Refinement of Midpoint and Trapezoid Type Inequalities for Multiplicatively Convex Functions

Amel Berhail and Badreddine Meftah

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 3
Pages: 267-278

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.2015455.1529



Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 3 (2024), 267-278
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.2015455.1529

Refinement of Midpoint and Trapezoid Type Inequalities for Multiplicatively Convex Functions

Amel Berhail[1*] and Badreddine Meftah[2]

Abstract. In this study, we first establish two new identities for multiplicative differentiable functions. Based on these identities, we derive the midpoint and trapezoid type inequalities. The acquired outcomes improve and refine upon Khan and Budak's findings. At the conclusion, some applications to special means are provided.

1. Introduction

Convexity is a fundamental idea that is used extensively and is closely related to the development of the theory of inequalities, which is a useful tool for analyzing the properties of some differential equation solutions and for estimating the errors in quadrature formulas. Let I be an interval of real numbers.

Definition 1.1 ([20])**.** A convex function $\beta : I \to \mathbb{R}$ is defined as follows:

$$\beta\left(tx + (1-t)\, y\right) \leq t\beta\left(x\right) + (1-t)\, \beta(y),$$

for all $x, y \in I$ and all $t \in [0,1]$.

Definition 1.2 ([20])**.** A positive function $f : I \to \mathbb{R}$ is considered multiplicatively or logarithmically convex if

$$\beta(tx + (1-t)y) \ \leq [\beta(x)]^t \left[\beta(y)\right]^{1-t},$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

2020 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.
Key words and phrases. Non-Newtonian calculus, Midpoint inequality, Trapezoid inequality, Multiplicatively convex functions.
Received: 11 November 2023, Accepted: 09 December 2023.
* Corresponding author.

The Hermite-Hadamard inequality, which is expressed as follows:

$$\beta\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int_a^b \beta(x)\,dx \leq \tfrac{\beta(a)+\beta(b)}{2},$$

is without a doubt the fundamental inequality for convex functions, where β is a convex function on the interval $[a,b]$ with $a<b$ (see [20]).

A trapezoid-type inequality for functions whose initial derivatives' absolute values are convex was developed by Dragomir and Agarwal in [5] as:

$$\left|\tfrac{\beta(a)+\beta(b)}{2} - \tfrac{1}{b-a}\int_a^b \beta(x)\,dx\right| \leq \tfrac{b-a}{8}\left(\left|\beta'(a)\right| + \left|\beta'(b)\right|\right).$$

Pearce and Pečarić demonstrated a midpoint type inequality in [19] for functions whose first derivatives' absolute values are convex as:

$$\left|\beta\left(\tfrac{a+b}{2}\right) - \tfrac{1}{b-a}\int_a^b \beta(x)\,dx\right| \leq \tfrac{b-a}{8}\left(\left|\beta'(a)\right| + \left|\beta'(b)\right|\right).$$

Regarding some papers addressing certain quadratures, look to [7, 9–12] and references therein.

In 1967, Grossman and Katz, created the first non-Newtonian calculation system, called geometric calculation. Over the next few years they had created an infinite family of non-Newtonian calculus, thus modifying the classical calculus introduced by Newton and Leibniz in the 17th century each of which differed markedly from the classical calculus of Newton and Leibniz known today as the non-Newtonian calculus or the multiplicative calculus, where the ordinary product and ratio are used respectively as sum and exponential difference over the domain of positive real numbers (see [6]). This calculation is useful for dealing with exponentially varying functions.

The complete mathematical description of multiplicative calculus was given by Bashirov et al. [4]. Also in the literature, there remains a trace of a similar calculation proposed by the mathematical biologists Volterra and Hostinsky [21] in 1938 called the Volterra calculation which is identified as a particular case of multiplicative calculation.

Recently, Ali et al. [1] gave the analogue of the Hermite-Hadamard inequality for multiplicatively convex functions as follows.

