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Abstract. In this paper, it is aimed to improvement the boundaries of fractional integral operators containing extended generalized Mittag-Leffler functions. The offered results enhance the previously known bounds of the distinct fractional integral operators for strongly p-convex functions of higher order. The acquired inequalities ensure boundedness, continuity, and Hadamard type inequality for fractional integrals containing an expanded Mittag-Leffler function. Furthermore, these results can be refined for various classes of strongly convex functions offered in the literature.

1. Introduction

In the pure and applied sciences, the theory of convexity has become a resource of motivation. This theory ensures an active area of study. Countless generalizations have emerged recently. For instance, strongly convexity is a significant concept. Convexity theory also played a crucial role in expanding and generalizing the theory of inequalities. To prove the mathematical inequalities, fractional integral operators play an essential role in various branches of mathematics and engineering. Many mathematicians have used fractional integrals to advance integral inequalities. The studies and major contributions of these two concepts are widely covered in the literature, see references [1–3, 7, 8, 12–14, 19–22, 27–29].

In recent years, there have been studies on the bounds of the fractional integral operators. We can list a few of these studies as follows:

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In 2020, Hong et al. obtained the fractional integral operator inequalities that provide the boundaries of these operators at an arbitrary point and derived these inequalities for s-exponentially-convex functions [13]. Farid et al. derived the bounds of the fractional integral operators containing Mittag-Leffler functions in the kernels by using (s, m)-convex functions [11]. In [26], Rahman et al. determined the bounds of the sums of general type fractional integrals and obtained some connected inequalities. Chen et al. demonstrated the bounds of the sum of fractional integral operators containing Mittag-Leffler functions for s-convex functions. They also proved that these operators are bounded and continuous [4]. Qiang et al. derived the fractional integral inequalities by defining exponentially (s, m)-convex functions. They demonstrated that these inequalities provide upper bounds, boundedness and continuity for fractional integrals containing an extended Mittag-Leffler function [25]. In [16], Kwun et al. obtained the bounds of the unified integral operators usage the convexity and conditions of these operators. Additionally, Chen et al. examined fractional integral inequalities for fractional integral operators containing expanded Mittag-Leffler functions. They also demonstrated that these inequalities (α, hm) provide upper bounds on left and right sided fractional integrals for convex functions [6].

In this paper, we examine the bounds of the fractional integral operators obtained by using higher order strongly p-convex functions.

Definition 1.1 (see [15])**.** The interval $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R}\setminus\{0\}$. A function $\varphi : I \to \mathbb{R}$ is said to be p-convex function, if

$$\varphi\left([\tau x^p + (1-\tau)y^p]^{\frac{1}{p}}\right) \leq \tau\varphi(x) + (1-\tau)\varphi(y), \tag{1.1}$$

holds for all $x, y \in I$ and $\tau \in [0, 1]$.

Definition 1.2 (see [24])**.** Let $I \subseteq \mathbb{R}$ be an interval and ϵ is a positive number. A function $\varphi : I = [\lambda, \kappa] \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex function with modulus $\epsilon > 0$, if

$$\varphi\left((1-\tau)x + \tau y\right) \leq (1-\tau)\varphi(x) + \tau\varphi(y) - \epsilon\tau(1-\tau)\|y - x\|^2, \tag{1.2}$$

holds for all $x, y \in I$, $\tau \in [0, 1]$.

Definition 1.3 (see [30])**.** Let $I \subset (0, \infty)$ be an interval and ϵ is a positive number. A function $\varphi : I \to \mathbb{R}$ is called strongly p-convex function with modulus $\epsilon > 0$, if

$$\varphi\left([(1-\tau)x^p + \tau y^p]^{\frac{1}{p}}\right) \leq (1-\tau)\varphi(x) + \tau\varphi(y) - \epsilon\tau(1-\tau)\|y^p - x^p\|^2, \tag{1.3}$$

for all $x, y \in I$, $\tau \in [0, 1]$.

Definition 1.4 (see [17]). Let I be a p-convex set. A function $\varphi : I \to \mathbb{R}$ is called to be strongly p-convex function of higher order, if

(1.4)
$$\varphi\left([\tau x^p + (1-\tau)y^p]^{\frac{1}{p}}\right) \leq \tau\varphi(x) + (1-\tau)\varphi(y) - \epsilon\phi(\tau)\|y^p - x^p\|^q,$$

for all $x, y \in I$, $\tau \in [0,1]$ and with $\epsilon \geq 0, q > 0$ and $\phi(\tau) = \tau^q(1-\tau) + \tau(1-\tau)^q$.

Remark 1.5.
(i) By taking $p = 1$ in Definition (1.4), we obtain definition of strongly convex function of higher order (see [28], Definition 9).
(ii) By taking $p = 1$ and $\phi(\tau) = \tau(1-\tau)$ in Definition (1.4), we obtain definition of strongly convex with order $q > 0$ (see [18], Definition 1.3).
(iii) By taking $q = 2$ and $\phi(\tau) = \tau(1-\tau)$ in Definition (1.4), we acquire description of strongly p-convex function.
(iv) By taking $p = -1$, $q = 2$ and $\phi(\tau) = \tau(1-\tau)$ in Definition (1.4), we obtain definition of strongly harmonically convex function (see [24], Definition 1.2).
(v) By taking $p = -1$, $q = 2$ and $\phi(\tau) = \tau(1-\tau)$ in Definition (1.4), we obtain definition of strongly harmonically convex function (see [24], Definition 1.2).

Definition 1.6 (see [9]). Let $\gamma, c, \omega, \xi, l \in \mathbb{C}$, and $\Re(l), \Re(\xi) > 0, \Re(c) > \Re(\gamma) > 0$ with $\tilde{q} \geq 0, \delta, \mu > 0$ and $0 < v \leq \delta + \mu$. Also let $\varphi \in L_1[\lambda, \kappa]$, and $x \in [\lambda, \kappa]$. In that case, the generalized fractional operators $\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\lambda+}\varphi$ and $\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\kappa-}\varphi$ are defined by:

(1.5)
$$\left(\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\lambda+}\varphi\right)(x;\tilde{q}) = \int_\lambda^x (x-\tau)^{\xi-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega(x-\tau)^\mu;\tilde{q}\right)\varphi(\tau)d\tau,$$
$$\left(\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\kappa-}\varphi\right)(x;\tilde{q}) = \int_x^\kappa (\tau-x)^{\xi-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega(\tau-x)^\mu;\tilde{q}\right)\varphi(\tau)d\tau.$$

where
$$E^{\gamma,\delta,v,c}_{\mu,\xi,l}(\tau;\tilde{q}) = \sum_{n=0}^{\infty} \frac{\beta_{\tilde{q}}(\gamma+nv, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nv}}{\Gamma(\mu n+\xi)} \frac{\tau^n}{(l)_{n\delta}}$$

is the generalized Mittag-Leffler function and $\beta_{\tilde{q}}$ is the enlargement of beta function is declared as noted below:

$$\beta_{\tilde{q}}(x,y) = \int_0^1 \tau^{x-1}(1-\tau)^{y-1} e^{-\left(\frac{\tilde{q}}{\tau(1-\tau)}\right)} d\tau,$$

where $\Re(x), \Re(y), \Re(c\tilde{q}) > 0$.

