Existence Analysis of Multi-Point Boundary Value Problems with Riesz-Caputo Fractional Derivatives

Takieddine Zeghida, Rabah Khaldi and Assia Guezane-Lakoud

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 3
Pages: 301-322

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2024.2006083.1394



Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 3 (2024), 301-322
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2024.2006083.1394

Existence Analysis of Multi-Point Boundary Value Problems with Riesz-Caputo Fractional Derivatives

Takieddine Zeghida[1*], Rabah Khaldi [2] and Assia Guezane-Lakoud[3]

Abstract. This paper explores the study of a specific category of nonlinear multi-point boundary value problems (BVPs) associated with Riesz-Caputo fractional differential equations and integral boundary conditions. The primary objective is to establish the existence of solutions under specific assumptions. We use Krasnoselskii's fixed point theorem and Leray-Schauder's nonlinear alternative to achieve this goal. Furthermore, numerical examples are presented and plotted to demonstrate the effectiveness of the obtained results.

1. Introduction

This paper aims to investigate the existence of solutions for a specific class of boundary value problems (BVPs) involving Riesz-Caputo derivative and multi-point integral boundary conditions. Our objective is to establish the existence results using Krasnoselskii's fixed point theorem and Leray-Schauder alternative to the following problem:

$$\begin{aligned} &{}_0^{RC}D_1^{\alpha}w(s) = A(s, w(s)), \quad 0 < s < 1, \\ &w(0) = 0, \\ &w(1) = \sum_{j=1}^{k} \lambda_j \int_0^{c_j} w(s)ds. \end{aligned} \tag{1.1}$$

Where $1 < \alpha \leq 2$, $\lambda_j > 0$, $0 < c_j < 1$. ${}_0^{RC}D_1^{\alpha}$ is the Riesz-Caputo derivatives, ${}_0^{C}D_1^{\alpha}$ is the left Caputo derivative of order α and the function

2020 *Mathematics Subject Classification.* 26A33, 34A08, 34B10.
Key words and phrases. Existence of solutions, Fixed point theorem, Riesz-Caputo
Received: 16 July 2023, Accepted: 30 December 2023.
* Corresponding author.

$A : [0,1]\times\mathbb{R} \to \mathbb{R}$ will be specified later and we assume that the constants λ_j and c_j satisfy the following condition

$$\sigma = \sum_{j=1}^{k} \lambda_j c_j^2 \neq 2.$$

Fractional calculus has emerged as a powerful tool for modeling complex systems characterized by nonlocality and memory effects. Fractional derivatives can capture long-range correlations and nonlocal interactions, which are prevalent in such systems. There has been a significant focus on examining this specific class of differential equations, as evidenced by the extensive references cited [3–10].

In the paper [10] the authors studied the following problem:

$$\begin{cases} {}^{C}D_{1^-}^{\theta}\phi_p\left({}^{C}D_{0^+}^{\alpha}z(t)\right) = f(t,z(t)), \\ z'(0) = 0, \\ z(1) - \zeta z(0) = \int_0^1 g(s,z(s))ds, \\ \phi_p\big({}^{C}D_{0^+}^{\alpha}z(1)\big)' = 0, \\ \phi_p\left({}^{C}D_{0^+}^{\alpha}z(1)\right) - \eta\phi_p\left({}^{C}D_{0^+}^{\alpha}z(0)\right) = 0, \\ \phi_p(s) = |s|^{p-2}s, \end{cases}$$

such that $0 < t < 1$, $1 < \theta$, $\gamma < 2$, $0 < \zeta < 1$, $\eta > 1$, $p > 1$ and $f, g \in C([0,1]\times\mathbb{R},\mathbb{R}^+)$.

Using the Guo-Krasnoselskii theorem, they established the existence of at least one positive solution.

In [8], the authors investigated a nonlinear fractional Euler-Lagrange equation incorporating fractional derivatives of the left Riemann Liouville and right Caputo types. They employed the lower and upper solutions method and the Laplace transform to analyze the equation.

In a related context, the nonlinear Euler-Bernoulli Beam type equations were explored, involving both left and right Caputo fractional derivatives in the work presented in [8]:

$${}^{C}D_{1^-}^{\alpha}\,{}^{C}D_{0^+}^{\beta}u(t) = f(t,u(t)), \quad 0 \le t \le 1,$$

with the boundary conditions:

$$u(0) = u'(0) = u(1) = D_{0^+}^{\beta+1}u(1) = 0$$

where $1 < \alpha,\ \beta < 2$, ${}^{C}D^{\alpha}_{1^-}$ and ${}^{C}D^{\beta}_{0^+}$ denote respectively the right and the left sides Caputo derivatives and $D^{\beta+1}_{0^+}$denotes the left Riemann–Liouville fractional derivative.

In the work [14], the following class of integral BVPs of fractional delayed differential equations was studied and proved to have at least two positive solutions:

$$\begin{cases} {}^{c}D^{\beta}z(t) + g(t, z_t) = 0, & t \in [0,1], \\ z(t) = \phi(t), & t \in [-\tau, 0], \\ z(0) = z(0) = z'''(0) = 0, \\ z(1) = k\int_0^1 z(t)dt, \end{cases}$$

here $3 < \beta \leq 4,\ 0 < k < 2,\ g : [0,1] \times C_\tau \longrightarrow [0,\infty)$ is a continuous function and $z_t = z(t+s)$ for $t \in [0,1]$ and $s \in [-\tau, 0]$.

Among the various types of fractional derivatives, the Riesz-Caputo fractional derivative has emerged as a prominent choice in numerous scientific fields, including viscoelastic materials, heat conduction, diffusion processes in complex systems, nonlocal constitutive laws, microscopic interaction forces, nonlocal stress-strain constitutive relations, nonlocal beams, control theory, and mechanics (see [4] for more details).