Theorem 1.3. *Let f be a positive and multiplicatively convex function on interval $[a,b]$, then the following inequalities hold*

$$\beta\left(\tfrac{a+b}{2}\right) \leq \left(\int_a^b \beta(x)^{dx}\right)^{\frac{1}{b-a}} \leq \sqrt{\beta(a)\,\beta(b)}.$$

They also established the quotient and product of two multiplicatively convex functions' inequalities. Ali et al. investigated the Hermite-Hadamard type inequalities for log-ϕ-convex and multiplicatively ϕ-convex functions in [2]. Özcan expanded on the findings from [1] for s-convex and h-convex functions, respectively, in [15] and [17]. Özcan provided the multiplicatively peinvex function counterpart of Hermite-Hadamard type inequalities in [16]. The Hermite-Hadamard type inequalities for multiplicatively h-peinvex functions were presented by Özcan in [18]. The Ostrowski and Simpson type inequalities for multiplicatively convex functions are examined by Ali et al. in [3]. Meftah [13] investigated multiplicatively convex functions and the Maclaurin inequality. The Dual Simpson type inequalities for multiplicatively convex functions were studied by Meftah and Lakhdari [14].

Recently, the midpoint and trapezium inequalities below were developed by Khan and Budak [8].

Theorem 1.4. *Let $\beta : I^\circ \subset \mathbb{R} \to \mathbb{R}^+$ be a * differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If β is increasing on $[a, b]$ and β^* is multiplicatively convex on $[a, b]$, then we have*

$$\left| \beta\left(\tfrac{a+b}{2}\right) \left(\int_a^b \beta(u)^{du} \right)^{\frac{1}{a-b}} \right| \leq \left(\beta^*(a)\, \beta^*(b)\right)^{\frac{b-a}{8}}.$$

Theorem 1.5. *Let $\beta : I^\circ \subset \mathbb{R} \to \mathbb{R}^+$ be a * differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If β is increasing on $[a, b]$ and β^* is multiplicatively convex on $[a, b]$, then we have*

$$\left| \sqrt{\beta(a)\, \beta(b)} \left(\int_a^b \beta(u)^{du} \right)^{\frac{1}{a-b}} \right| \leq \left(\beta^*(a)\, \beta^*(b)\right)^{\frac{b-a}{8}}.$$

Inspired by the aforementioned works, we put forth two novel identities for multiplicatively differentiable functions in this study. We prove midpoint and trapezoidal inequality for multiplicatively convex functions based on these principles. The outcomes refine and improve those obtained by Khan and Budak in [8]. Applications to special means are also given.

2. Preliminaries

In this section, we begin by recalling some definitions, properties and notions of derivation as well as multiplicative integration

Definition 2.1 ([4])**.** Let $\beta : \mathbb{R} \to \mathbb{R}^+$ be a positive function. The multiplicative derivative of the function β noted by β^* is defined as

follows

$$\frac{d^*\beta}{dt} = \beta^*(t) = \lim_{h\to 0}\left(\frac{\beta(t+h)}{\beta(t)}\right)^{\frac{1}{h}}.$$

Remark 2.2. If β has positive values and is differentiable at t, then β^* exists and the relation between β^* and ordinary derivative β' is as follows:

$$\beta^*(t) = e^{(\ln\beta(t))'} = e^{\frac{\beta'(t)}{\beta(t)}}.$$

The multiplicative derivative admits the following properties.

Theorem 2.3 ([4])**.** *Let β and φ be multiplicatively differentiable functions, and c be an arbitrary constant. Then functions $c\beta, \beta\varphi, \beta+\varphi, \beta/\varphi$ and β^{φ} are * differentiable and*

- $(c\beta)^*(t) = \beta^*(t)$,
- $(\beta\varphi)^*(t) = \beta^*(t)\,\varphi^*(t)$,
- $(\beta+\varphi)^*(t) = \beta^*(t)^{\frac{\beta(t)}{\beta(t)+\varphi(t)}}\,\varphi^*(t)^{\frac{\varphi(t)}{\varphi(t)+g(t)}}$,
- $\left(\frac{\beta}{\varphi}\right)^*(t) = \frac{\beta^*(t)}{\varphi^*(t)}$,
- $(\beta^{\varphi})^*(t) = \beta^*(t)^{\varphi(t)}\,\beta(t)^{\varphi'(t)}$.

In [4], Bashirov et al. introduced the concept of the * integral called multiplicative integral which is noted $\int_a^b (\beta(t))^{dt}$.

The relationship between the Riemann integral and the multiplicative integral is as follows.

