal{H})$, *called the pseudo-inverse of* U, *satisfying*

$$\begin{array}{cc} UU^\dagger U = U, & U^\dagger UU^\dagger = U^\dagger, \\ \left(UU^\dagger\right)^* = UU^\dagger, & \left(U^\dagger U\right)^* = U^\dagger U, \\ N\left(U^\dagger\right) = (R(U))^\perp = N\left(U^*\right), & R\left(U^\dagger\right) = (N(U))^\perp = R\left(U^*\right). \end{array}$$

The following useful result, so-called Douglas's theorem, will be used in the sequel.

Theorem 2.2 ([16])**.** *Let* $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$ *and* $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H})$. *The following statements are equivalent.*

(i) $R(T) \subseteq R(S)$.
(ii) *There exists* $\lambda > 0$ *such that* $TT^* \leq \lambda SS^*$.
(iii) *There exists* $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *such that* $T = SU$.

Corollary 2.3. *Assume that* $T, S \in \mathcal{B}(\mathcal{H})$. *Then, the following assertions are equivalent.*

(I) *There exists an invertible operator* $W \in \mathcal{B}(\mathcal{H})$ *such that* $T = SW$.
(II) $N(T) = N(S)$ *and* $R(T) = R(S)$.

Proposition 2.4 ([4])**.** *Let* $U \in \mathcal{B}(\mathcal{H})$. *The following statements are euivalent.*

(1) U *is injective and* $R(T)$ *is closed.*
(2) U *is left invertible.*

Definition 2.5. A sequence $\{f_i\}_{i\in\mathbb{I}}$ of elements in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}). \tag{2.1}$$

The numbers A, B are called frame bounds. If only the right inequality of (2.1) holds, $\{f_i\}_{i\in\mathbb{I}}$ is called a Bessel sequence.

K-frames which recently introduced by Găvruţa [20] are a generalization of frames, in the meaning that the lower frame bound only holds for that admits to reconstruct from the range of a linear and bounded operator K in a Hilbert space.

Definition 2.6. A sequence $\{f_i\}_{i\in\mathbb{I}}$ of elements in $\mathcal{H}$ is called a K-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}).$$

Because of the higher generality of K-frames, many properties for ordinary frames may not hold for K-frames, such as the corresponding synthesis operator for K-frames is not surjective, the frame operator for

K-frames is not invertible for all $f \in \mathcal{H}$, the alternate dual reconstruction pair for K-frames is not interchangeable in general, etc.

Controlled frames, as one of the newest generalizations of frames, have been introduced by Balazs et al. [5] to improve the numerical efficiency of iterative algorithms.

Definition 2.7. Let $C \in GL(\mathcal{H})$. A frame controlled by the operator C or C-controlled frame is a family $\{f_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$ such that there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} \langle f, f_i\rangle\langle Cf_i, f\rangle \leq B\|f\|^2, \quad (f \in \mathcal{H}).$$

Moreover, controlled frames were first studied by using two controller operators in [27]. The role of controller operators is like the role of preconditions matrices or operators in linear algebra.

Controlled K-frames, which are more general than K-frames and controlled frames were put forward in [25, 28].

Definition 2.8. Let $C, C' \in GL^+(\mathcal{H})$. A sequence $\{f_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$ is called a (C, C')-controlled K-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}} \langle f, Cf_i\rangle\langle C'f_i, f\rangle \leq B\|f\|^2, \quad (f \in \mathcal{H}). \tag{2.2}$$

If only the right inequality of (2.2) holds, then $\{f_i\}_{i\in\mathbb{I}}$ is called a (C, C')-controlled Bessel sequence.

We know that if $C, C' \in \mathcal{B}(\mathcal{H})$ are two positive operators, then the operator CC' is also positive if and only if C and C' commute with each other. From now on, we assume that $C, C' \in \mathcal{B}(\mathcal{H})$ are two positive operators with $CC' = C'C$. Suppose that $\{f_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled Bessel sequence for $\mathcal{H}$. The bounded and linear operator $T_{CC'} : \ell^2(\mathbb{I}) \to \mathcal{H}$ defined as

$$T_{CC'}\left(\{c_i\}_{i\in\mathbb{I}}\right) := \sum_{i\in\mathbb{I}} c_i(CC')^{\frac{1}{2}} f_i, \quad \left(\{c_i\}_{i\in\mathbb{I}} \in \ell^2\left(\mathbb{I}\right)\right),$$

is called the synthesis operator. The adjoint operator $T^*_{CC'} : \mathcal{H} \to \ell^2(\mathbb{I})$ that is obtained as

$$T^*_{CC'}(f) = \left\{\left\langle f, (CC')^{\frac{1}{2}} f_i\right\rangle\right\}_{i\in\mathbb{I}}, \quad (f \in \mathcal{H}),$$

is called the analysis operator. Composing $T_{CC'}$ and $T^*_{CC'}$, the operator $S_{CC'} : \mathcal{H} \to \mathcal{H}$ given by

$$S_{CC'}(f) = \sum_{i\in\mathbb{I}} \left\langle f, (CC')^{\frac{1}{2}} f_i\right\rangle (CC')^{\frac{1}{2}} f_i, \quad (f \in \mathcal{H}), \tag{2.3}$$

is called the (C, C')-controlled frame operator.