Compared to the Caputo derivative, the Riesz-Caputo fractional derivative offers several advantages. It can handle more general boundary conditions. It is more stable than Riemann-Liouville and it is useful for modeling systems with memory due to the appearance of both the left and right side fractional integral, making it a versatile tool for modeling diverse physical systems. Its successful application spans various domains, including anomalous diffusion, heat conduction, and wave propagation, where it has proven to be a practical mathematical framework.

The distinction between these two types of fractional derivatives lies in their treatment of the initial condition. While the Caputo derivative assumes the initial condition is expressed as a fractional-order derivative, the Riesz-Caputo derivative assumes the initial condition is a fractional-order integral.

In [5], the authors have investigated a new class of of fractional differential equations with the Riesz-Caputo derivative and have used fractional Gronwall inequalities and some fixed point theorems to prove the existence of solutions to the following BVP:

$$\begin{aligned} &{}_{0}^{RC}D_{T}^{\gamma}y(\tau) = g(\tau, y(\tau)), \quad \tau \in [0,1],\ 0 < \gamma < 1 \\ &y(0) = y_0 \\ &y(T) = y_T. \end{aligned} \tag{1.2}$$

In [6], the authors addressed the positivity of a problem similar to (2) (with $T = 1$), and they established the existence of positive solutions using Krasnoelskii's fixed-point theorem within a cone.

In this study, we aim to solve a specific problem incorporating integral boundary conditions and establish the existence of solutions using Krasnoselskii's fixed point theorem and the Leray-Schauder alternative. By analyzing the properties of the operator involved in our problem and satisfying the necessary conditions of the theorem, we demonstrate at least one fixed point that satisfies the given problem.

To illustrate the results obtained further, we include numerical examples and the trajectory of the solution of problem (1.1) using Python. These practical illustrations allow us to visualize the system's behavior, providing a more comprehensive understanding of its dynamics.

2. Preliminaries

In this section, we provide an overview of the essential definitions, lemmas, and preliminary facts that form the basis of this paper. These fundamental concepts will be crucial for our analysis and subsequent discussions.

Definition 2.1 (see [19])**.** The fractional left right Riemann-Liouville fractional integral of order $n-1 < \alpha \leq n$ are defined respectively as:

$$\left({}_0I_\tau^\alpha \omega\right)(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - \varsigma)^{\alpha-1} \omega(\varsigma) d\varsigma,$$

$$\left({}_\tau I_T^\alpha \omega\right)(\tau) = \frac{1}{\Gamma(\alpha)} \int_\tau^T (\varsigma - \tau)^{\alpha-1} \omega(\varsigma) d\varsigma,$$

where $n \in \mathbb{N}, 0 \leq \tau \leq T$, Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} \tau^{\alpha-1} e^{-\tau} d\tau$

Definition 2.2 (see [19])**.** The classical Riesz-Caputo derivative of order $\alpha > 0$ is given by:

$$\begin{aligned} {}_0^{RC}D_T^\alpha \omega(\tau) &= \frac{1}{\Gamma(n-\alpha)} \int_0^T \frac{\omega^{(n)}(\varsigma)}{|\tau - \varsigma|^{\alpha+1-n}} d\varsigma \\ &= \frac{1}{2} \left({}_0^C D_\tau^\alpha + (-1)^n \, {}_\tau^C D_T^\alpha \right) \omega(\tau), \quad n \in \mathbb{N},\ 0 \leq \tau \leq T, \end{aligned}$$

where ${}_0^C D_\tau^\alpha$ is the left hand side Caputo derivative, ${}_\tau^C D_T^\alpha$ is the right hand side Caputo derivative, which are respectively given by:

$${}_0^C D_\tau^\alpha \omega(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_0^\tau \frac{\omega^{(n)}(\varsigma)}{(\tau - \varsigma)^{\alpha+1-n}} d\varsigma, \quad n \in \mathbb{N},\ 0 \leq \tau \leq T,$$

$${}_\tau^C D_T^\alpha \omega(\tau) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_\tau^T \frac{\omega^{(n)}(\varsigma)}{(\varsigma - \tau)^{\alpha+1-n}} d\varsigma, \quad n \in \mathbb{N},\ 0 \leq \tau \leq T.$$

In addition, if $1 < \alpha \leq 2$ and $\omega(\tau) \in C[0,T]$, then

$$ {}_0^{RC}D_T^{\alpha}\omega(\tau) = \frac{1}{2}\left({}^{C}D_{\tau}^{\alpha} + {}_{\tau}^{C}D_T^{\alpha}\right)\omega(\tau). $$

Lemma 2.3. *For $n-1 < \alpha \leq n, n \in \mathbb{N}$, $w \in C^n([0,1])$, we have*

$$ {}_0I_s^{\alpha}\,{}_0^{C}D_s^{\alpha}w(s) = w(s) - \sum_{i=0}^{n-1}\frac{w^{(i)}(0)(s)^i}{i!} $$

and

$$ {}_sI_1^{\alpha}\,{}_0^{C}D_1^{\alpha}w(s) = w(s) - \sum_{i=0}^{n-1}\frac{(-1)^i w^{(i)}(1)}{i!}(1-s)^i. $$