Proposition 2.4 ([4])**.** *If β is Riemann integrable on $[a,b]$, then β is multiplicative integrable on $[a,b]$ and*

$$\int_a^b (\beta(t))^{dt} = \exp\left(\int_a^b \ln(\beta(t))\,dt\right).$$

Moreover, Bashirov et al. [4], show that the multiplicative integral has the following results and properties:

Theorem 2.5 ([4])**.** *Let β be a positive and Riemann integrable on $[a,b]$, then β is multiplicative integrable on $[a,b]$ and*

- $\int_a^b ((\beta(t))^p)^{dt} = \left(\int_a^b (\beta(t))^{dt}\right)^p$,
- $\int_a^b (\beta(t)\,\varphi(t))^{dt} = \int_a^b (\beta(t))^{dt}\int_a^b (\varphi(t))^{dt}$,
- $\int_a^b \left(\frac{\beta(t)}{\varphi(t)}\right)^{dt} = \frac{\int_a^b (\beta(t))^{dt}}{\int_a^b (\varphi(t))^{dt}}$,
- $\int_a^b (\beta(t))^{dt} = \int_a^c (\beta(t))^{dt}\int_c^b (\beta(t))^{dt}, \quad a<c<b$,
- $\int_a^a (\beta(t))^{dt} = 1$ and $\int_a^b (\beta(t))^{dt} = \left(\int_b^a (\beta(t))^{dt}\right)^{-1}$.

Theorem 2.6 (Multiplicative Integration by Parts [4]). *Let $\beta : [a,b] \to \mathbb{R}$ be multiplicative differentiable, let $\varphi : [a,b] \to \mathbb{R}$ be differentiable. Then the function β^{φ} is multiplicative integrable, and*

$$\int_a^b \left(\beta^* (t)^{g(t)}\right)^{dt} = \frac{\beta(b)^{\varphi(b)}}{\beta(a)^{\varphi(a)}} \times \frac{1}{\int_a^b \left(\beta(t)^{\varphi'(t)}\right)^{dt}}.$$

Lemma 2.7 ([3]). *Let $\beta : [a,b] \to \mathbb{R}$ be multiplicative differentiable, let $h : [a,b] \to \mathbb{R}$ and let $\varphi : J \subset \mathbb{R} \to \mathbb{R}$ be two differentiable functions. Then we have*

$$\int_a^b \left(\beta^* (h(t))^{h'(t)\varphi(t)}\right)^{dt} = \frac{\beta(h(b))^{\varphi(b)}}{\beta(h(a))^{\varphi(a)}} \times \frac{1}{\int_a^b \left(\beta(h(t))^{\varphi'(t)}\right)^{dt}}.$$

3. Main Results

The following lemmas are required in order to validate our results.

Lemma 3.1. *Assume $\beta : [a,b] \to \mathbb{R}^+$ be a mapping on $[a,b]$ with $a < b$ that is multiplicative differentiable. If β^* is multiplicatively integrable on $[a,b]$, then the following identity for multiplicative integrals holds.*

$$\beta\left(\tfrac{a+b}{2}\right)\left(\int_a^b \beta(u)^{du}\right)^{\frac{1}{a-b}} = \left(\int_0^1 \left(\beta^*\left((1-t)a + t\tfrac{a+b}{2}\right)^t\right)^{dt}\right)^{\frac{b-a}{4}} \times \left(\int_0^1 \left(\beta^*\left((1-t)\tfrac{a+b}{2} + tb\right)^{t-1}\right)^{dt}\right)^{\frac{b-a}{4}}.$$

Proof. Let

$$I_1 = \left(\int_0^1 \left(\beta^*\left((1-t)a + t\tfrac{a+b}{2}\right)^t\right)^{dt}\right)^{\frac{b-a}{4}},$$

and

$$I_2 = \left(\int_0^1 \left(\beta^*\left((1-t)\tfrac{a+b}{2} + tb\right)^{t-1}\right)^{dt}\right)^{\frac{b-a}{4}}.$$