Definition 1.7 (see [9]). Let $\varphi, \psi : [\lambda, \kappa] \to \mathbb{R}$, with $0 < \lambda < \kappa$, be the functions such that φ be positive, and $\varphi \in L_1[\lambda, \kappa]$ and ψ be differentiable and absolutely increasing. Also let $\frac{\phi}{x}$ be an increasing function on $[\lambda, \infty)$, and $\gamma, c, \omega, \xi, l, \in \mathbb{C}$, and $\Re(l), \Re(\xi) > 0, \Re(c) > \Re(\gamma) > 0$ with $\tilde{q} \geq 0, \delta, \mu > 0$, and $0 < v \leq \mu + \delta$. In that case, for $x \in [\lambda, \kappa]$ the fractional operators are described with

(1.6)

$$\left({}_{\psi}\chi^{\phi,\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\lambda+}\varphi\right)(x;\tilde{q}) = \int_{\lambda}^{x} \frac{\phi(\psi(x)-\psi(\tau))}{\psi(x)-\psi(\tau)} \times E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega(\psi(x)-\psi(\tau))^{\mu};\tilde{q}\right)\psi'(\tau)\varphi(\tau)d\tau,$$

$$\left({}_{\psi}\chi^{\phi,\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\kappa-}\varphi\right)(x;\tilde{q}) = \int_{x}^{\kappa} \frac{\phi(\psi(\tau)-\psi(x))}{\psi(\tau)-\psi(x)} \times E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega(\psi(\tau)-\psi(x))^{\mu};\tilde{q}\right)\psi'(\tau)\varphi(\tau)d\tau.$$

Definition 1.8 (see [32]). Let $\psi, \varphi : [\lambda, \kappa] \to \mathbb{R}$, with $0 < \lambda < \kappa$, be the functions such that φ be positive, and $\varphi \in L_1[\lambda, \kappa]$ and ψ be absolutely increasing and differentiable. Also, let $\xi, \gamma, \omega, l, c \in \mathbb{C}$, and $\Re(l), \Re(\xi) > 0, \Re(c) > \Re(\gamma) > 0$ with $\tilde{q} \geq 0, \delta, \mu > 0$, and $0 < v \leq \mu + \delta$. So then, for $x \in [\lambda, \kappa]$ the unified integral operators are declared with:

(1.7)

$$\left({}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\lambda+}\varphi\right)(x;\tilde{q}) = \int_{\lambda}^{x} (\psi(x)-\psi(\tau))^{\xi-1} \times E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega(\psi(x)-\psi(\tau))^{\mu};\tilde{q}\right)\psi'(\tau)\varphi(\tau)d\tau,$$

$$\left({}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\kappa-}\varphi\right)(x;\tilde{q}) = \int_{x}^{\kappa} (\psi(\tau)-\psi(x))^{\xi-1} \times E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega(\psi(\tau)-\psi(x))^{\mu};\tilde{q}\right)\psi'(\tau)\varphi(\tau)d\tau.$$

Definition 1.9 (see [31]). Let $\psi, \varphi : [\lambda, \kappa] \to \mathbb{R}$, with $0 < \lambda < \kappa$, be the functions, φ be positive, and $\varphi \in L_1[\lambda, \kappa]$ and ψ be absolutely increasing and differentiable. Let $\xi > k$ and $\xi, \gamma, \omega, l, c \in \mathbb{C}, \xi, l > 0$, and $c > \gamma > 0$, with $\tilde{q} \geq 0, \delta, \mu > 0$ and $0 < v \leq \mu + \delta$. So then, for $x \in [\lambda, \kappa]$, the left-right generalized k-fractional operators $\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\lambda+}\varphi\right)$ and $\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\kappa-}\varphi\right)$ are declared by

(1.8)

$$\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\lambda+}\varphi\right)(x;\tilde{q})$$

$$= \int_{\lambda}^{x} (\psi(x) - \psi(\tau))^{\frac{\xi}{k}-1} E_{\mu,\xi,l}^{\gamma,\delta,v,c} (\omega(\psi(x) - \psi(\tau))^{\mu}; \tilde{q}) \psi'(\tau)\varphi(\tau)d\tau$$

and

(1.9)

$$\left({}_{\psi}^{k}\chi_{\mu,\xi,l,\omega,\kappa-}^{\gamma,\delta,v,c}\varphi\right)(x;\tilde{q})$$
$$= \int_{x}^{\kappa} (\psi(\tau) - \psi(x))^{\frac{\xi}{k}-1} E_{\mu,\xi,l}^{\gamma,\delta,v,c} (\omega(\psi(\tau) - \psi(x))^{\mu}; \tilde{q}) \psi'(\tau)\varphi(\tau)d\tau.$$

2. Main Results

Theorem 2.1. *Let $\varphi : [\lambda, \kappa] \to \mathbb{R}$, $0 < \lambda < \kappa$, be a real-valued function. If φ is strongly p-convex function of higher order with modulus $\epsilon > 0$, $p \in \mathbb{R}\setminus\{0\}$, then for $\xi, \pi \geq 1$, then the following inequalities for k-fractional operators (1.8) and (1.9) holds:*

(2.1)

$$\left({}_{\psi}^{k}\chi_{\mu,\xi,l,\omega,\lambda+}^{\gamma,\delta,v,c}\varphi\circ\psi\right)(x;\tilde{q}) + \left({}_{\psi}^{k}\chi_{\mu,\pi,l,\omega,\kappa-}^{\gamma,\delta,v,c}\varphi\circ\psi\right)(x;\tilde{q})$$
$$\leq E_{\mu,\xi,l}^{\gamma,\delta,v,c} (\omega (\psi(x) - \psi(\lambda))^{\mu}; \tilde{q}) (\psi(x) - \psi(\lambda))^{\frac{\xi}{k}-1} \left[\frac{1}{\psi^p(x) - \psi^p(\lambda)}\right.$$
$$\times \left(\varphi(\psi(\lambda)) \left\{\psi^p(x)(\psi(x) - \psi(\lambda)) - \frac{\psi^{p+1}(x) - \psi^{p+1}(\lambda)}{p+1}\right\}\right.$$
$$\left.\left. + \varphi(\psi(x)) \left\{\frac{\psi^{p+1}(x) - \psi^{p+1}(\lambda)}{p+1} - \psi^p(\lambda)(\psi(x) - \psi(\lambda))\right\}\right)\right.$$
$$\left. - \epsilon\|\psi^p(x) - \psi^p(\lambda)\|^q D_1(\tau)\right]$$
$$+ E_{\mu,\pi,l}^{\gamma,\delta,v,c} (\omega (\psi(\kappa) - \psi(x))^{\mu}; \tilde{q}) (\psi(\kappa) - \psi(x))^{\frac{\pi}{k}-1} \left[\frac{1}{\psi^p(\kappa) - \psi^p(x)}\right.$$
$$\times \left(\varphi(\psi(\kappa)) \left\{\frac{\psi^{p+1}(\kappa) - \psi^{p+1}(x)}{p+1} - \psi^p(x)(\psi(\kappa) - \psi(x))\right\}\right.$$
$$\left.\left. + \varphi(\psi(x)) \left\{\psi^p(\kappa)(\psi(\kappa) - \psi(x)) - \frac{\psi^{p+1}(\kappa) - \psi^{p+1}(x)}{p+1}\right\}\right)\right.$$
$$\left. - \epsilon\|\psi^p(\kappa) - \psi^p(x)\|^q D_2(\tau)\right],$$

where

$$D_1(\tau) = \int_{\lambda}^{x} \phi\left(\frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \psi'(\tau)d\tau,$$

and

$$D_2(\tau) = \int_x^{\kappa} \phi\left(\frac{\psi^p(\tau) - \psi^p(x)}{\psi^p(\kappa) - \psi^p(x)}\right) \psi'(\tau) d\tau.$$

Proof. Let $x \in [\lambda, \kappa]$. Then for $\tau \in [\lambda, x)$ and$\xi \geq 1$, we obtain the following inequality:

$$\begin{aligned}(2.2)\qquad &(\psi(x) - \psi(\tau))^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l} \left(\omega \left(\psi(x) - \psi(\tau)\right)^{\mu}; \tilde{q}\right) \psi'(\tau) \\ &\leq (\psi(x) - \psi(\lambda))^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l} \left(\omega \left(\psi(x) - \psi(\lambda)\right)^{\mu}; \tilde{q}\right) \psi'(\tau).\end{aligned}$$

Since φ is strongly p-convex function of higher order, we get

$$\begin{aligned}(2.3)\qquad \varphi\left(\psi(\tau)\right) \leq &\left(\frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \varphi\left(\psi(\lambda)\right) \\ &+ \left(1 - \frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \varphi\left(\psi(x)\right) \\ &- \epsilon \phi\left(\frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \|\psi^p(x) - \psi^p(\lambda)\|^q.\end{aligned}$$

By multiplying (2.2) and (2.3), and then integrating over $[\lambda, x]$, we acquire

(2.4)

$$\begin{aligned}&\int_{\lambda}^{x} (\psi(x) - \psi(\tau))^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l} \left(\omega \left(\psi(x) - \psi(\tau)\right)^{\mu}; \tilde{q}\right) \varphi\left(\psi(\tau)\right) \psi'(\tau) d\tau \\ &\leq (\psi(x) - \psi(\lambda))^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l} \left(\omega \left(\psi(x) - \psi(\lambda)\right)^{\mu}; \tilde{q}\right) \\ &\quad \times \left[\varphi\left(\psi(\lambda)\right) \int_{\lambda}^{x} \frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)} \psi'(\tau) d\tau \right. \\ &\quad + \varphi\left(\psi(x)\right) \int_{\lambda}^{x} \left(1 - \frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \psi'(\tau) d\tau \\ &\quad \left. - \epsilon \|\psi^p(x) - \psi^p(\lambda)\|^q \int_{\lambda}^{x} \phi\left(\frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \psi'(\tau) d\tau\right].\end{aligned}$$