Hence, we obtain:

$$ \begin{aligned} {}_0I_1^{\alpha}\,{}_0^{RC}D_1^{\alpha}w(s) &= \frac{1}{2}\left[\left({}_0I_s^{\alpha}\,{}_0^{C}D_s^{\alpha} + {}_sI_s^{\alpha}\,{}_0^{C}D_s^{\alpha}\right)w(s)\right. \\ &\quad \left. + \left({}_0I_s^{\alpha}\,{}_s^{C}D_s^{\alpha} + {}_sI_1^{\alpha}\,{}_s^{C}D_1^{\alpha}\right)w(s)\right] \\ &= \frac{1}{2}\left[{}_0I_s^{\alpha C}\,{}_0D_s^{\alpha} + {}_sI_1^{\alpha C}\,{}_sD_1^{\alpha}\right]w(s). \end{aligned} $$

Based on the given conditions of $1 < \alpha \leq 2$ and $w \in C^1([0,1])$, we can derive the following:

(2.1)

$$ \begin{aligned} {}_0I_1^{\alpha}\,{}_0^{RC}D_1^{\alpha}w(s) &= \frac{1}{2}\left(2w(s) - w(0) - w(1) - w'(0)s + w'(1)(1-s)\right) \\ &= w(s) - \frac{1}{2}\left(w(0) + w(1)\right) - \frac{1}{2}(w'(0)s - w'(1)(1-s)). \end{aligned} $$

Lemma 2.4. *Given a function $f \in L^1[0,1]$, the following linear boundary value problem:*

$$ \begin{aligned} &{}_0^{RC}D_1^{\alpha}w(s) = f(s), \quad s \in [0,1] \\ &w(0) = 0, \\ &w(1) = \sum_{j=1}^{k}\lambda_j\int_0^{c_j} w(s)ds, \end{aligned} $$

has a unique solution, which is given by:

(2.2)

$$ \begin{aligned} w(s) &= \int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}dx \\ &\quad + \frac{2s}{\sigma-2}\left[\int_0^1 B(x)\,f(x)(1-x)^{\alpha-2}dx\right. \end{aligned} $$

$$+\sum_{j=1}^{k}\lambda_j\left(-\int_0^{c_j}\frac{(c_j-x)^{\alpha}}{\Gamma(\alpha+1)}f(x)dx+\int_{c_j}^{1}\frac{(x-c_j)^{\alpha}}{\Gamma(\alpha+1)}f(x)dx\right)\Bigg],$$

where

$$k(s,x)=\frac{1}{\Gamma(\alpha)}\left((1-x)-(\alpha-1)(1-s)+|s-x|^{\alpha-1}(1-x)^{2-\alpha}\right)$$

and

$$B(x)=\frac{1}{\Gamma(\alpha)}\times\Bigg[2(1-x)+\sum_{j=1}^{k}\lambda_j\left(c_j(1-x)-(\alpha-1)(c_j-\frac{c_j^2}{2})+\frac{(1-x)^{2-\alpha}x^{\alpha}}{\alpha}\right)\Bigg].$$

Proof. Applying the operator ${}_0I_1^{\alpha}$ to both sides of the equation under lemma 2.4 and utilizing (2.1), we obtain:

$$w(s)=\frac{1}{2}\left(w(0)+w(1)\right)+\frac{1}{2}(w'(0)s-w'(1)(1-s))+\frac{1}{\Gamma(\alpha)}\int_0^1|s-x|^{\alpha-1}f(x)dx.$$

As a consequence of the initial condition $w(0)=0$, we can deduce

$$w\left(s\right)=\frac{w(1)}{2}+\frac{w'(0)s-w'(1)(1-s)}{2}+\int_0^1\frac{|s-x|^{\alpha-1}}{\Gamma(\alpha)}f(x)dx, \tag{2.3}$$

hence,

$$w'(s)=\frac{w'(0)+w'(1)}{2}+\int_0^s\frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)}f(x)dx+\int_s^1\frac{(x-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(x)dx.$$

Therefore, we can derive:

$$\frac{w(1)}{2}=\frac{1}{2}w'(0)+\int_0^1\frac{(1-x)^{\alpha-1}}{\Gamma(\alpha)}f(x)dx, \tag{2.4}$$

$$\frac{w'(1)}{2}=\frac{w'(0)}{2}+\int_0^1\frac{(1-x)^{\alpha-2}}{\Gamma(\alpha-1)}f(x)dx. \tag{2.5}$$

Substituting the derived expressions from equations (2.4) and (2.5) into equation (2.3), we arrive at the following:

$$w(s)=w'(0)s+\int_0^1\frac{(1-x)^{\alpha-1}}{\Gamma(\alpha)}f(x)dx-\int_0^1\frac{(1-x)^{\alpha-2}}{\Gamma(\alpha-1)}(1-s)f(x)dx+\int_0^1\frac{|s-x|^{\alpha-1}}{\Gamma(\alpha)}f(x)dx.$$

Thus

$$w(s) = w'(0)s + \int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}dx, \tag{2.6}$$

where

$$k(s,x) = \frac{1}{\Gamma(\alpha)}\left((1-x) - (\alpha-1)(1-s) + |s-x|^{\alpha-1}(1-x)^{2-\alpha}\right).$$

Now, Taking into account the condition $w(1) = \sum_{j=1}^{k} \lambda_j \int_0^{c_j} w(s)ds$, we can deduce the following expression:

$$\begin{aligned} &w'(0) + \int_0^1 \frac{2(1-x)^{\alpha-1}}{\Gamma(\alpha)} f(x)\,dx \\ &= \sum_{j=1}^{k} \lambda_j \int_0^{c_j} w'(0)\,s\,ds + \sum_{j=1}^{k} \lambda_j \int_0^{c_j}\int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}\,dx\,ds \\ &= \frac{w'(0)}{2}\sum_{j=1}^{k} \lambda_j c_j^2 + \sum_{j=1}^{k} \lambda_j \int_0^{c_j}\int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}\,dx\,ds. \end{aligned}$$