Using the integration by parts for multiplicative integrals, from I_1 we have

$$\begin{aligned} I_1 &= \left(\int_0^1 \left(\beta^*\left((1-t)a + t\tfrac{a+b}{2}\right)^t\right)^{dt}\right)^{\frac{b-a}{4}} \\ &= \int_0^1 \left(\beta^*\left((1-t)a + t\tfrac{a+b}{2}\right)^{\frac{b-a}{4}t}\right)^{dt} \\ &= \int_0^1 \left(\beta^*\left((1-t)a + t\tfrac{a+b}{2}\right)^{\left(\frac{b-a}{2}\right)\frac{1}{2}t}\right)^{dt} \end{aligned}$$

$$= \frac{\left(\beta\left(\frac{a+b}{2}\right)\right)^{\frac{1}{2}}}{1} \cdot \frac{1}{\int_0^1 \left(\left(\beta\left((1-t)\frac{a+b}{2}+tb\right)\right)^{\frac{1}{2}}\right)^{dt}}$$

$$= \left(\beta\left(\frac{a+b}{2}\right)\right)^{\frac{1}{2}} \frac{1}{\left(\left(\int_0^1 \beta\left((1-t)\frac{a+b}{2}+tb\right)^{\frac{1}{2}}\right)^{dt}\right)}$$

$$= \left(\beta\left(\frac{a+b}{2}\right)\right)^{\frac{1}{2}} \left(\int_a^{\frac{a+b}{2}} \beta\left(u\right)^{du}\right)^{\frac{1}{a-b}}.$$

Similarly, we have

$$I_2 = \left(\int_0^1 \left(\beta^*\left((1-t)\,\frac{a+b}{2}+tb\right)^{t-1}\right)^{dt}\right)^{\frac{b-a}{4}}$$

$$= \int_0^1 \left(\beta^*\left((1-t)\,\frac{a+b}{2}+tb\right)^{\frac{b-a}{4}(t-1)}\right)^{dt}$$

$$= \frac{1}{\left(\beta\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{2}}} \cdot \frac{1}{\int_0^1 \left(\beta\left((1-t)\frac{a+b}{2}+tb\right)\right)^{dt}}$$

$$= \left(\beta\left(\frac{a+b}{2}\right)\right)^{\frac{1}{2}} \cdot \frac{1}{\left(\int_0^1 \left(\left(\beta\left((1-t)\frac{a+b}{2}+tb\right)\right)^{\frac{1}{2}}\right)^{dt}\right)}$$

$$= \left(\beta\left(\frac{a+b}{2}\right)\right)^{\frac{1}{2}} \left(\int_0^1 \beta f\left((1-t)\,\frac{a+b}{2}+tb\right)^{dt}\right)^{-\frac{1}{2}}$$

$$= \left(\beta\left(\frac{a+b}{2}\right)\right)^{\frac{1}{2}} \left(\int_{\frac{a+b}{2}}^b \beta\left(u\right)^{du}\right)^{\frac{1}{a-b}}.$$

Multiplying above equalities we get the desired result. The proof is completed. □

Lemma 3.2. *Assume $\beta : [a,b] \to \mathbb{R}^+$ be a mapping on $[a,b]$ with $a < b$ that is multiplicative differentiable. If β^* is multiplicatively integrable on $[a,b]$, then the following identity for multiplicative integrals holds:*

(3.1)

$$G\left(\beta(a),\beta(b)\right)\left(\int_a^b \beta\left(u\right)^{du}\right)^{\frac{1}{a-b}} = \left(\int_0^1 \left(\beta^*\left((1-t)\,a+t\frac{a+b}{2}\right)^{t-1}\right)^{dt}\right)^{\frac{b-a}{4}} \times \left(\int_0^1 \left(\beta^*\left((1-t)\,\frac{a+b}{2}+tb\right)^{t}\right)^{dt}\right)^{\frac{b-a}{4}},$$

where G is the geometric mean, i.e., $G\left(\alpha,\delta\right) = \sqrt{\alpha\delta}$ for $\alpha,\delta > 0$.