From which we acquire that the following inequality:

(2.5)

$$\begin{aligned}&\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\lambda+} \varphi \circ \psi\right)(x; \tilde{q}) \\ &\leq (\psi(x) - \psi(\lambda))^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l} \left(\omega \left(\psi(x) - \psi(\lambda)\right)^{\mu}; \tilde{q}\right) \\ &\quad \times \left[\frac{\varphi\left(\psi(\lambda)\right)}{\psi^p(x) - \psi^p(\lambda)} \left(\psi^p(x) \left(\psi(x) - \psi(\lambda)\right) - \frac{\psi^{p+1}(x) - \psi^{p+1}(\lambda)}{p+1}\right)\right.\end{aligned}$$

$$+\frac{\varphi\left(\psi(x)\right)}{\psi^{p}(x)-\psi^{p}(\lambda)}\left(\frac{\psi^{p+1}(x)-\psi^{p+1}(\lambda)}{p+1}-\psi^{p}(\lambda)\left(\psi(x)-\psi(\lambda)\right)\right)$$
$$-\epsilon\|\psi^{p}(x)-\psi^{p}(\lambda)\|^{q}D_{1}(\tau)\Bigg].$$

Now, on the other hand for $\tau\in(x,\kappa]$ and $\pi\geq 1$, we acquire the following inequality:

$$\begin{aligned}(2.6)\qquad &\left(\psi(\tau)-\psi(x)\right)^{\frac{\pi}{k}-1}E_{\mu,\pi,l}^{\gamma,\delta,v,c}\left(\omega\left(\psi(\tau)-\psi(x)\right)^{\mu};\tilde{q}\right)\psi'(\tau)\\&\leq\left(\psi(\kappa)-\psi(x)\right)^{\frac{\pi}{k}-1}E_{\mu,\pi,l}^{\gamma,\delta,v,c}\left(\omega\left(\psi(\kappa)-\psi(x)\right)^{\mu};\tilde{q}\right)\psi'(\tau).\end{aligned}$$

Again since φ is strongly p-convex function of higher order, we get

$$\begin{aligned}(2.7)\qquad \varphi\left(\psi(\tau)\right)\leq&\left(\frac{\psi^{p}(\tau)-\psi^{p}(x)}{\psi^{p}(\kappa)-\psi^{p}(x)}\right)\varphi\left(\psi(\kappa)\right)\\&+\left(1-\frac{\psi^{p}(\tau)-\psi^{p}(x)}{\psi^{p}(\kappa)-\psi^{p}(x)}\right)\varphi\left(\psi(x)\right)\\&-\epsilon\phi\left(\frac{\psi^{p}(\tau)-\psi^{p}(x)}{\psi^{p}(\kappa)-\psi^{p}(x)}\right)\|\psi^{p}(\kappa)-\psi^{p}(x)\|^{q}.\end{aligned}$$

By multiplying (2.6) and (2.7), and then integrating over $[x,\kappa]$, we obtain

(2.8)

$$\begin{aligned}&\int_{x}^{\kappa}\left(\psi(\tau)-\psi(x)\right)^{\frac{\pi}{k}-1}E_{\mu,\pi,l}^{\gamma,\delta,v,c}\left(\omega\left(\psi(\tau)-\psi(x)\right)^{\mu};\tilde{q}\right)\varphi\left(\psi(\tau)\right)\psi'(\tau)d\tau\\&\leq\left(\psi(\kappa)-\psi(x)\right)^{\frac{\pi}{k}-1}E_{\mu,\pi,l}^{\gamma,\delta,v,c}\left(\omega\left(\psi(\kappa)-\psi(x)\right)^{\mu};\tilde{q}\right)\\&\quad\times\Bigg[\varphi\left(\psi(\kappa)\right)\int_{x}^{\kappa}\frac{\psi^{p}(\tau)-\psi^{p}(x)}{\psi^{p}(\kappa)-\psi^{p}(x)}\psi'(\tau)d\tau\\&\quad+\varphi\left(\psi(x)\right)\int_{x}^{\kappa}\left(1-\frac{\psi^{p}(\tau)-\psi^{p}(x)}{\psi^{p}(\kappa)-\psi^{p}(x)}\right)\psi'(\tau)d\tau\\&\quad-\epsilon\|\psi^{p}(\kappa)-\psi^{p}(x)\|^{q}\int_{x}^{\kappa}\phi\left(\frac{\psi^{p}(\tau)-\psi^{p}(x)}{\psi^{p}(\kappa)-\psi^{p}(x)}\right)\psi'(\tau)d\tau\Bigg].\end{aligned}$$

The right side satisfies the following inequality:

(2.9)

$$\begin{aligned}&\left({}_{\psi}^{k}\chi_{\mu,\pi,l,\omega,\lambda+}^{\gamma,\delta,v,c}\varphi\circ\psi\right)(x;\tilde{q})\\&\leq\left(\psi(\kappa)-\psi(x)\right)^{\frac{\pi}{k}-1}E_{\mu,\pi,l}^{\gamma,\delta,v,c}\left(\omega\left(\psi(\kappa)-\psi(x)\right)^{\mu};\tilde{q}\right)\end{aligned}$$

$$\times \left[\frac{\varphi\left(\psi(\kappa)\right)}{\psi^{p}(\kappa)-\psi^{p}(x)} \left(\frac{\psi^{p+1}(\kappa)-\psi^{p+1}(x)}{p+1} - \psi^{p}(x)\left(\psi(\kappa)-\psi(x)\right) \right) \right.$$
$$+ \frac{\varphi\left(\psi(x)\right)}{\psi^{p}(\kappa)-\psi^{p}(x)} \left(\psi^{p}(\kappa)\left(\psi(\kappa)-\psi(x)\right) - \frac{\psi^{p+1}(\kappa)-\psi^{p+1}(x)}{p+1} \right)$$
$$\left. - \epsilon \|\psi^{p}(\kappa)-\psi^{p}(x)\|^{q} D_{2}(\tau) \right].$$

By adding (2.5) and (2.9), the required inequality (2.1) is acquired. □

Corollary 2.2. *By setting $\xi = \pi$ in (2.1), the following inequality is acquired.*

$$\begin{aligned}
(2.10)\quad & \left({}_{\psi}^{k}\chi_{\mu,\xi,l,\omega,\lambda+}^{\gamma,\delta,v,c}\varphi\circ\psi\right)(x;\tilde{q}) + \left({}_{\psi}^{k}\chi_{\mu,\xi,l,\omega,\kappa-}^{\gamma,\delta,v,c}\varphi\circ\psi\right)(x;\tilde{q}) \\
&\leq E_{\mu,\xi,l}^{\gamma,\delta,v,c}\left(\omega\left(\psi(x)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)\left(\psi(x)-\psi(\lambda)\right)^{\frac{\xi}{k}-1} \\
&\quad \times \left[\frac{1}{\psi^{p}(x)-\psi^{p}(\lambda)} \left(\varphi\left(\psi(\lambda)\right) \left\{ \psi^{p}(x)\left(\psi(x)-\psi(\lambda)\right) \right.\right.\right. \\
&\quad \left. - \frac{\psi^{p+1}(x)-\psi^{p+1}(\lambda)}{p+1} \right\} + \varphi\left(\psi(x)\right) \left\{ \frac{\psi^{p+1}(x)-\psi^{p+1}(\lambda)}{p+1} \right. \\
&\quad \left.\left.\left. - \psi^{p}(\lambda)\left(\psi(x)-\psi(\lambda)\right) \right\} \right) - \epsilon\|\psi^{p}(x)-\psi^{p}(\lambda)\|^{q} D_{1}(\tau) \right] \\
&\quad + E_{\mu,\xi,l}^{\gamma,\delta,v,c}\left(\omega\left(\psi(\kappa)-\psi(x)\right)^{\mu};\tilde{q}\right)\left(\psi(\kappa)-\psi(x)\right)^{\frac{\xi}{k}-1} \\
&\quad \times \left[\frac{1}{\psi^{p}(\kappa)-\psi^{p}(x)} \left(\varphi\left(\psi(\kappa)\right) \left\{ \frac{\psi^{p+1}(\kappa)-\psi^{p+1}(x)}{p+1} \right.\right.\right. \\
&\quad \left. - \psi^{p}(x)\left(\psi(\kappa)-\psi(x)\right) \right\} + \varphi\left(\psi(x)\right) \left\{ \psi^{p}(\kappa)\left(\psi(\kappa)-\psi(x)\right) \right. \\
&\quad \left.\left.\left. - \frac{\psi^{p+1}(\kappa)-\psi^{p+1}(x)}{p+1} \right\} \right) - \epsilon\|\psi^{p}(\kappa)-\psi^{p}(x)\|^{q} D_{2}(\tau) \right].
\end{aligned}$$