So,

$$\begin{aligned} \frac{w'(0)}{2}\left(2 - \sum_{j=1}^{k} \lambda_j c_j^2\right) &= \sum_{j=1}^{k} \lambda_j \int_0^{c_j}\int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}dxds \\ &\quad - \int_0^1 \frac{2(1-x)^{\alpha-1}}{\Gamma(\alpha)} f(x)dx, \end{aligned}$$

Therefore, it follows:

$$\begin{aligned} w'(0) = \frac{2}{\sigma-2}\Bigg[&\int_0^1 \frac{2(1-x)^{\alpha-1}}{\Gamma(\alpha)} f(x)\,dx \\ &- \sum_{j=1}^{k} \lambda_j \int_0^{c_j}\int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}\,dx\,ds\Bigg], \end{aligned} \tag{2.7}$$

where $\sigma = \sum_{j=1}^{k} \lambda_j c_j^2$. Substituting (2.7) in (2.6), we get

$$\begin{aligned} w(s) = \frac{2s}{\sigma-2}\Bigg[&\int_0^1 \frac{2(1-x)^{\alpha-1}}{\Gamma(\alpha)} f(x)\,dx \\ &- \sum_{j=1}^{k} \lambda_j \int_0^{c_j}\int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}\,dx\,ds\Bigg] \end{aligned} \tag{2.8}$$

$$+\int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}\,dx.$$

Next, we evaluate the double integral in equation (2.8), resulting in:

$$\begin{aligned}(2.9)\quad &\int_0^{c_j}\int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}dxds\\ &=\left(\int_0^1\left[c_j(1-x)^{\alpha-1}-(\alpha-1)(1-x)^{\alpha-2}(c_j-\frac{c_j^2}{2})\right]f(x)dx\right.\\ &\quad\left.+\int_0^{c_j}\int_0^1|s-x|^{\alpha-1}f(x)dxds\right).\end{aligned}$$

By interchanging the order of integration in the second double integral on the right-hand side of equation (2.9), we arrive at:

$$\begin{aligned}\int_0^{c_j}\int_0^1|s-x|^{\alpha-1}f(x)\,dx\,ds&=\int_0^{c_j}\int_0^s(s-x)^{\alpha-1}f(x)\,dx\,ds\\ &\quad+\int_0^{c_j}\int_s^1(x-s)^{\alpha-1}f(x)\,dx\,ds\\ &=\int_0^{c_j}\int_x^{c_j}(s-x)^{\alpha-1}f(x)\,ds\,dx\\ &\quad+\int_0^{c_j}\int_0^x(x-s)^{\alpha-1}f(x)\,ds\,dx\\ &\quad+\int_{c_j}^1\int_0^{c_j}(x-s)^{\alpha-1}f(x)\,ds\,dx\\ &=\int_0^1\frac{x^\alpha}{\alpha}f(x)\,dx+\int_0^{c_j}\frac{(c_j-x)^\alpha}{\alpha}f(x)\,dx\\ &\quad-\int_{c_j}^1\frac{(x-c_j)^\alpha}{\alpha}f(x)\,dx.\end{aligned}$$

Hence

$$\begin{aligned}w(s)&=\int_0^1 k(s,x)f(x)(1-x)^{\alpha-2}dx\\ &\quad+\frac{2s}{\sigma-2}\left[\int_0^1 B(x)f(x)(1-x)^{\alpha-2}dx\right.\\ &\quad\left.-\sum_{j=1}^k\lambda_j\left(\int_0^{c_j}\frac{(c_j-x)^\alpha}{\Gamma(\alpha+1)}f(x)dx-\int_{c_j}^1\frac{(x-c_j)^\alpha}{\Gamma(\alpha+1)}f(x)dx\right)\right].\end{aligned}$$

where

$$B(x) = \frac{1}{\Gamma(\alpha)}\Bigg[2(1-x) - \sum_{j=1}^{k}\lambda_j\left(c_j(1-x) - (\alpha-1)(c_j - \frac{c_j^2}{2}) + \frac{(1-x)^{2-\alpha}x^{\alpha}}{\alpha}\right)\Bigg]. \square$$

Lemma 2.5. *The functions k and B possess the following properties:*

1) $|k(s,x)| \leq \frac{\alpha+1}{\Gamma(\alpha)}$.
2) $|B(x)| \leq \frac{2}{\Gamma(\alpha)} + \sum_{j=1}^{k}\lambda_j\left(\frac{\alpha^2 c_j+1}{\Gamma(\alpha+1)}\right)$.

Proof. The proof is simple and will be omitted. □

3. Existence of Solution

Define the Banach spaces $X = C([0,1])$ and $L^1_{\gamma}[0,1]$, $0 < \gamma < 1$ endowed with the following norms respectively:

$$||w||_X = \max_{s\in[0,1]}|w(s)|,$$
$$||w||_{L^1_{\gamma}} = \int_0^1 |w(s)|\,(1-s)^{\gamma-1}\,ds.$$

In order to establish the existence of solutions of the boundary value problem (1.1), we make the following assumptions:

(C) The function $A(.,t)$ is continuous and not identically null on $[0,1]$. Moreover, there exists a positive function $\psi \in L^1_{\alpha-1}[0,1]$ that satisfies the condition:

$$|A(x,t) - A(x,r)| \leq \psi(x)\,|t-r|, \quad 0 \leq x \leq 1,\ t,r \in \mathbb{R} \tag{3.1}$$

where

$$\|\psi\|_{L^1_{\alpha-1}} < \frac{1}{2L}, \tag{3.2}$$

$$L = \max\left\{\frac{\alpha+1}{\Gamma(\alpha)}, \frac{2}{|\sigma-2|}\left(\frac{2}{\Gamma(\alpha)} + \sum_{j=1}^{k}\lambda_j\frac{\alpha^2 c_j+2}{\Gamma(\alpha+1)}\right)\right\}. \tag{3.3}$$