Proof. Using integration by parts for multiplicative integrals, on the right side of (3.1), we have

$$I_1 = \left(\int_0^1 \left(\beta^* \left((1-t)\, a + t \tfrac{a+b}{2} \right)^{t-1} \right)^{dt} \right)^{\frac{b-a}{4}},$$

and

$$I_2 = \left(\int_0^1 \left(\beta^* \left((1-t)\, \tfrac{a+b}{2} + tb \right)^{t} \right)^{dt} \right)^{\frac{b-a}{4}}.$$

Using the integration by parts for multiplicative integrals, from I_1 we have

$$\begin{aligned}
I_1 &= \left(\int_0^1 \left(\beta^* \left((1-t)\, a + t \tfrac{a+b}{2} \right)^{t-1} \right)^{dt} \right)^{\frac{b-a}{4}} \\
&= \int_0^1 \left(\beta^* \left((1-t)\, a + t \tfrac{a+b}{2} \right)^{\frac{b-a}{4}(t-1)} \right)^{dt} \\
&= \int_0^1 \left(\beta^* \left((1-t)\, a + t \tfrac{a+b}{2} \right)^{\left(\frac{b-a}{2}\right)\frac{1}{2}(t-1)} \right)^{dt} \\
&= \frac{1}{\beta(a)^{-\frac{1}{2}}} \cdot \frac{1}{\int_0^1 \left(\left(\beta\left((1-t) \frac{a+b}{2} + tb \right) \right)^{\frac{1}{2}} \right)^{dt}} \\
&= (\beta\,(a))^{\frac{1}{2}}\, \frac{1}{\left(\left(\int_0^1 \beta\left((1-t) \frac{a+b}{2} + tb \right)^{\frac{1}{2}} \right)^{dt} \right)} \\
&= (\beta\,(a))^{\frac{1}{2}} \left(\int_a^{\frac{a+b}{2}} \beta\,(u)^{du} \right)^{\frac{1}{a-b}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_2 &= \left(\int_0^1 \left(\beta^* \left((1-t)\, \tfrac{a+b}{2} + tb \right)^{t} \right)^{dt} \right)^{\frac{b-a}{4}} \\
&= \int_0^1 \left(\beta^* \left((1-t)\, \frac{a+b}{2} + tb \right)^{\frac{b-a}{4}t} \right)^{dt} \\
&= \frac{(\beta(b))^{\frac{1}{2}}}{1} \cdot \frac{1}{\int_0^1 \left(\beta\left((1-t) \frac{a+b}{2} + tb \right) \right)^{dt}} \\
&= (\beta\,(b))^{\frac{1}{2}} \cdot \frac{1}{\left(\int_0^1 \left(\left(\beta\left((1-t) \frac{a+b}{2} + tb \right) \right)^{\frac{1}{2}} \right)^{dt} \right)} \\
&= (\beta\,(b))^{\frac{1}{2}} \left(\int_0^1 \beta\left((1-t)\, \tfrac{a+b}{2} + tb \right)^{dt} \right)^{-\frac{1}{2}}
\end{aligned}$$

$$= (\beta(b))^{\frac{1}{2}} \left(\int_{\frac{a+b}{2}}^{b} \beta(u)^{du} \right)^{\frac{1}{a-b}}.$$

Multiplying above equalities we get the desired result. The proof is completed. □

Theorem 3.3. *Assume $\beta : [a,b] \to \mathbb{R}^+$ be a mapping on $[a,b]$ with $a < b$ that is multiplicative differentiable. If β^* is multiplicative convex on $[a,b]$, then we have*

$$\left| \beta\left(\tfrac{a+b}{2}\right) \left(\int_a^b \beta(u)^{du} \right)^{\frac{1}{a-b}} \right| \leq \left((\beta^*(a)) \left(\beta^*\left(\tfrac{a+b}{2}\right)\right)^4 (\beta^*(b)) \right)^{\frac{b-a}{24}}.$$

Proof. From Lemma 3.1, properties of the multiplicative integral and the multiplicative convexity of β^*, we have