Remark 2.9. Assume that $\{f_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled Bessel sequence for $\mathcal{H}$, $S_F:\mathcal{H}\to\mathcal{H}$ is the frame operator

$$S_Ff=\sum_{i\in\mathbb{I}}\langle f,f_i\rangle f_i,$$

and C and C' commute with S_F. Then, the (C,C')-controlled K-frame inequality (2.2) can be represented in the form

$$\begin{aligned}A\|K^*f\|^2&\leq\sum_{i\in\mathbb{I}}\langle f,Cf_i\rangle\langle C'f_i,f\rangle\\&=\left\langle C'\sum_{i\in\mathbb{I}}\langle Cf,f_i\rangle f_i,f\right\rangle\\&=\left\langle C'S_FCf,f\right\rangle\\&=\left\langle (CC')^{\frac{1}{2}}S_F(CC')^{\frac{1}{2}}f,f\right\rangle\\&=\sum_{i\in\mathbb{I}}\left\langle f,(CC')^{\frac{1}{2}}f_i\right\rangle\left\langle (CC')^{\frac{1}{2}}f_i,f\right\rangle\\&=\sum_{i\in\mathbb{I}}\left|\left\langle f,(CC')^{\frac{1}{2}}f_i\right\rangle\right|^2\\&\leq B\|f\|^2,\end{aligned}$$

for every $f\in\mathcal{H}$. In the sequel, we assume that C,C' and S_F mutually commute.

3. Controlled K-Orthonormal and Controlled K-Riesz Bases

Orthonormal bases in Hilbert spaces are the abstract (infinite dimensional) counterparts of the canonical bases in $\mathbb{C}^n$ and have many similar properties. Orthonormal bases are widely used in mathematics as well as physics, signal processing, and many other areas where expansions in terms of **convenient building blocks** are needed. There are several characterizations of frames and Riesz bases in terms of orthonormal bases. For example, we know that frames and Riesz bases are the image of orthonormal bases under surjective and bijective maps, respectively. Orthonormal bases have some extentions such as g-orthonormal bases [33] and K-orthonormal bases [1].

Now, we introduce the notion of controlled K-orthonormal bases and then, we give some characterizations of them.

Definition 3.1. Suppose that $K\in\mathcal{B}(\mathcal{H})$ and $C,C'\in\mathcal{GL}^+(\mathcal{H})$. A sequence $\{f_i\}_{i\in\mathbb{I}}$ of $\mathcal{H}$ is called a (C,C')-controlled K-orthonormal basis, or simply a (C,C')-controlled K-ONB if the following conditions hold.

a) The sequence $\{f_i\}_{i\in\mathbb{I}}$ is an orthonormal system, that is $\langle f_i, f_j\rangle = \delta_{ij}$.
b) For every $f \in \mathcal{H}$,

$$\sum_{i\in\mathbb{I}} \left|\left\langle f, (CC')^{\frac{1}{2}} f_i\right\rangle\right|^2 = \|K^* f\|^2. \tag{3.1}$$

Example 3.2. Let $\{e_i\}_{i\in\mathbb{I}}$ be an orthonormal basis for $\ell^2(\mathbb{I})$. Define the operator K on $\ell^2(\mathbb{I})$ by $Ke_i = e_{i+1}$. Moreover, define the operators C and C' as:

$$C : \ell^2(\mathbb{I}) \to \ell^2(\mathbb{I}),\ C(a_1, a_2, a_3, a_4, ...) = (2a_1, a_2, 2a_3, a_4, ...),$$
$$C' : \ell^2(\mathbb{I}) \to \ell^2(\mathbb{I}),\ C'(a_1, a_2, a_3, a_4, ...) = (\frac{1}{2}a_1, a_2, \frac{1}{2}a_3, a_4, ...).$$

Clearly, $C, C' \in \mathcal{GL}^+(\mathcal{H})$ and $CC' = C'C$. Put $\{f_i\}_{i\in\mathbb{I}} = \{Ke_i\}_{i\in\mathbb{I}}$. Then, $\{f_i\}_{i\in\mathbb{I}}$ is an orthonormal system, furthermore,

$$\begin{aligned}\sum_{i\in\mathbb{I}} \left|\left\langle f, (CC')^{\frac{1}{2}} f_i\right\rangle\right|^2 &= \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \\ &= \sum_{i\in\mathbb{I}} |\langle f, Ke_i\rangle|^2 \\ &= \sum_{i\in\mathbb{I}} |\langle K^* f, e_i\rangle|^2 \\ &= \|K^* f\|^2.\end{aligned}$$

In the first theorem, we show that (C, C')-controlled K-orthonormal bases are the image of orthonormal bases under some isometry operators.