Let us define the operators Π and Λ as follows:

$$\Pi w(s) = \int_0^1 k(s,x)A(x,w(x))\,(1-x)^{\alpha-2}dx,$$
$$\Lambda w(s) = \frac{2s}{\sigma-2}\left[\int_0^1 B(x)A(x,w(x))\,(1-x)^{\alpha-2}dx\right.$$

$$- \sum_{j=1}^{k} \lambda_j \left(\int_0^{c_j} \frac{(c_j - x)^{\alpha}}{\Gamma(\alpha+1)} A\left(x, w(x)\right) dx \right.$$
$$\left. - \int_{c_j}^{1} \frac{(x - c_j)^{\alpha}}{\Gamma(\alpha+1)} A\left(x, w(x)\right) dx \right) \Bigg].$$

Clearly, a fixed point of the operator $\Pi + \Lambda$ corresponds to a solution of the problem (1.1). Hence, to prove the existence of a solution to the problem (1.1), we shall establish a fixed point for the operator $\Pi + \Lambda$. To achieve this, we will employ the Krasnoselskii fixed point theorem, which will provide the necessary conditions for the existence of a fixed point.

Theorem 3.1. *Under the condition (C), the problem* (1.1) *possesses at least one solution in* X.

Proof. Define the ball $\mathcal{B}$ as follow:

$$\mathcal{B} := \{w \in X : ||w||_X \leq R\},$$

where R is chosen such

$$R \geq \left(\frac{1}{2L} - \|\psi\|_{L^1_{\alpha-1}} \right)^{-1} \frac{M}{\alpha - 1}, \tag{3.4}$$

where L be defined by condition (C) and

$$M = \max \left\{ \left| A\left(x, 0\right) \right|, 0 \leq x \leq 1 \right\}.$$

It can be verified that the ball $\mathcal{B}$ is a convex, closed, and bounded set. The proof will be presented in three steps.

First step. The operator Π is completely continuous.

a) The mapping Π is continuous on $\mathcal{B}$. In fact, let $(w_n)_n$ be a sequence such that $w_n \to w$ in $\mathcal{B}$, we will show that $\Pi w_n \to \Pi w$ in $\mathcal{B}$. To do this, we have the following:

$$\begin{aligned} |\Pi w_n(s) - \Pi w(s)| &\leq \int_0^1 |k(s,x)| \left| A\left(x, w_n(x)\right) - A\left(x, w(x)\right) \right| (1-x)^{\alpha-2} dx \\ &\leq \frac{\alpha+1}{\Gamma(\alpha)} \int_0^1 \left| A\left(x, w_n(x)\right) - A\left(x, w(x)\right) \right| (1-x)^{\alpha-2} dx \\ &\leq \frac{\alpha+1}{\Gamma(\alpha)} \int_0^1 \psi\left(x\right) \left| w_n\left(x\right) - w\left(x\right) \right| (1-x)^{\alpha-2} dx \\ &\leq \frac{\alpha+1}{\Gamma(\alpha)} \|\psi\|_{L^1_{\alpha-1}} \|w_n - w\|_X \to 0, \qquad \text{as } n \to +\infty, \end{aligned}$$

Hence $\|\Pi w_n - \Pi w\|_X \to 0$, as $n \to +\infty$.

b) The mapping Λ is uniformly bounded on $\mathcal{B}$. Let $w \in \mathcal{B}$, then by utilizing Lemma 2.5 and condition (C), we obtain:

(3.5)

$$\begin{aligned}
|\Pi w(s)| &\leq \int_0^1 |k(s,x)|\,|A\,(x,w(x)))|\,(1-x)^{\alpha-2}dx \\
&\leq \frac{\alpha+1}{\Gamma(\alpha)}\int_0^1 \left[|A\,(x,w(x))-A\,(x,0)|+|A\,(x,0)|\right](1-x)^{\alpha-2}dx \\
&\leq \frac{\alpha+1}{\Gamma(\alpha)}\int_0^1 \left[\psi\,(x)\,|w\,(x)|+M\right](1-x)^{\alpha-2}dx \\
&\leq \frac{\alpha+1}{\Gamma(\alpha)}\left(R\,\|\psi\|_{L^1_{\alpha-1}}+\frac{M}{\alpha-1}\right).
\end{aligned}$$

Thus Π is uniformly bounded on the ball $\mathcal{B}$.

c) The mapping Π is equicontinuous on $\mathcal{B}$. Let $w \in \mathcal{B}$ and $0 \leq s_1 \leq s_2 \leq 1$, we have

$$\begin{aligned}
&|\Pi w(s_1)-\Pi w(s_2)| \\
&\leq \int_0^1 |k(s_1,x)-k(s_2,x)|\,|A\,(x,w(x))|\,(1-x)^{\alpha-2}dx \\
&\leq \frac{(\alpha-1)\,(s_2-s_1)}{\Gamma(\alpha)}\int_0^1 |A\,(x,w(x))|\,(1-x)^{\alpha-2}dx \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^{s_1}\left[(s_2-x)^{\alpha-1}-(s_1-x)^{\alpha-1}\right]|A\,(x,w(x))|\,(1-x)^{\alpha-2}dx \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}\left||s_1-x|^{\alpha-1}-|s_2-x|^{\alpha-1}\right||A\,(x,w(x))|\,(1-x)^{\alpha-2}dx \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{s_2}^1\left[(x-s_1)^{\alpha-1}-(x-s_2)^{\alpha-1}\right]|A\,(x,w(x))|\,(1-x)^{\alpha-2}dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&|\Pi w(s_1)-\Pi w(s_2)| \\
&\leq \frac{(\alpha-1)\,(s_2-s_1)}{\Gamma(\alpha)}\int_0^1 |A\,(x,w(x))|\,(1-x)^{\alpha-2}dx \\
&\quad+\frac{(s_2-s_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{s_1}|A\,(x,w(x))|\,(1-x)^{\alpha-2}dx \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}\left|(x-s_1)^{\alpha-1}-(s_2-x)^{\alpha-1}\right||A\,(x,w(x))|\,(1-x)^{\alpha-2}dx \\
&\quad+\frac{(s_2-s_1)^{\alpha-1}}{\Gamma(\alpha)}\int_{s_2}^1|A\,(x,w(x))|\,(1-x)^{\alpha-2}dx
\end{aligned}$$