$$\begin{aligned}
&\left| \beta\left(\tfrac{a+b}{2}\right) \left(\int_a^b \beta(u)^{du} \right)^{\frac{1}{a-b}} \right| \\
&\leq \left(\exp \tfrac{b-a}{4} \int_0^1 \left| \ln \left(\beta^* \left((1-t)\, a + t \tfrac{a+b}{2} \right)^t \right) \right| dt \right) \\
&\quad \times \left(\exp \tfrac{b-a}{4} \int_0^1 \left| \ln \left(\beta^* \left((1-t)\, \tfrac{a+b}{2} + tb \right)^{t-1} \right) \right| dt \right) \\
&= \left(\exp \tfrac{b-a}{4} \int_0^1 t \left| \ln \left(\beta^* \left((1-t)\, a + t \tfrac{a+b}{2} \right)^t \right) \right| dt \right) \\
&\quad \times \left(\exp \frac{b-a}{4} \int_0^1 (1-t) \left| \ln \left(\beta^* \left((1-t)\, \frac{a+b}{2} + tb \right) \right) \right| dt \right) \\
&\leq \left(\exp \tfrac{b-a}{4} \int_0^1 t \left| \ln \left(\beta^*(a) \right)^{(1-t)} \left(\beta^* \left(\tfrac{a+b}{2} \right) \right)^t \right| dt \right) \\
&\quad \times \left(\exp \tfrac{b-a}{4} \int_0^1 (1-t) \left| \ln \left(\beta^* \left(\tfrac{a+b}{2} \right) \right)^{(1-t)} (\beta^*(b))^t \right| dt \right) \\
&= \left(\exp \tfrac{b-a}{4} \int_0^1 \left(t(1-t) \ln (\beta^*(a)) + t^2 \ln \left(\beta^* \left(\tfrac{a+b}{2} \right) \right) \right) dt \right) \\
&\quad \times \left(\exp \tfrac{b-a}{4} \int_0^1 \left((1-t)^2 \ln \left(\beta^* \left(\frac{a+b}{2} \right) \right) + t(1-t) \ln (\beta^*(b)) \right) dt \right) \\
&= \left(\exp \frac{b-a}{4} \left(\ln \beta^*(a) \int_0^1 t(1-t)\, dt + \ln \beta^* \left(\tfrac{a+b}{2} \right) \int_0^1 t^2 dt \right) \right)
\end{aligned}$$

$$\times \left(\exp \tfrac{b-a}{4} \left(\ln \beta^* \left(\tfrac{a+b}{2} \right) \int_0^1 (1-t)^2 \, dt + \ln \beta^* (b) \int_0^1 t (1-t) \, dt \right) \right)$$

$$= \left(\exp \tfrac{b-a}{4} \left(\tfrac{1}{6} \ln \beta^* (a) + \frac{1}{3} \ln \beta^* \left(\frac{a+b}{2} \right) \right) \right)$$

$$\times \left(\exp \frac{b-a}{4} \left(\tfrac{1}{3} \ln \beta^* \left(\tfrac{a+b}{2} \right) + \tfrac{1}{6} \ln \beta^* (b) \right) \right)$$

$$= \left((\beta^* (a))^{\frac{1}{6}} \left(\beta^* \left(\tfrac{a+b}{2} \right) \right)^{\frac{2}{3}} (\beta^* (b))^{\frac{1}{6}} \right)^{\frac{b-a}{4}}$$

$$= \left((\beta^* (a)) \left(\beta^* \left(\tfrac{a+b}{2} \right) \right)^4 (\beta^* (b)) \right)^{\frac{b-a}{24}}.$$

The proof is completed. □

Corollary 3.4. *In Theorem 3.3, if we assume that $\beta^* \leq M$, we get*

$$\left| \beta \left(\tfrac{a+b}{2} \right) \left(\int_a^b f(u)^{du} \right)^{\frac{1}{a-b}} \right| \leq M^{\frac{b-a}{4}}.$$

Remark 3.5. In Theorem 3.3, using the multiplicative convexity of β^*i.e. $\beta^* \left(\tfrac{a+b}{2} \right) \leq \sqrt{\beta^* (a) \beta^* (b)}$, we obtain Theorem 3.3 from [8].

Theorem 3.6. *Assume $\beta : [a,b] \to \mathbb{R}^+$ be a mapping on $[a,b]$ with $a < b$ that is multiplicative differentiable. If β^* is multiplicative convex on $[a,b]$, then we have*

$$\left| G(\beta(a), \beta(b)) \left(\int_a^b \beta(u)^{du} \right)^{\frac{1}{a-b}} \right| \leq \left((\beta^* (a)) \left(\beta^* \left(\tfrac{a+b}{2} \right) \right) (\beta^* (b)) \right)^{\frac{b-a}{12}},$$

where G is the geometric mean.