Theorem 3.3. *Let $(CC')^{-\frac{1}{2}}K$ be an isometry operator. Then, (C, C')-controlled K-orthonormal bases are precisely the sequences $\{Ue_i\}_{i\in\mathbb{I}}$, where $\{e_i\}_{i\in\mathbb{I}}$ is an arbitrary orthonormal basis for $\mathcal{H}$ and $U \in \mathcal{B}(\mathcal{H})$ is an isometry operator with $R(U) = R\left((CC')^{-\frac{1}{2}}K\right)$.*

Proof. First, assume that there exists an isometry operator U such that $R(U) = R\left((CC')^{-\frac{1}{2}}K\right)$. Then, the sequence $\{Ue_i\}_{i\in\mathbb{I}}$ is an othonormal system, since

$$\langle Ue_i, Ue_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}.$$

Now, since $N(U) = N\left((CC')^{-\frac{1}{2}}K\right)$ and $R(U) = R\left((CC')^{-\frac{1}{2}}K\right)$, so by Corollary 2.3, there exists an invertible operator $W \in \mathcal{B}(\mathcal{H})$ such that $U = (CC')^{-\frac{1}{2}}KW$. Clearly, W is also isometry, because

$$\begin{aligned}\langle Wf, Wf\rangle &= \left\langle (CC')^{-\frac{1}{2}}KWf, (CC')^{-\frac{1}{2}}KWf\right\rangle \\ &= \langle Uf, Uf\rangle\end{aligned}$$

$$= \langle f, f\rangle.$$

So, by the fact that W is an invertible and isometry operator, it can be concluded that it is a unitary operator. Hence,

$$\begin{aligned}\sum_{i\in\mathbb{I}} \left|\left\langle f, (CC')^{\frac{1}{2}} U e_i\right\rangle\right|^2 &= \left\|U^*(CC')^{\frac{1}{2}} f\right\|^2 \\ &= \|W^* K^* f\|^2 \\ &= \|K^* f\|^2.\end{aligned}$$

Conversely, let $\{f_i\}_{i\in\mathbb{I}}$ be a (C, C')-controlled K-ONB for $\mathcal{H}$. Set $e_i^{'} = K^*(CC')^{-\frac{1}{2}} f_i$. Then, we can show that the sequence $\{e_i^{'}\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}$. First, note that since $(CC')^{-\frac{1}{2}} K$ is unitary on $R\left((CC')^{-\frac{1}{2}} K\right)$, so we have

$$\begin{aligned}\langle e_i^{'}, e_j^{'}\rangle &= \left\langle K^*(CC')^{-\frac{1}{2}} f_i, K^*(CC')^{-\frac{1}{2}} f_j\right\rangle \\ &= \langle f_i, f_j\rangle \\ &= \delta_{ij}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i\in\mathbb{I}} |\langle f, e_i^{'}\rangle|^2 &= \sum_{i\in\mathbb{I}} \left|\left\langle f, K^*(CC')^{-\frac{1}{2}} f_i\right\rangle\right|^2 \\ &= \sum_{i\in\mathbb{I}} \left|\left\langle (CC')^{-\frac{1}{2}} K f, f_i\right\rangle\right|^2 \\ &= \sum_{i\in\mathbb{I}} \left|\left\langle (CC')^{-1} K f, (CC')^{\frac{1}{2}} f_i\right\rangle\right|^2 \\ &= \left\|K^*(CC')^{-1} K f\right\|^2 \\ &= \|f\|^2.\end{aligned}$$

Therefore, $\{e_i^{'}\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}$. So, by Theorem 3.4.7 of [13], there exists a unitary operator $W \in \mathcal{B}(\mathcal{H})$ such that $e_i^{'} = W e_i$. Put $U = (CC')^{-\frac{1}{2}} K W$. Then, U is an isometry operator and by Corollary 2.3, $R(U) = R\left((CC')^{-\frac{1}{2}} K\right)$. The proof is complete. □

The following proposition provides some conditions on the operators $K, U \in \mathcal{B}(\mathcal{H})$ under which a (C, C')-controlled K-ONB becomes a (C, C')-controlled U-ONB.

Proposition 3.4. *Let $K \in \mathcal{B}(\mathcal{H})$ be a co-isometry operator and $\{f_i\}_{i\in\mathbb{I}}$ be a (C, C')-controlled K-ONB for $\mathcal{H}$. If U is an isometry operator*

which commutes with C and C', then $\{Uf_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled U-ONB for $\mathcal{H}$.

Proof. The sequence $\{Uf_i\}_{i\in\mathbb{I}}$ is an orthonormal system, because

$$\begin{aligned}\langle Uf_i, Uf_j\rangle &= \langle U^*Uf_i, f_j\rangle\\ &= \langle f_i, f_j\rangle\\ &= \delta_{ij}.\end{aligned}$$

Moreover,

$$\begin{aligned}\sum_{i\in\mathbb{I}}\left|\left\langle f,(CC')^{\frac{1}{2}}Uf_i\right\rangle\right|^2 &= \sum_{i\in\mathbb{I}}\left|\left\langle U^*f,(CC')^{\frac{1}{2}}f_i\right\rangle\right|^2\\ &= \|K^*U^*f\|^2\\ &= \|U^*f\|^2.\end{aligned}$$

□

In the next theorem, we give a characterization of (C,C')-controlled K-orthonormal bases.