Corollary 2.3. *By setting $\varphi \in L_{\infty}[\lambda,\kappa]$ in 2.1, the following inequality is acquired.*

$$\begin{aligned}
(2.11)\quad & \left({}_{\psi}^{k}\chi_{\mu,\xi,l,\omega,\lambda+}^{\gamma,\delta,v,c}\varphi\circ\psi\right)(x;\tilde{q}) + \left({}_{\psi}^{k}\chi_{\mu,\pi,l,\omega,\kappa-}^{\gamma,\delta,v,c}\varphi\circ\psi\right)(x;\tilde{q}) \\
&\leq \|\varphi\|_{\infty}\left(\psi(x)-\psi(\lambda)\right)^{\frac{\xi}{k}} E_{\mu,\xi,l}^{\gamma,\delta,v,c}\left(\omega\left(\psi(x)-\psi(\lambda)\right)^{\mu};\tilde{q}\right) \\
&\quad - \epsilon\frac{\|\psi^{p}(x)-\psi^{p}(\lambda)\|^{q}}{\psi^{p}(x)-\psi^{p}(\lambda)} D_{1}(\tau) \\
&\quad + \|\varphi\|_{\infty}\left(\psi(\kappa)-\psi(x)\right)^{\frac{\pi}{k}} E_{\mu,\pi,l}^{\gamma,\delta,v,c}\left(\omega\left(\psi(x)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)
\end{aligned}$$

$$- \epsilon \frac{\|\psi^p(\kappa) - \psi^p(x)\|^q}{\psi^p(\kappa) - \psi^p(x)} D_2(\tau).$$

Corollary 2.4. *By setting $\xi = \pi$ in (2.11), the following inequality is acquired.*

$$\begin{aligned}
(2.12) \qquad & \left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\lambda+}\varphi \circ \psi\right)(x;\tilde{q}) + \left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\kappa-}\varphi \circ \psi\right)(x;\tilde{q}) \\
& \leq \|\varphi\|_\infty \left(\psi(x) - \psi(\lambda)\right)^{\frac{\xi}{k}} E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega\left(\psi(x) - \psi(\lambda)\right)^\mu; \tilde{q}\right) \\
& \quad - \epsilon \frac{\|\psi^p(x) - \psi^p(\lambda)\|^q}{\psi^p(x) - \psi^p(\lambda)} D_1(\tau) \\
& \quad + \|\varphi\|_\infty \left(\psi(\kappa) - \psi(x)\right)^{\frac{\xi}{k}} E^{\gamma,\delta,v,c}_{\mu,\pi,l}\left(\omega\left(\psi(x) - \psi(\lambda)\right)^\mu; \tilde{q}\right) \\
& \quad - \epsilon \frac{\|\psi^p(\kappa) - \psi^p(x)\|^q}{\psi^p(\kappa) - \psi^p(x)} D_2(\tau).
\end{aligned}$$

Remark 2.5.
(i) By setting $\psi = I, k = p = 1$ and $q = \epsilon = 0$ in (2.1), then ([5], Corollary 1) is obtained.
(ii) By setting $\psi = I, k = p = 1$, $q = \epsilon = 0$ and $\omega = \tilde{q} = 0$ in (2.1), then ([10], Theorem 1) is obtained.

Theorem 2.6. *Let $\varphi : [\lambda, \kappa] \to \mathbb{R}$, $0 < \lambda < \kappa$, be a real-valued function. If φ is differentiable and $|\varphi'|$ is strongly p-convex function of higher order with modulus $\epsilon > 0$, $p \in \mathbb{R}\backslash\{0\}$, then for $\xi, \pi \geq 1$, then the following inequalities for k-fractional operators (1.8) and (1.9) holds:*

(2.13)

$$\begin{aligned}
& \left|\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\lambda+}\varphi' \circ \psi\right)(x;\tilde{q}) + \left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\pi,l,\omega,\kappa-}\varphi' \circ \psi\right)(x;\tilde{q})\right| \\
& \leq \left(\psi(x) - \psi(\lambda)\right)^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega\left(\psi(x) - \psi(\lambda)\right)^\mu; \tilde{q}\right) \\
& \quad \times \left[\frac{|\varphi'\left(\psi(\lambda)\right)|}{\psi^p(x) - \psi^p(\lambda)}\left(\psi^p(x)\left(\psi(x) - \psi(\lambda)\right) - \frac{\psi^{p+1}(x) - \psi^{p+1}(\lambda)}{p+1}\right)\right. \\
& \quad \left. - \frac{|\varphi'\left(\psi(x)\right)|}{\psi^p(x) - \psi^p(\lambda)}\left(\frac{\psi^{p+1}(x) - \psi^{p+1}(\lambda)}{p+1} - \psi^p(\lambda)\left(\psi(x) - \psi(\lambda)\right)\right)\right. \\
& \quad - \epsilon\|\psi^p(x) - \psi^p(\lambda)\|^q D_1(\tau) \\
& \quad + \left(\psi(\kappa) - \psi(x)\right)^{\frac{\pi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\pi,l}\left(\omega\left(\psi(\kappa) - \psi(x)\right)^\mu; \tilde{q}\right) \\
& \quad \times \left[\frac{|\varphi'\left(\psi(\kappa)\right)|}{\psi^p(\kappa) - \psi^x(\lambda)}\left(\frac{\psi^{p+1}(\kappa) - \psi^{p+1}(x)}{p+1} - \psi^p(x)\left(\psi(\kappa) - \psi(x)\right)\right)\right. \\
& \quad \left. - \frac{|\varphi'\left(\psi(x)\right)|}{\psi^p(\kappa) - \psi^p(x)}\left(\psi^p(\kappa)\left(\psi(\kappa) - \psi(x)\right) - \frac{\psi^{p+1}(\kappa) - \psi^{p+1}(x)}{p+1}\right)\right.
\end{aligned}$$

$$- \epsilon \|\psi^p(\kappa) - \psi^p(x)\|^q D_2(\tau) \Bigg].$$

Here $D_1(\tau)$ and $D_2(\tau)$ are defined in Theorem 2.1.

Proof. As $x \in [\lambda, \kappa]$ and $\tau \in [\lambda, x)$, by usage strongly p-convexity of higher order of $|\varphi'|$, we acquire

$$\begin{aligned} (2.14) \qquad \left|\varphi'\left(\psi(\tau)\right)\right| \leq & \left(\frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \left|\varphi'\left(\psi(\kappa)\right)\right| \\ & + \left(1 - \frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \left|\varphi'\left(\psi(x)\right)\right| \\ & - \epsilon\phi\left(\frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \|\psi^p(x) - \psi^p(\lambda)\|^q. \end{aligned}$$

From (2.14), we get

$$\begin{aligned} (2.15) \qquad \varphi'\left(\psi(\tau)\right) \leq & \left(\frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \left|\varphi'\left(\psi(\kappa)\right)\right| \\ & + \left(1 - \frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \left|\varphi'\left(\psi(x)\right)\right| \\ & - \epsilon\phi\left(\frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \|\psi^p(x) - \psi^p(\lambda)\|^q. \end{aligned}$$

The product of (2.2) and (2.15) gives the following inequality:

$$\begin{aligned} (2.16) \quad & \left(\psi(x) - \psi(\tau)\right)^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega\left(\psi(x) - \psi(\tau)\right)^{\mu}; \tilde{q}\right) \varphi'\left(\psi(\tau)\right) \psi'(\tau) \\ & \leq \left(\psi(x) - \psi(\lambda)\right)^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega\left(\psi(x) - \psi(\lambda)\right)^{\mu}; \tilde{q}\right) \\ & \quad \times \Bigg[\left(\frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \left|\varphi'\left(\psi(\lambda)\right)\right| \psi'(\tau) \\ & \quad + \left(1 - \frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \left|\varphi'\left(\psi(x)\right)\right| \psi'(\tau) \\ & \quad - \epsilon\phi\left(\frac{\psi^p(x) - \psi^p(\tau)}{\psi^p(x) - \psi^p(\lambda)}\right) \|\psi^p(x) - \psi^p(\lambda)\|^q \psi'(\tau) \Bigg]. \end{aligned}$$