Proof. From Lemma 3.2, properties of the multiplicative integral and the multiplicative convexity of β^*, we have

$$\left| G(\beta(a), \beta(b)) \left(\int_a^b \beta(u)^{du} \right)^{\frac{1}{a-b}} \right|$$

$$\leq \left(\exp \tfrac{b-a}{4} \int_0^1 \left| \ln \left(\beta^* \left((1-t) a + t \tfrac{a+b}{2} \right)^{t-1} \right) \right| dt \right)$$

$$\times \left(\exp \tfrac{b-a}{4} \int_0^1 \left| \ln \left(\beta^* \left((1-t) \tfrac{a+b}{2} + tb \right)^t \right) \right| dt \right)$$

$$= \left(\exp \tfrac{b-a}{4} \int_0^1 (1-t) \left| \ln \left(\beta^* \left((1-t) a + t \tfrac{a+b}{2} \right)^t \right) \right| dt \right)$$

$$
\begin{aligned}
&\times \left(\exp \tfrac{b-a}{4} \int_0^1 t \left|\ln\left(\beta^*\left((1-t)\tfrac{a+b}{2}+tb\right)\right)\right| dt\right) \\
&\leq \left(\exp \tfrac{b-a}{4} \int_0^1 (1-t) \left|\ln\left(\beta^*(a)\right)^{(1-t)}\left(\beta^*\left(\tfrac{a+b}{2}\right)\right)^t\right| dt\right) \\
&\times \left(\exp \tfrac{b-a}{4} \int_0^1 t \left|\ln\left(\beta^*\left(\tfrac{a+b}{2}\right)\right)^{(1-t)}\left(\beta^*(b)\right)^t\right| dt\right) \\
&= \left(\exp \tfrac{b-a}{4} \int_0^1 \left((1-t)^2 \ln\left(\beta^*(a)\right) + (1-t)\, t \ln\left(\beta^*\left(\tfrac{a+b}{2}\right)\right)\right) dt\right) \\
&\times \left(\exp \tfrac{b-a}{4} \int_0^1 \left(t(1-t)\ln\left(\beta^*\left(\tfrac{a+b}{2}\right)\right) + t^2 \ln\left(\beta^*(b)\right)\right) dt\right) \\
&= \left(\exp \tfrac{b-a}{4} \left(\ln \beta^*(a) \int_0^1 (1-t)^2\, dt + \ln \beta^*\left(\tfrac{a+b}{2}\right) \int_0^1 t(1-t)\, dt\right)\right) \\
&\times \left(\exp \tfrac{b-a}{4} \left(\ln \beta^*\left(\tfrac{a+b}{2}\right) \int_0^1 t(1-t)\, dt + \ln \beta^*(b) \int_0^1 t^2 dt\right)\right) \\
&= \left(\exp \tfrac{b-a}{4} \left(\tfrac{1}{3} \ln \beta^*(a) + \tfrac{1}{6} \ln \beta^*\left(\tfrac{a+b}{2}\right)\right)\right) \\
&\times \left(\exp \tfrac{b-a}{4} \left(\tfrac{1}{6} \ln \beta^*\left(\tfrac{a+b}{2}\right) + \tfrac{1}{3} \ln \beta^*(b)\right)\right) \\
&= \left(\left(\beta^*(a)\right)^{\frac{1}{3}} \left(\beta^*\left(\tfrac{a+b}{2}\right)\right)^{\frac{2}{6}} \left(\beta^*(b)\right)^{\frac{1}{3}}\right)^{\frac{b-a}{4}} \\
&= \left(\left(\beta^*(a)\right)\left(\beta^*\left(\tfrac{a+b}{2}\right)\right)\left(\beta^*(b)\right)\right)^{\frac{b-a}{12}}.
\end{aligned}
$$

The proof is completed. □

Remark 3.7. In Theorem 3.6, using the multiplicative convexity of β^*, i.e., $\beta^*\left(\frac{a+b}{2}\right) \leq \sqrt{\beta^*(a)\,\beta^*(b)}$, we obtain Theorem 4.2 from [8].