$$\leq \frac{(\alpha-1)(s_2-s_1)+2(s_2-s_1)^{\alpha-1}}{\Gamma(\alpha)}\left[R\|\psi\|_{L^1_{\alpha-1}}+\frac{M}{\alpha-1}\right]$$
$$+\frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}\left|(x-s_1)^{\alpha-1}-(s_2-x)^{\alpha-1}\right||A(x,w(x))|(1-x)^{\alpha-2}dx.$$

that tends to 0 as $s_1 \to s_2$. Therefore, the set $\Pi(\mathcal{B})$ is relatively compact. Applying the Arzela-Ascoli theorem, we can deduce that Π is a compact operator on $\mathcal{B}$.

Second step. The operator Λ is a contraction. In fact, for $w, w_1 \in \mathcal{B}$

$$\begin{aligned}|\Lambda w(s)-\Lambda w_1(s)| &\leq \frac{2}{|\sigma-2|}\int_0^1\left(|B(x)|+\sum_{j=1}^k\lambda_j\frac{|c_j-x|^\alpha}{\Gamma(\alpha+1)}\right)\\ &\quad\times|A(x,w(x))-A(x,w_1(x))|(1-x)^{\alpha-2}dx\\ &\leq \frac{2}{|\sigma-2|}\left(\frac{2}{\Gamma(\alpha)}+\sum_{j=1}^k\lambda_j\left(\frac{\alpha^2c_j+1}{\Gamma(\alpha+1)}\right)+\frac{\lambda_j}{\Gamma(\alpha+1)}\right)\\ &\quad\times\int_0^1\psi(x)|w(x)-w_1(x)|(1-x)^{\alpha-2}dx\\ &\leq \frac{2}{|\sigma-2|}\left(\frac{2}{\Gamma(\alpha)}+\sum_{j=1}^k\lambda_j\frac{\alpha^2c_j+2}{\Gamma(\alpha+1)}\right)\\ &\quad\times\|\psi\|_{L^1_{\alpha-1}}\|w-w_1\|_X,\end{aligned}$$

taking (3.3) into account, we get

$$\|\Lambda w-\Lambda w_1\|_X \leq L\|\psi\|_{L^1_{\alpha-1}}\|w-w_1\|_X,$$

with $L\|\psi\|_{L^1_{\alpha-1}} < \frac{1}{2}$ by condition (3.2), we conclude that Λ is a contraction.

Third step. Let us show that $\Pi w + \Lambda w_1 \in \mathcal{B}$ for every $w, w_1 \in \mathcal{B}$. By utilizing (3.5),(3.3) and (3.4), we obtain

$$\begin{aligned}|\Pi w(s)| &\leq \frac{\alpha+1}{\Gamma(\alpha)}\left(R\|\psi\|_{L^1_{\alpha-1}}+\frac{M}{\alpha-1}\right)\\ &\leq L\left(R\|\psi\|_{L^1_{\alpha-1}}+\frac{M}{\alpha-1}\right)\\ &\leq \frac{R}{2}.\end{aligned}$$

On the other hand, by following a similar approach as in (3.5) step 2 and utilizing Lemma 2.5, (3.3) and (3.4), we can derive the following

estimate:

$$
\begin{aligned}
|\Lambda w_1(s)| &\leq \frac{2s}{|\sigma-2|}\left[\int_0^1 |B(x)|\,|A(x,w_1(x))|\,(1-x)^{\alpha-2}\,dx\right.\\
&\quad+\sum_{j=1}^{k}\lambda_j\left(\int_0^{c_j}\frac{(c_j-x)^{\alpha}}{\Gamma(\alpha+1)}\,|A(x,w_1(x))|\,dx\right.\\
&\quad\left.\left.+\int_{c_j}^1\frac{(x-c_j)^{\alpha}}{\Gamma(\alpha+1)}\,|A(x,w_1(x))|\,dx\right)\right]\\
&=\frac{2s}{|\sigma-2|}\left(\int_0^1\left[|B(x)|+\sum_{j=1}^{k}\lambda_j\frac{|c_j-x|^{\alpha}\,(1-x)^{2-\alpha}}{\Gamma(\alpha+1)}\right]\right.\\
&\quad\left.\times\,|A(x,w_1(x))|\,(1-x)^{\alpha-2}\,dx\right)\\
&\leq\frac{2}{|\sigma-2|}\left(\frac{2}{\Gamma(\alpha)}+\sum_{j=1}^{k}\lambda_j\frac{\alpha^2c_j+1}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\lambda_j\frac{1}{\Gamma(\alpha+1)}\right)\\
&\quad\times\int_0^1|A(x,w_1(x))|\,(1-x)^{\alpha-2}\,dx\\
&\leq\frac{2}{|\sigma-2|}\left(\frac{2}{\Gamma(\alpha)}+\sum_{j=1}^{k}\lambda_j\frac{\alpha^2c_j+2}{\Gamma(\alpha+1)}\right)\\
&\quad\times\left(R\,\|\psi\|_{L^1_{\alpha-1}}+\frac{M}{\alpha-1}\right)\\
&\leq L\left(R\,\|\psi\|_{L^1_{\alpha-1}}+\frac{M}{\alpha-1}\right)\leq\frac{R}{2}.
\end{aligned}
$$

thus, $\|\Pi w+\Lambda w_1\|_X\leq R$. By invoking Krasnoselskii's fixed point theorem, we can conclude that the BVP problem (1.1) has at least one solution in X. □