After integrating above inequality over $[\lambda, x]$, we obtain

(2.17)

$$\begin{aligned} & \int_{\lambda}^{x} \left(\psi(x) - \psi(\tau)\right)^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega\left(\psi(x) - \psi(\tau)\right)^{\mu}; \tilde{q}\right) \varphi'\left(\psi(\tau)\right) \psi'(\tau) d\tau \\ & \quad \leq \left(\psi(x) - \psi(\lambda)\right)^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega\left(\psi(x) - \psi(\lambda)\right)^{\mu}; \tilde{q}\right) \end{aligned}$$

$$
\begin{aligned}
&\times \left[\left|\varphi'\left(\psi(\lambda)\right)\right| \int_{\lambda}^{x} \left(\frac{\psi^p(x)-\psi^p(\tau)}{\psi^p(x)-\psi^p(\lambda)} \right) \psi'(\tau) d\tau \right. \\
&+ \left|\varphi'\left(\psi(x)\right)\right| \int_{\lambda}^{x} \left(1 - \frac{\psi^p(x)-\psi^p(\tau)}{\psi^p(x)-\psi^p(\lambda)} \right) \psi'(\tau) d\tau \\
&\left. - \epsilon \|\psi^p(x)-\psi^p(\lambda)\|^q \int_{\lambda}^{x} \phi \left(\frac{\psi^p(x)-\psi^p(\tau)}{\psi^p(x)-\psi^p(\lambda)} \right) \psi'(\tau) d\tau \right].
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\psi(x)-\psi(\lambda)\right)^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l} \left(\omega \left(\psi(x)-\psi(\lambda)\right)^{\mu}; \tilde{q}\right) \\
&\times \left[\frac{\left|\varphi'\left(\psi(\lambda)\right)\right|}{\psi^p(x)-\psi^p(\lambda)} \left(\psi^p(x)\left(\psi(x)-\psi(\lambda)\right) - \frac{\psi^{p+1}(x)-\psi^{p+1}(\lambda)}{p+1} \right) \right. \\
&+ \frac{\left|\varphi'\left(\psi(x)\right)\right|}{\psi^p(x)-\psi^p(\lambda)} \left(\frac{\psi^{p+1}(x)-\psi^{p+1}(\lambda)}{p+1} - \psi^p(\lambda)\left(\psi(x)-\psi(\lambda)\right) \right) \\
&\left. - \epsilon \|\psi^p(x)-\psi^p(\lambda)\|^q D_1(\tau) \right].
\end{aligned}
$$

Also from (2.14), we get

$$
\begin{aligned}
\varphi'\left(\psi(\tau)\right) \geq -\Bigg[& \left(\frac{\psi^p(x)-\psi^p(\tau)}{\psi^p(x)-\psi^p(\lambda)} \right) \left|\varphi'\left(\psi(\lambda)\right)\right| \\
& + \left(1 - \frac{\psi^p(x)-\psi^p(\tau)}{\psi^p(x)-\psi^p(\lambda)} \right) \left|\varphi'\left(\psi(x)\right)\right| \\
& - \epsilon \phi \left(\frac{\psi^p(x)-\psi^p(\tau)}{\psi^p(x)-\psi^p(\lambda)} \right) \|\psi^p(x)-\psi^p(\lambda)\|^q \Bigg].
\end{aligned} \tag{2.18}
$$

Following the identical process as we did for (2.15), we get

(2.19)

$$
\begin{aligned}
&- \int_{\lambda}^{x} \left(\psi(x)-\psi(\tau)\right)^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l} \left(\omega \left(\psi(x)-\psi(\tau)\right)^{\mu}; \tilde{q}\right) \varphi'\left(\psi(\tau)\right) \psi'(\tau) d\tau \\
&\leq \left(\psi(x)-\psi(\lambda)\right)^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l} \left(\omega \left(\psi(x)-\psi(\lambda)\right)^{\mu}; \tilde{q}\right) \\
&\times \left[\frac{\left|\varphi'\left(\psi(\lambda)\right)\right|}{\psi^p(x)-\psi^p(\lambda)} \left(\psi^p(x)\left(\psi(x)-\psi(\lambda)\right) - \frac{\psi^{p+1}(x)-\psi^{p+1}(\lambda)}{p+1} \right) \right. \\
&+ \frac{\left|\varphi'\left(\psi(x)\right)\right|}{\psi^p(x)-\psi^p(\lambda)} \left(\frac{\psi^{p+1}(x)-\psi^{p+1}(\lambda)}{p+1} - \psi^p(\lambda)\left(\psi(x)-\psi(\lambda)\right) \right) \\
&\left. - \epsilon \|\psi^p(x)-\psi^p(\lambda)\|^q D_1(\tau) \right].
\end{aligned}
$$

From (2.17) and (2.19), we acquire

(2.20)

$$\begin{aligned}
&\left|\left({}_{\psi}^{k}\chi_{\mu,\xi,l,\omega,\lambda+}^{\gamma,\delta,v,c}\varphi'\circ\psi\right)(x;\tilde{q})\right| \\
&\quad\leq(\psi(x)-\psi(\lambda))^{\frac{\xi}{k}-1}E_{\mu,\xi,l}^{\gamma,\delta,v,c}\left(\omega\left(\psi(x)-\psi(\lambda)\right)^{\mu};\tilde{q}\right) \\
&\qquad\times\left[\frac{|\varphi'(\psi(\lambda))|}{\psi^{p}(x)-\psi^{p}(\lambda)}\left(\psi^{p}(x)\left(\psi(x)-\psi(\lambda)\right)-\frac{\psi^{p+1}(x)-\psi^{p+1}(\lambda)}{p+1}\right)\right. \\
&\qquad+\frac{|\varphi'(\psi(x))|}{\psi^{p}(x)-\psi^{p}(\lambda)}\left(\frac{\psi^{p+1}(x)-\psi^{p+1}(\lambda)}{p+1}-\psi^{p}(\lambda)\left(\psi(x)-\psi(\lambda)\right)\right) \\
&\qquad\left.-\epsilon\|\psi^{p}(x)-\psi^{p}(\lambda)\|^{q}D_{1}(\tau)\right].
\end{aligned}$$

Now, let $x\in[\lambda,\kappa]$ and $t\in(x,\kappa]$. By usage strongly p-convexity of higher order of $|\varphi'|$, we acquire

$$\begin{aligned}
\left|\varphi'(\psi(\tau))\right|&\leq\left(\frac{\psi^{p}(\tau)-\psi^{p}(x)}{\psi^{p}(\kappa)-\psi^{p}(x)}\right)\left|\varphi'(\psi(\kappa))\right| \\
&\quad+\left(1-\frac{\psi^{p}(\tau)-\psi^{p}(x)}{\psi^{p}(\kappa)-\psi^{p}(x)}\right)\left|\varphi'(\psi(x))\right| \\
&\quad-\epsilon\phi\left(\frac{\psi^{p}(\tau)-\psi^{p}(x)}{\psi^{p}(\kappa)-\psi^{p}(x)}\right)\|\psi^{p}(\kappa)-\psi^{p}(x)\|^{q}.
\end{aligned}\tag{2.21}$$

On the identical process as we acquire for (2.2), (2.15) and (2.18), we get, from (2.6), the following inequality holds:

(2.22)

$$\begin{aligned}
&\left|\left({}_{\psi}^{k}\chi_{\mu,\pi,l,\omega,\kappa-}^{\gamma,\delta,v,c}\varphi'\circ\psi\right)(x;\tilde{q})\right| \\
&\quad\leq(\psi(\kappa)-\psi(x))^{\frac{\pi}{k}-1}E_{\mu,\pi,l}^{\gamma,\delta,v,c}\left(\omega\left(\psi(\kappa)-\psi(x)\right)^{\mu};\tilde{q}\right) \\
&\qquad\times\left[\frac{|\varphi'(\psi(\kappa))|}{\psi^{p}(\kappa)-\psi^{p}(x)}\left(\frac{\psi^{p+1}(\kappa)-\psi^{p+1}(x)}{p+1}-\psi^{p}(x)\left(\psi(\kappa)-\psi(x)\right)\right)\right. \\
&\qquad+\frac{|\varphi'(\psi(x))|}{\psi^{p}(\kappa)-\psi^{p}(x)}\left(\psi^{p}(\kappa)\left(\psi(\kappa)-\psi(x)\right)-\frac{\psi^{p+1}(\kappa)-\psi^{p+1}(x)}{p+1}\right) \\
&\qquad\left.-\epsilon\|\psi^{p}(\kappa)-\psi^{p}(x)\|^{q}D_{2}(\tau)\right].
\end{aligned}$$