Corollary 3.8. *In Theorem 3.6, if we assume that $\beta^* \leq M$, we get*

$$
\left| G\left(\beta(a), \beta(b)\right) \left(\int_a^b \beta(u)^{du}\right)^{\frac{1}{a-b}} \right| \leq M^{\frac{b-a}{4}}.
$$

References

1. M.A. Ali, M. Abbas, Z. Zhang, I.B. Sial and R. Arif, *On integral inequalities for product and quotient of two multiplicatively convex functions*, Asian research journal of mathematics, 12 (3) (2019), pp. 1-11.
2. M.A. Ali, M. Abbas and A.A. Zafer, *On some Hermite-Hadamard integral inequalities in multiplicative calculus*, J. Ineq. Special Func., 10 (1) (2019), pp. 111-122.

3. M.A. Ali, H. Budak, M.Z. Sarikaya and Z. Zhang, *Ostrowski and Simpson type inequalities for multiplicative integrals*, Proyecciones, 40 (3) (2021), pp.743-763.
4. A.E. Bashirov, E.M. Kurpinar and A. *Özyapici, Multiplicative calculus and its applications*, J. Math. Anal. Appl., 337 (1) (2008), pp. 36-48.
5. S.S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett., 11 (5) (1998), pp. 91-95.
6. M. Grossman and R. Katz, *Non-Newtonian calculus*, Lee Press, Pigeon Cove, Mass., 1972.
7. A. Kashuri, B. Meftah and P.O. Mohammed, *Some weighted Simpson type inequalities for differentiable s-convex functions and their applications*, J. Frac. Calc. & Nonlinear Sys., 1 (1) (2021), pp. 75-94.
8. S. Khan and H. Budak, *On midpoint and trapezoid type inequalities for multiplicative integrals*, Mathematica, 64 (87) (2022), pp. 95-108.
9. B. Meftah, *Ostrowski's inequality for functions whose first derivatives are s-preinvex in the second sense*, Khayyam J. Math., 3 (1) (2017), pp. 61-80.
10. B. Meftah, M. Merad, N. Ouanas and A. Souahi, *Some new Hermite-Hadamard type inequalities for functions whose n^{th} derivatives are convex*, Acta Comment. Univ. Tartu. Math., 23 (2) (2019), pp. 163-178.
11. B. Meftah and K. Mekalfa, *Some weighted trapezoidal type inequalities via h-preinvexity*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan., 24 (2020), 81-97.
12. B. Meftah and K. Mekalfa, *Some weighted trapezoidal inequalities for differentiable log-convex functions*, J. Interdiscip. Math., 24 (3) (2021), pp. 505-517.
13. B. Meftah, *Maclaurin type inequalities for multiplicatively convex functions*, Proc. Amer. Math. Soc., 151 (5) (2023), pp. 2115-2125.
14. B. Meftah and A. Lakhdari, *Dual Simpson type inequalities for multiplicatively convex functions*, Filomat, 37 (2023), bo. 22, 7673-7683.
15. S. Özcan, *Hermite-hadamard type inequalities for multiplicatively s-convex functions*, Cumhuriyet Science Journal, 41 (1) (2020), pp. 245-259.
16. S. Özcan, *Some integral inequalities of Hermite-Hadamard type for multiplicatively preinvex functions*, AIMS Math., 5 (2) (2020), pp. 1505-1518.

17. S. Özcan, *Hermite-Hadamard type inequalities for multiplicatively h-convex functions*, Konuralp J. Math., 8 (1) (2020),pp. 158-164.
18. S. Özcan, *Hermite-Hadamard type inequalities for multiplicatively h-preinvex functions*, Turk. J. Anal. Number Theory., 9 (3) (2021), pp. 65-70.
19. C.E.M. Pearce and J. Pečarić, *Inequalities for differentiable mappings with application to special means and quadrature formulæ*, Appl. Math. Lett., 13 (2) (2000), pp. 51-55.
20. J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.
21. V. Volterra and B. Hostinsky, *Operations Infinitesimales Lineaires,* Gauthier-Villars, Paris, 1938.

---

[1] Department of Mathematics, Faculty of Mathematics, Computer Science and Sciences Matter, 08 May 1945 University, Guelma, Algeria.
Email address: berhail.amel@@univ-guelma.dz

[2] Department of Mathematics, Faculty of Mathematics, Computer Science and Sciences Matter, 08 May 1945 University, Guelma, Algeria.
Email address: meftah.badreddine@univ-guelma.dz