Theorem 3.5. *Let $\{\delta_i\}_{i\in\mathbb{I}}$ be the canonical orthonormal basis for $\ell^2(\mathbb{I})$. Then, the sequence $\{f_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled K-ONB for $\mathcal{H}$ if and only if there exists an isometry operator $U:\ell^2(\mathbb{I})\to\mathcal{H}$ such that $\{f_i\}_{i\in\mathbb{I}}=\{U\delta_i\}_{i\in\mathbb{I}}$ and for every $f\in\mathcal{H}$, $\left\|U^*(CC')^{\frac{1}{2}}f\right\|=\|K^*f\|$.*

Proof. First, suppose that $\{f_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled K-ONB for $\mathcal{H}$. Define the operator $U:\ell^2(\mathbb{I})\to\mathcal{H}$ as $U(\{f_i\}_{i\in\mathbb{I}}):=\sum_{i\in\mathbb{I}}c_if_i$. Then, U is an isometry operator, since for every $c=\{c_i\}_{i\in\mathbb{I}}\in\ell^2(\mathbb{I})$,

$$\begin{aligned}\langle Uc,Uc\rangle &= \left\langle\sum_{i\in\mathbb{I}}c_if_i,\sum_{i\in\mathbb{I}}c_if_i\right\rangle\\ &= \sum_{i\in\mathbb{I}}|c_i|^2\\ &= \langle c,c\rangle.\end{aligned}$$

Moreover, we have

$$U\delta_i = f_i.$$

Finally,

$$\begin{aligned}\left\|U^*(CC')^{\frac{1}{2}}f\right\|^2 &= \left\|\left\{\left\langle(CC')^{\frac{1}{2}}f,f_i\right\rangle\right\}_{i\in\mathbb{I}}\right\|^2\\ &= \sum_{i\in\mathbb{I}}\left|\left\langle(CC')^{\frac{1}{2}}f,f_i\right\rangle\right|^2\\ &= \|K^*f\|^2.\end{aligned}$$

Conversely, assume that there exists an isometry operator $U : \ell^2(\mathbb{I}) \to \mathcal{H}$ such that $\{f_i\}_{i\in\mathbb{I}} = \{U\delta_i\}_{i\in\mathbb{I}}$ and for every $f \in \mathcal{H}$, $\left\|U^*(CC')^{\frac{1}{2}}f\right\| = \|K^*f\|$. Then, $\{f_i\}_{i\in\mathbb{I}}$ is an orthonormal system, because

$$\begin{aligned}\langle f_i, f_j\rangle &= \langle U\delta_i, U\delta_j\rangle \\ &= \langle \delta_i, \delta_j\rangle \\ &= \delta_{ij}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\sum_{i\in\mathbb{I}}\left|\left\langle f, (CC')^{\frac{1}{2}}f_i\right\rangle\right|^2 &= \sum_{i\in\mathbb{I}}\left|\left\langle f, (CC')^{\frac{1}{2}}U\delta_i\right\rangle\right|^2 \\ &= \sum_{i\in\mathbb{I}}\left|\left\langle U^*(CC')^{\frac{1}{2}}f, \delta_i\right\rangle\right|^2 \\ &= \left\|U^*(CC')^{\frac{1}{2}}f\right\|^2 \\ &= \|K^*f\|^2.\end{aligned}$$

Hence, we conclude that $\{f_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled K-ONB for $\mathcal{H}$ and the proof is complete. □

It is worth noting that by Theorem 5.5.4 of [13], the frames are the image of orthonormal bases under bounded surjective operators on $\mathcal{H}$. The following theorem shows that the image of a (C, C')-controlled K-orthonormal basis under some closed range operators is a (C, C')-controlled K-frame.

Theorem 3.6. *Suppose that $\{f_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled K-ONB for $\mathcal{H}$, $U : \mathcal{H} \to \mathcal{H}$ is a closed range operator which commutes with operators C, C' and K and $R(K^*) \subset R(U)$. Then, $\{Uf_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled K-frame for $\mathcal{H}$.*

Proof. For each $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{i\in\mathbb{I}}\langle f, CUf_i\rangle\langle C'Uf_i, f\rangle &= \sum_{i\in\mathbb{I}}\left|\left\langle U^*f, (CC')^{\frac{1}{2}}f_i\right\rangle\right|^2 \\ &= \|K^*U^*f\|^2 \\ &\leq \|K^*\|^2\|U^*\|^2\|f\|^2.\end{aligned}$$