From inequalities (2.20) and (2.22), (2.13) is obtained. □

Corollary 2.7. *By setting $\xi = \pi$ in (2.13), the following inequality is acquired.*

(2.23)

$$\begin{aligned}
&\left|\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\lambda+}\varphi'\circ\psi\right)(x;\tilde{q}) + \left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi,l,\omega,\kappa-}\varphi'\circ\psi\right)(x;\tilde{q})\right| \\
&\quad\leq (\psi(x)-\psi(\lambda))^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega\left(\psi(x)-\psi(\lambda)\right)^{\mu};\tilde{q}\right) \\
&\qquad\times\left[\frac{|\varphi'(\psi(\lambda))|}{\psi^p(x)-\psi^p(\lambda)}\left(\psi^p(x)(\psi(x)-\psi(\lambda)) - \frac{\psi^{p+1}(x)-\psi^{p+1}(\lambda)}{p+1}\right)\right. \\
&\qquad-\frac{|\varphi'(\psi(x))|}{\psi^p(x)-\psi^p(\lambda)}\left(\frac{\psi^{p+1}(x)-\psi^{p+1}(\lambda)}{p+1} - \psi^p(\lambda)(\psi(x)-\psi(\lambda))\right) \\
&\qquad-\epsilon\|\psi^p(x)-\psi^p(\lambda)\|^q D_1(\tau) \\
&\qquad+(\psi(\kappa)-\psi(x))^{\frac{\xi}{k}-1} E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega\left(\psi(\kappa)-\psi(x)\right)^{\mu};\tilde{q}\right) \\
&\qquad\times\left[\frac{|\varphi'(\psi(\kappa))|}{\psi^p(\kappa)-\psi^x(\lambda)}\left(\frac{\psi^{p+1}(\kappa)-\psi^{p+1}(x)}{p+1} - \psi^p(x)(\psi(\kappa)-\psi(x))\right)\right. \\
&\qquad-\frac{|\varphi'(\psi(x))|}{\psi^p(\kappa)-\psi^p(x)}\left(\psi^p(\kappa)(\psi(\kappa)-\psi(x)) - \frac{\psi^{p+1}(\kappa)-\psi^{p+1}(x)}{p+1}\right) \\
&\qquad\left.\left.-\epsilon\|\psi^p(\kappa)-\psi^p(x)\|^q D_2(\tau)\right].\right.
\end{aligned}$$

Remark 2.8. By setting $\psi = I, k = p = 1$ and $q = \epsilon = 0$ in (2.13), then ([5], Corollary 2) is obtained.

Lemma 2.9. *Let $\varphi : [\lambda,\kappa] \to \mathbb{R}$, $0 < \lambda < \kappa$ is strongly p-convex function of higher order with modulus $\epsilon > 0$, $p \in \mathbb{R}\setminus\{0\}$. If*

$$\varphi\left(\psi^p(\lambda)+\psi^p(\kappa)-\psi^p(x)\right) = \varphi\left(\psi(x)\right),$$

then the following inequality holds:

$$\begin{aligned}
&\varphi\left(\left[\frac{\psi^p(\lambda)+\psi^p(\kappa)}{2}\right]^{\frac{1}{p}}\right) \\
&\qquad\leq \varphi\left(\psi(x)\right) - \frac{\epsilon}{2^q}\|\psi^p(\lambda)+\psi^p(\kappa)-\psi^p(x)-\psi^p(x)\|^q.
\end{aligned} \tag{2.24}$$

Proof. Since φ is strongly p-convex function of higher order, we get

(2.25)

$$\varphi\left(\left[\frac{\psi^p(\lambda)+\psi^p(\kappa)}{2}\right]^{\frac{1}{p}}\right)$$

$$\leq \frac{\varphi\left(\left[\tau\psi^p(\lambda)+(1-\tau)\psi^p(\kappa)\right]^{\frac{1}{p}}\right)+\varphi\left(\left[\tau\psi^p(\kappa)+(1-\tau)\psi^p(\lambda)\right]^{\frac{1}{p}}\right)}{2}$$
$$-\frac{\epsilon}{2^q}\|\tau\psi^p(\kappa)+(1-\tau)\psi^p(\lambda)-(\tau\psi^p(\lambda)+(1-\tau)\psi^p(\kappa))\|^q.$$

Let $\psi(x)=[\tau\psi^p(\lambda)+(1-\tau)\psi^p(\kappa)]^{\frac{1}{p}}$ and $\psi^p(\lambda)+\psi^p(\kappa)-\psi^p(x)=\tau\psi^p(\kappa)+(1-\tau)\psi^p(\lambda)$,

(2.26)
$$\varphi\left(\left[\frac{\psi^p(\lambda)+\psi^p(\kappa)}{2}\right]^{\frac{1}{p}}\right)\leq\frac{\varphi\left(\psi(x)\right)+\varphi\left(\psi^p(\lambda)+\psi^p(\kappa)-\psi^p(x)\right)}{2}$$
$$-\frac{\epsilon}{2^q}\|\psi^p(\lambda)+\psi^p(\kappa)-\psi^p(x)-\psi^p(x)\|^q.$$

By usage $\varphi\left(\psi^p(\lambda)+\psi^p(\kappa)-\psi^p(x)\right)=\varphi\left(\psi(x)\right)$, inequality (2.24) is can be acquired □

Theorem 2.10. *Let $\varphi:[\lambda,\kappa]\to\mathbb{R}$, $0<\lambda<\kappa$, be a real-valued function. If φ is differentiable and $|\varphi'|$ is strongly p-convex function of higher order modulus $\epsilon>0$, $p\in\mathbb{R}\setminus\{0\}$ and $\varphi\left(\left[\psi^p(\lambda)+\psi^p(\kappa)-\psi(x)\right]^{\frac{1}{p}}\right)=\varphi\left(\psi^{\frac{1}{p}}(x)\right)$ then for $\xi,\pi>0$, then the following inequalities for k-fractional integral operators (1.8) and (1.9) holds:*

(2.27)
$$\varphi\left(\left[\frac{\psi^p(\lambda)+\psi^p(\kappa)}{2}\right]^{\frac{1}{p}}\right)$$
$$\times\left[\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\pi+k,l,\omega,\kappa-}1\right)(\lambda;\tilde{q})+\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi+k,l,\omega,\lambda+}1\right)(\kappa;\tilde{q})\right]$$
$$\leq\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\pi+k,l,\omega,\kappa-}\varphi\circ\psi\right)(\lambda;\tilde{q})+\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi+k,l,\omega,\lambda+}\varphi\circ\psi\right)(\kappa;\tilde{q})$$
$$\leq\left(\psi(\kappa)-\psi(\lambda)\right)^{\frac{\pi}{k}}E^{\gamma,\delta,v,c}_{\mu,\pi,l}\left(\omega\left(\psi(\kappa)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)$$
$$\times\left[\frac{\varphi\left(\psi(\kappa)\right)}{\psi^p(\kappa)-\psi^p(\lambda)}\left\{\frac{\psi^{p+1}(\kappa)-\psi^{p+1}(\lambda)}{p+1}-\psi^p(\lambda)\left(\psi(\kappa)-\psi(\lambda)\right)\right\}\right.$$
$$+\frac{\varphi\left(\psi(\lambda)\right)}{\psi^p(\kappa)-\psi^p(\lambda)}\left\{\psi^p(\kappa)\left(\psi(\kappa)-\psi(\lambda)\right)-\frac{\psi^{p+1}(\kappa)-\psi^{p+1}(\lambda)}{p+1}\right\}$$
$$\left.-\epsilon\|\psi^p(\kappa)-\psi^p(\lambda)\|^qD_3(\tau)\right]$$
$$+\left(\psi(\kappa)-\psi(\lambda)\right)^{\frac{\xi}{k}}E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega\left(\psi(\kappa)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)$$