Next, we provide an additional existence result for the problem (1.1) using the Leray-Schauder nonlinear alternative. Let us recall Leray-Schauder's nonlinear alternative:

Theorem 3.2. *Let X be a Banach space and $\mathcal{O}$ be a bounded open subset of X, $0\in\mathcal{O}$. Suppose that $\mathcal{T}:\overline{\mathcal{O}}\to X$ be a completely continuous mapping. Then, either $\mathcal{T}$ has a fixed point in $\overline{\mathcal{O}}$ or there exists $w\in\partial\mathcal{O}$ (the boundary of $\mathcal{O}$ in X), such that $w=\lambda\mathcal{T}w$ for some $0<\lambda<1$.*

Let's express the integral equation (2.2) in the following form:

$$\begin{aligned} w(s) = &\int_0^1 \mathcal{H}(s,x)A(x,w(x))(1-x)^{\alpha-2}\,dx \\ &+ \frac{2s}{\sigma-2}\sum_{j=1}^{k}\lambda_j\left(-\int_0^{c_j}\frac{(c_j-x)^{\alpha}}{\Gamma(\alpha+1)}A(x,w(x))\,dx\right. \\ &\left.+\int_{c_j}^1\frac{(x-c_j)^{\alpha}}{\Gamma(\alpha+1)}A(x,w(x))\,dx\right) \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}(s,x) = \frac{1}{\Gamma(\alpha)}\Bigg[&\left(\frac{4s}{\sigma-2}+1\right)(1-x)-(\alpha-1)(1-s)+|s-x|^{\alpha-1}(1-x)^{2-\alpha} \\ &+\frac{2s}{\sigma-2}\sum_{j=1}^{k}\lambda_j\left(c_j(1-x)-(\alpha-1)\left(c_j-\frac{c_j^2}{2}\right)+\frac{(1-x)^{2-\alpha}x^{\alpha}}{\alpha}\right)\Bigg] \end{aligned}$$

Lemma 3.3. *The inequality $|\mathcal{H}(s,x)| \leq \kappa$ holds, where κ is defined as*

$$\kappa = \frac{1+\alpha}{\Gamma(\alpha)} + \frac{2}{|\sigma-2|}\left(2\alpha+\sum_{j=1}^{k}\lambda_j\left(\alpha^2c_j+1\right)\right).$$

Let us define the integral operator $\mathcal{T}$ as follows:

$$\begin{aligned} \mathcal{T}w(s) = &\int_0^1 \mathcal{H}(s,x)A(x,w(x))(1-x)^{\alpha-2}\,dx \\ &+ \frac{2s}{\sigma-2}\sum_{j=1}^{k}\lambda_j\left(-\int_0^{c_j}\frac{(c_j-x)^{\alpha}}{\Gamma(\alpha+1)}A(x,w(x))\,dx\right. \\ &\left.+\int_{c_j}^1\frac{(x-c_j)^{\alpha}}{\Gamma(\alpha+1)}A(x,w(x))\,dx\right), \end{aligned}$$

we aim to show that the integral operator $\mathcal{T}$ possesses a fixed point by applying the Leray-Schauder nonlinear alternative. The existence of such a fixed point would then correspond to a solution to the problem (1.1).

Theorem 3.4. *Assume that $A(.,0)$ nonidentically null and there exist two nonnegative functions $\psi \in L^1_{\alpha-1}[0,1]$, $\varphi \in C[0,+\infty)$ with φ nondecreasing and $\eta > 0$, such that*

$$|A(x,t)| \leq \psi(x)\,\varphi(|t|),$$

for all $0 \leq x \leq 1$ *and* $t \in \mathbb{R}$, *with*

$$\left(\kappa + \sum_{j=1}^{k} \frac{2\lambda_j}{|\sigma - 2|}\right) \|\psi\|_{L^1_{\alpha-1}} \varphi(\eta) < \eta. \tag{3.6}$$

Then the nonlinear problem (1.1) *admits at least one nontrivial solution in* X.

Proof. Let $\mathcal{B}_\eta = \{w \in X, \|w\|_X < \eta\}, \eta > 0$. The set $\mathcal{T}(\mathcal{B}_\eta)$ is uniformly bounded. Indeed, let $w \in \mathcal{B}_\eta$, we have

$$\begin{aligned}
|\mathcal{T}w(s)| &\leq \int_0^1 |\mathcal{H}(s,x)| \, |A(x, w(x))| \, (1-x)^{\alpha-2} dx \\
&\quad + \frac{2}{|\sigma - 2|} \sum_{j=1}^{k} \lambda_j \left(\int_0^{c_j} \frac{(c_j - x)^\alpha}{\Gamma(\alpha+1)} |A(x, w(x))| \, dx \right. \\
&\quad \left. + \int_{c_j}^1 \frac{(x - c_j)^\alpha}{\Gamma(\alpha+1)} |A(x, w(x))| \, dx \right) \\
&\leq \int_0^1 \left(|\mathcal{H}(s,x)| + \frac{2}{|\sigma - 2|} \sum_{j=1}^{k} \lambda_j \frac{|c_j - x|^\alpha}{\Gamma(\alpha+1)} \right) \\
&\quad \times |A(x, w(x))| \, (1-x)^{\alpha-2} dx \\
&\leq \left(\kappa + \sum_{j=1}^{k} \frac{2\lambda_j}{|\sigma - 2|} \right) \int_0^1 \Psi(x) \, \varphi(|w(x)|) \, (1-x)^{\alpha-2} dx \\
&\leq \left(\kappa + \sum_{j=1}^{k} \frac{2\lambda_j}{|\sigma - 2|} \right) \|\psi\|_{L^1_{\alpha-1}} \varphi(\eta) \\
&< +\infty.
\end{aligned}$$