Now, we prove the lower K-frame bound condition. First, note that for each $f \in \mathcal{H}$,

$$\begin{aligned}\|K^*f\|^2 &= \|(U^\dagger)^*U^*K^*f\|^2 \\ &\leq \|(U^\dagger)^*\|^2\|U^*K^*f\|^2,\end{aligned}$$

and so

$$\begin{aligned}\|K^*U^*f\|^2 &= \|U^*K^*f\|^2 \\ &\geq \|(U^\dagger)^*\|^{-2}\|K^*f\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{i\in\mathbb{I}}\langle f, CUf_i\rangle\langle C'Uf_i, f\rangle &= \sum_{i\in\mathbb{I}}\left|\left\langle U^*f, (CC')^{\frac{1}{2}}f_i\right\rangle\right|^2 \\ &= \|K^*U^*f\|^2 \\ &\geq \|(U^\dagger)^*\|^{-2}\|K^*f\|^2.\end{aligned}$$

□

In [22], Y. Huang et al. introduced a general version of bases in Hilbert spaces, known as K-Riesz bases. First, we recall that a sequence $\{f_i\}_{i\in\mathbb{I}}$ is called $\overline{R(K)}$-complete if

$$\{f\in\mathcal{H} \ : \ \langle f, f_i\rangle = 0, i\in\mathbb{I}\} \subset \left(\overline{R(K)}\right)^{\perp} = N(K^*).$$

Definition 3.7. A sequence $\{f_i\}_{i\in\mathbb{I}}$ is a K-Riesz basis for $\mathcal{H}$ if it is $\overline{R(K)}$-complete and there exist constants $A, B > 0$ such that for all finite sequence $\{c_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})$,

$$A\sum_{i\in\mathbb{I}}|c_i|^2 \leq \left\|\sum_{i\in\mathbb{I}} c_i f_i\right\|^2 \leq B\sum_{i\in\mathbb{I}}|c_i|^2.$$

As a generalization of the above definition, we introduce the concept of (C, C')-controlled K-Riesz basis and then, we investigate some properties of them.

Definition 3.8. A sequence $\{f_i\}_{i\in\mathbb{I}}$ is called a (C, C')-controlled K-Riesz basis, or simply a (C, C')-controlled K-RB, if it is $\overline{R(K)}$-complete and there is $0 < A \leq B < \infty$ such that for every finite scalar sequence $\{c_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})$, we have

$$A\sum_{i\in\mathbb{I}}|c_i|^2 \leq \left\|\sum_{i\in\mathbb{I}} c_i(CC')^{\frac{1}{2}}f_i\right\|^2 \leq B\sum_{i\in\mathbb{I}}|c_i|^2. \tag{3.2}$$

Example 3.9. Suppose $\{e_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}$ and $C, C' \in GL^+(\mathcal{H})$ are the operators which is defined in Example 3.2. Define the operator $K \in \mathcal{B}(\mathcal{H})$ by

$$Kf := \sum_{i=2}^{\infty}\langle f, e_i\rangle e_i.$$

Then, $K^*f := \sum_{i=2}^{\infty}\langle f, e_i\rangle e_i$. Now, for $\{f_i\}_{i\in\mathbb{I}} = \{e_i\}_{i\in\mathbb{I}}$, we have

$$\begin{aligned}\|K^*f\|^2 &= \left\|\sum_{i=2}^{\infty}\langle f, e_i\rangle e_i\right\|^2 \\ &= \sum_{i=2}^{\infty}|\langle f, f_i\rangle|^2 .\end{aligned}$$

Hence, it is concluded that if $\langle f, f_i\rangle = 0$, for each $i \in \mathbb{I}$, then $K^*f = 0$ and so $f \in N(K^*)$. This means that $\{f_i\}_{i\in\mathbb{I}}$ is $\overline{R(K)}$-complete. Furtheremore,

$$\begin{aligned}\left\|\sum_{i\in\mathbb{I}} c_i(CC')^{\frac{1}{2}} f_i\right\|^2 &= \left\|\sum_{i\in\mathbb{I}} c_i f_i\right\|^2 \\ &= \left\|\sum_{i\in\mathbb{I}} c_i e_i\right\|^2 \\ &= \sum_{i\in\mathbb{I}}|c_i|^2 .\end{aligned}$$

Therefore, $\{f_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled K-Riesz basis for $\mathcal{H}$.

The following theorem gives an equivalent and practical condition for a sequence $\{f_i\}_{i\in\mathbb{I}}$ being a (C, C')-controlled K-Riesz basis.