$$\times \left[\frac{\varphi\left(\psi(\kappa)\right)}{\psi^p(\kappa)-\psi^p(\lambda)} \left\{ \frac{\psi^{p+1}(\kappa)-\psi^{p+1}(\lambda)}{p+1} - \psi^p(\lambda)\left(\psi(\kappa)-\psi(\lambda)\right) \right\} \right.$$
$$+ \frac{\varphi\left(\psi(\lambda)\right)}{\psi^p(\kappa)-\psi^p(\lambda)} \left\{ \psi^p(\kappa)\left(\psi(\kappa)-\psi(\lambda)\right) - \frac{\psi^{p+1}(\kappa)-\psi^{p+1}(\lambda)}{p+1} \right\}$$
$$\left. - \epsilon\|\psi^p(\kappa)-\psi^p(\lambda)\|^q D_3(\tau) \right]$$
$$+ \frac{\epsilon}{2^q} \left[\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\pi+k,l,\omega,\kappa-} \|\psi^p(\lambda)+\psi^p(\kappa)-\psi^p(x)-\psi^p(x)\|^q \right)(\lambda;\tilde{q}) \right.$$
$$\left. + \left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi+k,l,\omega,\lambda+} \|\psi^p(\lambda)+\psi^p(\kappa)-\psi^p(x)-\psi^p(x)\|^q \right)(\kappa;\tilde{q}) \right],$$

where

$$D_3(\tau) = \int_\lambda^\kappa \phi\left(\frac{\psi^p(x)-\psi^p(\lambda)}{\psi^p(\kappa)-\psi^p(\lambda)} \right) \psi'(x)dx.$$

Proof. For $x \in [\lambda,\kappa]$, we acquire

$$\left(\psi(x)-\psi(\lambda)\right)^{\frac{\pi}{k}} E^{\gamma,\delta,v,c}_{\mu,\pi,l}\left(\omega\left(\psi(x)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)\psi'(x)$$
$$\leq \left(\psi(\kappa)-\psi(\lambda)\right)^{\frac{\pi}{k}} E^{\gamma,\delta,v,c}_{\mu,\pi,l}\left(\omega\left(\psi(\kappa)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)\psi'(x). \tag{2.28}$$

Since φ is strongly p-convex function of higher order, $x \in [\lambda,\kappa]$, we get

$$\varphi\left(\psi(x)\right) \leq \left(\frac{\psi^p(x)-\psi^p(\lambda)}{\psi^p(\kappa)-\psi^p(\lambda)} \right)\varphi\left(\psi(\kappa)\right)$$
$$+ \left(1 - \frac{\psi^p(x)-\psi^p(\lambda)}{\psi^p(\kappa)-\psi^p(\lambda)} \right)\varphi\left(\psi(\lambda)\right)$$
$$- \epsilon\phi\left(\frac{\psi^p(x)-\psi^p(\lambda)}{\psi^p(\kappa)-\psi^p(\lambda)} \right)\|\psi^p(\kappa)-\psi^p(\lambda)\|^q. \tag{2.29}$$

Multiplying (2.28) and (2.29), then integrating over $[\lambda,\kappa]$, we obtain

(2.30)

$$\int_\lambda^\kappa \left(\psi(x)-\psi(\lambda)\right)^{\frac{\pi}{k}} E^{\gamma,\delta,v,c}_{\mu,\pi,l}\left(\omega\left(\psi(x)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)\varphi\left(\psi(x)\right)\psi'(x)dx$$
$$\leq \left(\psi(\kappa)-\psi(\lambda)\right)^{\frac{\pi}{k}} E^{\gamma,\delta,v,c}_{\mu,\pi,l}\left(\omega\left(\psi(\kappa)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)$$
$$\times \left[\varphi\left(\psi(\kappa)\right) \int_\lambda^\kappa \frac{\psi^p(x)-\psi^p(\lambda)}{\psi^p(\kappa)-\psi^p(\lambda)}\psi'(x)dx \right.$$
$$\left. + \varphi\left(\psi(\lambda)\right) \int_\lambda^\kappa \left(1 - \frac{\psi^p(x)-\psi^p(\lambda)}{\psi^p(\kappa)-\psi^p(\lambda)} \right)\psi'(x)dx \right.$$

$$- \epsilon \|\psi^p(\kappa) - \psi^p(\lambda)\|^q \int_\lambda^\kappa \phi\left(\frac{\psi^p(x) - \psi^p(\lambda)}{\psi^p(\kappa) - \psi^p(\lambda)}\right) \psi'(x) dx \Bigg].$$

From which we have

(2.31)

$$\begin{aligned}
&\left({}_{\psi}^{k}\chi_{\mu,\pi+k,l,\omega,\kappa-}^{\gamma,\delta,v,c} \varphi \circ \psi\right)(\lambda; \tilde{q}) \\
&\quad \leq (\psi(\kappa) - \psi(\lambda))^{\frac{\pi}{k}} E_{\mu,\pi,l}^{\gamma,\delta,v,c} \left(\omega \left(\psi(\kappa) - \psi(\lambda)\right)^{\mu}; \tilde{q}\right) \\
&\qquad \times \Bigg[\frac{\varphi\left(\psi(\kappa)\right)}{\psi^p(\kappa) - \psi^p(\lambda)} \left\{ \frac{\psi^{p+1}(\kappa) - \psi^{p+1}(\lambda)}{p+1} - \psi^p(\lambda) \left(\psi(\kappa) - \psi(\lambda)\right) \right\} \\
&\qquad + \frac{\varphi\left(\psi(\lambda)\right)}{\psi^p(\kappa) - \psi^p(\lambda)} \left\{ \psi^p(\kappa) \left(\psi(\kappa) - \psi(\lambda)\right) - \frac{\psi^{p+1}(\kappa) - \psi^{p+1}(\lambda)}{p+1} \right\} \\
&\qquad - \epsilon \|\psi^p(\kappa) - \psi^p(\lambda)\|^q D_3(\tau) \Bigg].
\end{aligned}$$

On the contrary, for $x \in [\lambda, \kappa]$, we obtain

$$\begin{aligned}
&(\psi(\kappa) - \psi(x))^{\frac{\xi}{k}} E_{\mu,\xi,l}^{\gamma,\delta,v,c} \left(\omega \left(\psi(\kappa) - \psi(x)\right)^{\mu}; \tilde{q}\right) \psi'(x) \\
&\qquad \leq (\psi(\kappa) - \psi(\lambda))^{\frac{\xi}{k}} E_{\mu,\xi,l}^{\gamma,\delta,v,c} \left(\omega \left(\psi(\kappa) - \psi(\lambda)\right)^{\mu}; \tilde{q}\right) \psi'(x).
\end{aligned} \tag{2.32}$$

By multiplying (2.29) and (2.32), and then integrating over $[\lambda, \kappa]$, we get

(2.33)

$$\begin{aligned}
&\int_\lambda^\kappa (\psi(\kappa) - \psi(x))^{\frac{\xi}{k}} E_{\mu,\xi,l}^{\gamma,\delta,v,c} \left(\omega \left(\psi(\kappa) - \psi(x)\right)^{\mu}; \tilde{q}\right) \varphi\left(\psi(x)\right) \psi'(x) dx \\
&\quad \leq (\psi(\kappa) - \psi(\lambda))^{\frac{\xi}{k}} E_{\mu,\xi,l}^{\gamma,\delta,v,c} \left(\omega \left(\psi(\kappa) - \psi(\lambda)\right)^{\mu}; \tilde{q}\right) \\
&\qquad \times \Bigg[\varphi\left(\psi(\kappa)\right) \int_\lambda^\kappa \frac{\psi^p(x) - \psi^p(\lambda)}{\psi^p(\kappa) - \psi^p(\lambda)} \psi'(x) dx \\
&\qquad + \varphi\left(\psi(\lambda)\right) \int_\lambda^\kappa \left(1 - \frac{\psi^p(x) - \psi^p(\lambda)}{\psi^p(\kappa) - \psi^p(\lambda)}\right) \psi'(x) dx \\
&\qquad - \epsilon \|\psi^p(\kappa) - \psi^p(\lambda)\|^q \int_\lambda^\kappa \phi\left(\frac{\psi^p(x) - \psi^p(\lambda)}{\psi^p(\kappa) - \psi^p(\lambda)}\right) \psi'(x) dx \Bigg].
\end{aligned}$$