Theorem 3.10. *Suppose that $C, C' \in GL^+(\mathcal{H})$. The following are equivalent.*

(I) *$\{f_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled K-RB.*

(II) *There exists a bounded left invertible operator $M \in \mathcal{B}(\mathcal{H})$ such that $R(K) \subset R\left((CC')^{-\frac{1}{2}}M\right)$ and $\{f_i\}_{i\in\mathbb{I}} = \left\{(CC')^{-\frac{1}{2}}Me_i\right\}_{i\in\mathbb{I}}$, where $\{e_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}$.*

Proof. $(I) \Rightarrow (II)$ Let $\{f_i\}_{i\in\mathbb{I}}$ be a (C, C')-controlled K-RB in $\mathcal{H}$. Define the operator $M : \mathcal{H} \to \mathcal{H}$ as

$$M(f) := \sum_{i\in\mathbb{I}}\langle f, e_i\rangle(CC')^{\frac{1}{2}} f_i, \quad f \in \mathcal{H}. \tag{3.3}$$

Since the sequence $\{\langle f, e_i\rangle_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})$, so by (3.2), we have

$$\begin{aligned}A\|f\|^2 &= A\sum_{i\in\mathbb{I}}|\langle f, e_i\rangle|^2 \\ &\leq \left\|\sum_{i\in\mathbb{I}}\langle f, e_i\rangle(CC')^{\frac{1}{2}} f_i\right\|^2\end{aligned}$$

$$\begin{aligned}&= \|M(f)\|^2\\ &\leq B\sum_{i\in\mathbb{I}}|\langle f, e_i\rangle|^2\\ &= B\|f\|^2.\end{aligned}$$

Therefore it can be concluded that M is a bounded, injective and closed range operator. Consequently, by Proposition 2.4, M is left invertible with the adjoint operator

$$M^*(f) = \sum_{i\in\mathbb{I}}\left\langle f, (CC')^{\frac{1}{2}}f_i\right\rangle e_i.$$

Moreover,

$$\begin{aligned}(CC')^{-\frac{1}{2}}Me_j &= (CC')^{-\frac{1}{2}}\sum_{i\in\mathbb{I}}\langle e_j, e_i\rangle(CC')^{\frac{1}{2}}f_i\\ &= \sum_{i\in\mathbb{I}}\langle e_j, e_i\rangle f_i\\ &= f_j.\end{aligned}$$

Now, we show that $R(K)\subset R\Big((CC')^{-\frac{1}{2}}M\Big)$. Let $f\in N\left(\left((CC')^{-\frac{1}{2}}M\right)^*\right)$. Then,

$$\begin{aligned}0 &= M^*(CC')^{-\frac{1}{2}}f\\ &= \sum_{i\in\mathbb{I}}\left\langle (CC')^{-\frac{1}{2}}f, (CC')^{\frac{1}{2}}f_i\right\rangle e_i\\ &= \sum_{i\in\mathbb{I}}\langle f, f_i\rangle e_i.\end{aligned}$$

Hence, by Definition 3.8, $f\in N(K^*)$ and so $N\left(\left((CC')^{-\frac{1}{2}}M\right)^*\right)\subset N(K^*)$. It is then concluded that $R(K)\subset R\Big((CC')^{-\frac{1}{2}}M\Big)$.

$(II)\Rightarrow(I)$ Assume that there exists a left invertible operator $M\in\mathcal{B}(\mathcal{H})$, with the left inverse M_L, such that $R(K)\subset R\left((CC')^{-\frac{1}{2}}M\right)$ and $\{f_i\}_{i\in\mathbb{I}}=\Big\{(CC')^{-\frac{1}{2}}Me_i\Big\}_{i\in\mathbb{I}}$. We show that $\{f_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled K-RB. For the upper inequality, we have

$$\left\|\sum_{i\in\mathbb{I}}c_i(CC')^{\frac{1}{2}}f_i\right\|^2 = \left\|\sum_{i\in\mathbb{I}}c_iMe_i\right\|^2$$

$$\leq \|M\|^2 \sum_{i\in\mathbb{I}} |c_i|^2.$$

Now, we prove the lower inequality.

$$\begin{aligned}\sum_{i\in\mathbb{I}} |c_i|^2 &= \left\| \sum_{i\in\mathbb{I}} c_i e_i \right\|^2 \\ &= \left\| M_L M \left(\sum_{i\in\mathbb{I}} c_i e_i \right) \right\|^2 \\ &\leq \|M_L\|^2 \left\| \sum_{i\in\mathbb{I}} c_i M e_i \right\|^2 \\ &= \|M_L\|^2 \left\| \sum_{i\in\mathbb{I}} c_i (CC')^{\frac{1}{2}} f_i \right\|^2.\end{aligned}$$

Therefore, $\|M_L\|^{-2} \sum_{i\in\mathbb{I}} |c_i|^2 \leq \left\| \sum_{i\in\mathbb{I}} c_i \, (CC')^{\frac{1}{2}} \, f_i \right\|^2$. Now, we show that $\{f_i\}_{i\in\mathbb{I}}$ is $\overline{R(K)}$-complete. Let there is $f \in \mathcal{H}$ such that for each $i \in \mathbb{I}$, $\langle f, f_i\rangle = 0$. Thus, for each $i \in \mathbb{I}$, $0 = \left\langle f, (CC')^{-\frac{1}{2}} M e_i \right\rangle = \left\langle M^*(CC')^{-\frac{1}{2}} f, e_i \right\rangle$. Since $\{e_i\}_{i\in\mathbb{I}}$ is an orthonormal basis, $M^*(CC')^{-\frac{1}{2}} f = 0$ implying $f \in N\left((CC')^{-\frac{1}{2}} M\right)$. On the other hand, by assumption, $R(K) \subset R\left((CC')^{-\frac{1}{2}} M\right)$ indicating that $N\left((CC')^{-\frac{1}{2}} M\right) \subset N(K^*)$. Hence $f \in N(K^*)$ and thus $\{f_i\}_{i\in\mathbb{I}}$ is $\overline{R(K)}$-complete. □

It is concluded, from Proposition 3.2 in [5], that the controlled frame operator $S_{CC'}$ given by (2.3) is positive, self-adjoint and invertible, but the case of invertibility does not hold for a controlled K-frame operator, in general [34]. Therefore, the classical canonical dual for a controlled K-frame is missing. However, we can find some (C, C')-controlled Bessel sequences that provide the reconstruction formula for the elements of $R(K)$.

Proposition 3.11. *Let $\{f_i\}_{i\in\mathbb{I}}$ be a (C, C')-controlled K-RB in $\mathcal{H}$ and the operators K, C and C' commute with each other. Then, there exists a unique (C, C')-controlled Bessel sequence $\{g_i\}_{i\in\mathbb{I}} \in \mathcal{H}$ such that*

$$Kf = \sum_{i\in\mathbb{I}} \langle f, (CC')^{\frac{1}{2}} g_i\rangle (CC')^{\frac{1}{2}} f_i. \tag{3.4}$$

Proof. Let $\{f_i\}_{i\in\mathbb{I}}$ be a (C, C')-controlled K-RB in $\mathcal{H}$. So, there exists a left invertible operator M, defined as (3.3) with $R(K) \subset R\left((CC')^{-\frac{1}{2}} M\right)$

and $\{f_i\}_{i\in\mathbb{I}}=\left\{(CC')^{-\frac{1}{2}}Me_i\right\}_{i\in\mathbb{I}}$. Since $R(K)\subset R\left((CC')^{-\frac{1}{2}}M\right)$, by Theorem 2.2, there is $U\in\mathcal{B}(\mathcal{H})$ such that $K=(CC')^{-\frac{1}{2}}MU$. Now, for each $f\in\mathcal{H}$,

$$\begin{aligned}Kf&=(CC')^{-\frac{1}{2}}MUf\\&=(CC')^{-\frac{1}{2}}\sum_{i\in\mathbb{I}}\langle Uf,e_i\rangle(CC')^{\frac{1}{2}}f_i\\&=\sum_{i\in\mathbb{I}}\langle Uf,e_i\rangle f_i,\end{aligned}$$

and so

$$\begin{aligned}K(CC')^{\frac{1}{2}}f&=(CC')^{\frac{1}{2}}Kf\\&=\sum_{i\in\mathbb{I}}\langle Uf,e_i\rangle(CC')^{\frac{1}{2}}f_i.\end{aligned}$$

Therefore,

$$\begin{aligned}Kf&=\sum_{i\in\mathbb{I}}\left\langle U(CC')^{-\frac{1}{2}}f,e_i\right\rangle(CC')^{\frac{1}{2}}f_i\\&=\sum_{i\in\mathbb{I}}\left\langle f,(CC')^{-\frac{1}{2}}U^*e_i\right\rangle(CC')^{\frac{1}{2}}f_i.\end{aligned}$$

Put $g_i=(CC')U^*e_i,\ i\in\mathbb{I}$. Then, we have

$$Kf=\sum_{i\in\mathbb{I}}\left\langle f,(CC')^{\frac{1}{2}}g_i\right\rangle(CC')^{\frac{1}{2}}f_i.$$

Clearly, $\{g_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled Bessel sequence, since

$$\begin{aligned}\sum_{i\in\mathbb{I}}\langle f,Cg_i\rangle\langle C'g_i,f\rangle&=\sum_{i\in\mathbb{I}}\left|\left\langle f,(CC')^{\frac{1}{2}}g_i\right\rangle\right|^2\\&=\sum_{i\in\mathbb{I}}\left|\left\langle f,(CC')^{\frac{1}{2}}(CC')U^*e_i\right\rangle\right|^2\\&=\sum_{i\in\mathbb{I}}\left|\left\langle U(CC')(CC')^{\frac{1}{2}}f,e_i\right\rangle\right|^2\\&\leq\|U\|^2\|C\|^2\|C'\|^2\|(CC')^{\frac{1}{2}}\|^2\|f\|^2.\end{aligned}$$

The uniqueness of $\{g_i\}_{i\in\mathbb{I}}$ is obvious and so the proof is complete. □

The (C,C')-controlled Bessel sequence $\{g_i\}_{i\in\mathbb{I}}$ satisfying in (3.4) is called the (C,C')-controlled K-dual of $\{f_i\}_{i\in\mathbb{I}}$. The sequences $\{f_i\}_{i\in\mathbb{I}}$

and $\{g_i\}_{i\in\mathbb{I}}$ in (3.4) are not interchangeable in general. More precisely, from (3.4), it follows that

$$K^*f = \sum_{i\in\mathbb{I}} \langle f, (CC')^{\frac{1}{2}} f_i\rangle (CC')^{\frac{1}{2}} g_i.$$

Hence, $\{f_i\}_{i\in\mathbb{I}}$ and $\{g_i\}_{i\in\mathbb{I}}$ are interchangeable if and only if K is self adjoint.