From which, we get

(2.34)

$$\left({}_{\psi}^{k}\chi_{\mu,\xi+k,l,\omega,\lambda+}^{\gamma,\delta,v,c} \varphi \circ \psi\right)(\kappa; \tilde{q})$$

$$\leq (\psi(\kappa)-\psi(\lambda))^{\frac{\xi}{k}} E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega\left(\psi(\kappa)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)$$
$$\times\left[\frac{\varphi\left(\psi(\kappa)\right)}{\psi^{p}(\kappa)-\psi^{p}(\lambda)}\left\{\frac{\psi^{p+1}(\kappa)-\psi^{p+1}(\lambda)}{p+1}-\psi^{p}(\lambda)\left(\psi(\kappa)-\psi(\lambda)\right)\right\}\right.$$
$$+\frac{\varphi\left(\psi(\lambda)\right)}{\psi^{p}(\kappa)-\psi^{p}(\lambda)}\left\{\psi^{p}(\kappa)\left(\psi(\kappa)-\psi(\lambda)\right)-\frac{\psi^{p+1}(\kappa)-\psi^{p+1}(\lambda)}{p+1}\right\}$$
$$\left.-\epsilon\|\psi^{p}(\kappa)-\psi^{p}(\lambda)\|^{q}D_{3}(\tau)\right].$$

From (2.31) and (2.34), we obtain the second inequality of (2.27). Multiplying (2.24) by $(\psi(x)-\psi(\lambda))^{\frac{\pi}{k}} E^{\gamma,\delta,v,c}_{\mu,\pi,l}\left(\omega\left(\psi(x)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)\psi'(x)$ and integrating over $[\lambda,\kappa]$, we acquire

(2.35)

$$\varphi\left(\left[\frac{\psi^{p}(\lambda)+\psi^{p}(\kappa)}{2}\right]^{\frac{1}{p}}\right)$$
$$\times\int_{\lambda}^{\kappa}(\psi(x)-\psi(\lambda))^{\frac{\pi}{k}} E^{\gamma,\delta,v,c}_{\mu,\pi,l}\left(\omega\left(\psi(x)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)\psi'(x)dx$$
$$\leq\int_{\lambda}^{\kappa}(\psi(x)-\psi(\lambda))^{\frac{\pi}{k}} E^{\gamma,\delta,v,c}_{\mu,\pi,l}\left(\omega\left(\psi(x)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)\varphi\left(\psi(x)\right)\psi'(x)dx$$
$$-\frac{\epsilon}{2^{q}}\int_{\lambda}^{\kappa}(\psi(x)-\psi(\lambda))^{\frac{\pi}{k}} E^{\gamma,\delta,v,c}_{\mu,\pi,l}\left(\omega\left(\psi(x)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)$$
$$\times\|\psi^{p}(\lambda)+\psi^{p}(\kappa)-\psi^{p}(x)-\psi^{p}(x)\|^{q}.$$

By using (1.9), we get

(2.36)

$$\varphi\left(\left[\frac{\psi^{p}(\lambda)+\psi^{p}(\kappa)}{2}\right]^{\frac{1}{p}}\right)\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\pi+k,l,\omega,\kappa-}1\right)(\lambda;\tilde{q})$$
$$\leq\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\pi+k,l,\omega,\kappa-}\varphi\circ\psi\right)(\lambda;\tilde{q})$$
$$-\frac{\epsilon}{2^{q}}\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\pi+k,l,\omega,\kappa-}\|\psi^{p}(\lambda)+\psi^{p}(\kappa)-\psi^{p}(x)-\psi^{p}(x)\|^{q}\right)(\lambda;\tilde{q}).$$

By multiplying (2.24) with

$$(\psi(\kappa)-\psi(x))^{\frac{\xi}{k}} E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega\left(\psi(\kappa)-\psi(x)\right)^{\mu};\tilde{q}\right)\psi'(x)$$

and integrating over $[\lambda,\kappa]$, also usage (1.8), we acquire

(2.37)

$$\varphi\left(\left[\frac{\psi^{p}(\lambda)+\psi^{p}(\kappa)}{2}\right]^{\frac{1}{p}}\right)\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi+k,l,\omega,\lambda+}1\right)(\kappa;\tilde{q})$$

$$\leq \left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi+k,l,\omega,\lambda+}\varphi\circ\psi\right)(\kappa;\tilde{q}) - \frac{\epsilon}{2^q}\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi+k,l,\omega,\lambda+}\|\psi^p(\lambda)+\psi^p(\kappa)-\psi^p(x)-\psi^p(x)\|^q\right)(\kappa;\tilde{q}).$$

By adding (2.36) and (2.37), the first inequality of (2.27) can be acquired. □

Corollary 2.11. *By setting $\xi=\pi$ in* (2.27)*, the following inequality is acquired.*

(2.38)

$$\begin{aligned}
&\varphi\left(\left[\frac{\psi^p(\lambda)+\psi^p(\kappa)}{2}\right]^{\frac{1}{p}}\right)\\
&\quad\times\left[\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi+k,l,\omega,\kappa-}1\right)(\lambda;\tilde{q})+\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi+k,l,\omega,\lambda+}1\right)(\kappa;\tilde{q})\right]\\
&\leq\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi+k,l,\omega,\kappa-}\varphi\circ\psi\right)(\lambda;\tilde{q})+\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi+k,l,\omega,\lambda+}\varphi\circ\psi\right)(\kappa;\tilde{q})\\
&\leq(\psi(\kappa)-\psi(\lambda))^{\frac{\xi}{k}}\,E^{\gamma,\delta,v,c}_{\mu,\xi,l}\left(\omega\left(\psi(\kappa)-\psi(\lambda)\right)^{\mu};\tilde{q}\right)\\
&\quad\times\left(\left[\frac{\varphi\left(\psi(\kappa)\right)}{\psi^p(\kappa)-\psi^p(\lambda)}\left\{\frac{\psi^{p+1}(\kappa)-\psi^{p+1}(\lambda)}{p+1}-\psi^p(\lambda)\left(\psi(\kappa)-\psi(\lambda)\right)\right\}\right.\right.\\
&\quad+\frac{\varphi\left(\psi(\lambda)\right)}{\psi^p(\kappa)-\psi^p(\lambda)}\left\{\psi^p(\kappa)\left(\psi(\kappa)-\psi(\lambda)\right)-\frac{\psi^{p+1}(\kappa)-\psi^{p+1}(\lambda)}{p+1}\right\}\\
&\quad\left.-\epsilon\|\psi^p(\kappa)-\psi^p(\lambda)\|^q D_3(\tau)\right]\\
&\quad+\left[\frac{\varphi\left(\psi(\kappa)\right)}{\psi^p(\kappa)-\psi^p(\lambda)}\left\{\frac{\psi^{p+1}(\kappa)-\psi^{p+1}(\lambda)}{p+1}-\psi^p(\lambda)\left(\psi(\kappa)-\psi(\lambda)\right)\right\}\right.\\
&\quad+\frac{\varphi\left(\psi(\lambda)\right)}{\psi^p(\kappa)-\psi^p(\lambda)}\left\{\psi^p(\kappa)\left(\psi(\kappa)-\psi(\lambda)\right)-\frac{\psi^{p+1}(\kappa)-\psi^{p+1}(\lambda)}{p+1}\right\}\\
&\quad\left.\left.-\epsilon\|\psi^p(\kappa)-\psi^p(\lambda)\|^q D_3(\tau)\right]\right)\\
&\quad+\frac{\epsilon}{2^q}\left[\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi+k,l,\omega,\kappa-}\|\psi^p(\lambda)+\psi^p(\kappa)-\psi^p(x)-\psi^p(x)\|^q\right)(\lambda;\tilde{q})\right.\\
&\quad\left.+\left({}^{k}_{\psi}\chi^{\gamma,\delta,v,c}_{\mu,\xi+k,l,\omega,\lambda+}\|\psi^p(\lambda)+\psi^p(\kappa)-\psi^p(x)-\psi^p(x)\|^q\right)(\kappa;\tilde{q})\right].
\end{aligned}$$

3. Conclusions

In this paper, we examined the bounds of fractional integral operators containing the expanded Mittag-Leffler function in their kernel. For this, we used higher-order strongly p-convex functions. The defined bounds are compact formulas that produce fractional integral inequalities for various well-known integral operators. This topic studied before for different classes of convex functions, but strongly convex functions have not been addressed. This study is important since strongly convex functions are a significant part of the theory of convex functions. The obtained results can be extended for different classes of convex and strongly convex functions in the literature. Additionally, various conclusions can achieved by giving specific values to the parameters of the Mittag-Leffler function.

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