We know that every Riesz basis is a frame. It is very natural to ask whether every (C, C')-controlled K-Riesz basis is a (C, C')-controlled K-frame. The following proposition addresses this question. Before proceeding, we need the following lemma.

Lemma 3.12. *Let $\{f_i\}_{i\in\mathbb{I}}$ and $\{g_i\}_{i\in\mathbb{I}}$ be two (C, C')-controlled Bessel sequences as in* (3.4)*. Then, $\{f_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled K-frame in $\mathcal{H}$.*

Proof. Using (3.4), for every $f \in \mathcal{H}$,

$$\begin{aligned}
\|K^*f\|^4 &= |\langle K^*f, K^*f\rangle|^2 \\
&= \left|\left\langle \sum_{i\in\mathbb{I}} \langle f, (CC')^{\frac{1}{2}} f_i\rangle (CC')^{\frac{1}{2}} g_i, K^*f \right\rangle\right|^2 \\
&\leq \sum_{i\in\mathbb{I}} \left|\langle f, (CC')^{\frac{1}{2}} f_i\rangle\right|^2 \sum_{i\in\mathbb{I}} \left|\langle K^*f, (CC')^{\frac{1}{2}} g_i\rangle\right|^2 \\
&\leq \sum_{i\in\mathbb{I}} \left|\langle f, (CC')^{\frac{1}{2}} f_i\rangle\right|^2 B_G \|K^*f\|^2,
\end{aligned}$$

where B_G is the upper bound of $\{g_i\}_{i\in\mathbb{I}}$. Therefore,

$$\begin{aligned}
\|K^*f\|^2 B_G^{-1} &\leq \sum_{i\in\mathbb{I}} \left|\langle f, (CC')^{\frac{1}{2}} f_i\rangle\right|^2 \\
&= \sum_{i\in\mathbb{I}} \langle f, Cf_i\rangle \langle C'f_i, f\rangle.
\end{aligned}$$

□

Proposition 3.13. *Let $\{f_i\}_{i\in\mathbb{I}}$ be a (C, C')-controlled K-RB in $\mathcal{H}$ and the operators K, C and C' commute with each other. Then, $\{f_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled K-frame in $\mathcal{H}$.*

Proof. The proof is immediately obtained by Proposition 3.11 and Lemma 3.12. □

Finally, we give a necessary and sufficient condition for a sequence $\{g_i\}_{i\in\mathbb{I}}$ to be a (C, C')-controlled K-dual for a given (C, C')-controlled K-Riesz basis $\{f_i\}_{i\in\mathbb{I}}$.

Theorem 3.14. *Let $\{f_i\}_{i\in\mathbb{I}}$ be a (C,C')-controlled K-RB with the synthesis operator $T_{CC'(F)}$ in $\mathcal{H}$. Then $\{g_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled K-dual of $\{f_i\}_{i\in\mathbb{I}}$ if and only if $\{g_i\}_{i\in\mathbb{I}}=\left\{(CC')^{-\frac{1}{2}}V\delta_i\right\}_{i\in\mathbb{I}}$, where $\{\delta_i\}_{i\in\mathbb{I}}$ is the standard orthonormal basis of $\ell^2(I)$ and $V:\ell^2(I)\to\mathcal{H}$ is a bounded operator such that $T_{CC'(F)}V^*=K$.*

Proof. Assume that $\{g_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled K-dual of $\{f_i\}_{i\in\mathbb{I}}$ with the synthesis operator $T_{CC'(G)}$. Put $V=T_{CC'(G)}$. Then, from (3.4) we have $K=T_{CC'(F)}T^*_{CC'(G)}=T_{CC'(F)}V^*$. Conversely, if there is $V:\ell^2(I)\to\mathcal{H}$ such that $T_{CC'(F)}V^*=K$ and $\{g_i\}_{i\in\mathbb{I}}=\left\{(CC')^{-\frac{1}{2}}V\delta_i\right\}_{i\in\mathbb{I}}$, then

$$\begin{aligned}\sum_{i\in\mathbb{I}}\langle f,(CC')^{\frac{1}{2}}g_i\rangle(CC')^{\frac{1}{2}}f_i&=\sum_{i\in\mathbb{I}}\langle f,V\delta_i\rangle(CC')^{\frac{1}{2}}f_i\\&=\sum_{i\in\mathbb{I}}\langle V^*f,\delta_i\rangle(CC')^{\frac{1}{2}}f_i\\&=T_{CC'(F)}(V^*f)\\&=Kf,\end{aligned}$$

which shows that $\{g_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled K-dual of $\{f_i\}_{i\in\mathbb{I}}$. □

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