The set $\mathcal{T}(\mathcal{B}_\eta)$ is equicontinuous. In fact, let $w \in \mathcal{B}_\eta$ and $0 \leq s_1 \leq s_2 \leq 1$, we have

$$\begin{aligned}
&|\mathcal{T}w(s_1) - \mathcal{T}w(s_2)| \\
&\leq \int_0^1 |\mathcal{H}(s_1, x) - \mathcal{H}(s_2, x)| \, |A(x, w(x))| \, (1-x)^{\alpha-2} \, dx \\
&\quad + \frac{2(s_2 - s_1)}{|\sigma - 2|} \sum_{j=1}^{k} \lambda_j \int_0^1 \frac{|c_j - x|^\alpha}{\Gamma(\alpha+1)} |A(x, w(x))| \, (1-x)^{\alpha-2} \, dx \\
&\leq \left[\frac{4(s_2 - s_1)}{|\sigma - 2| \, \Gamma(\alpha)} + \frac{(\alpha - 1)(s_2 - s_1)}{\Gamma(\alpha)} \right] \int_0^1 |A(x, w(x))| \, (1-x)^{\alpha-2} \, dx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^1\left||s_2-x|^{\alpha-1}-|s_1-x|^{\alpha-1}\right||A(x,w(x))|\,(1-x)^{\alpha-2}\,dx\\
&+\frac{2\,(s_2-s_1)}{|\sigma-2|}\sum_{j=1}^{k}\lambda_j\int_0^1|A(x,w(x))|\,(1-x)^{\alpha-2}\,dx\\
\leq&\left(\frac{4}{|\sigma-2|\,\Gamma(\alpha)}+\frac{(\alpha-1)}{\Gamma(\alpha)}+\frac{2}{|\sigma-2|}\sum_{j=1}^{k}\lambda_j\right)(s_2-s_1)\,\|\psi\|_{L^1_{\alpha-1}}\,\varphi(\eta)\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{s_1}\left[(s_2-x)^{\alpha-1}-(s_1-x)^{\alpha-1}\right]|A(x,w(x))|\,(1-x)^{\alpha-2}\,dx\\
&+\frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}\left||s_1-x|^{\alpha-1}-|s_2-x|^{\alpha-1}\right||A(x,w(x))|\,(1-x)^{\alpha-2}\,dx\\
&+\frac{1}{\Gamma(\alpha)}\int_{s_2}^{1}\left[(x-s_1)^{\alpha-1}-(x-s_2)^{\alpha-1}\right]|A(x,w(x))|\,(1-x)^{\alpha-2}\,dx.
\end{aligned}
$$

that tends to 0 as $s_1 \to s_2$.

Therefore, we can deduce that $\mathcal{T}(\mathcal{B}_\eta)$ is relatively compact. By Arzela-Ascoli theorem, it follows that $\mathcal{T}$ is compact.

Now, denote $\mathcal{O}$=$\mathcal{B}_\eta$, then for $w \in \partial O$, such that $w = \lambda\mathcal{T}w$ with $0 < \lambda < 1$, we obtain

$$
\begin{aligned}
|w(x)| &\leq |\mathcal{T}w(s)|\\
&\leq \left(\kappa+\sum_{j=1}^{k}\frac{2\lambda_j}{|\sigma-2|}\right)\|\psi\|_{L^1_{\alpha-1}}\,\varphi(\eta),
\end{aligned}
$$

taking (3.6) into account, we get

$$\|w\|_X = \eta < \eta,$$

therefore, applying the Leray-Schauder nonlinear alternative, we can conclude that the operator $\mathcal{T}$ has a fixed point w^* in the set $\overline{\mathcal{O}}$. Furthermore, since $A(.,0)$ is not identically zero on $[0,1]$, the fixed point w^* is a nontrivial solution of problem (1.1) in the Banach space X. Thus, the existence of a nontrivial solution for problem (1.1) has been established. The proof is now complete. □

4. Examples

Example 4.1. Consider the problem (1.1) with $\alpha = \frac{3}{2}$, $\lambda_j = 2^{j+1}$, $c_j = \frac{1}{j+1}$ for $j = 1, 2, 3$ and

$$A(x,t) = \frac{1}{60}\sqrt{1-x}e^{-10x}\left(t-\frac{x}{2\,(t^2+1)}\right),$$

we have that the function $A(.,0) = -\frac{x}{120}\sqrt{1-x}e^{-10x}$ is continuous and not identically null on $[0,1]$, moreover

$$\begin{aligned}|A(x,t) - A(x,r)| &= \frac{1}{60}\sqrt{1-x}e^{-10x}\left|t - r + \frac{x}{2\left(r^2+1\right)} - \frac{x}{2\left(t^2+1\right)}\right| \\ &\leq \frac{1}{60}\sqrt{1-x}e^{-10x}\left|t-r\right|\left|1 + \frac{x}{2}\frac{t+r}{\left(r^2+1\right)\left(t^2+1\right)}\right| \\ &\leq \psi\left(x\right)\left|t-r\right|,\end{aligned}$$

where $\psi\left(x\right) = \frac{1}{40}\sqrt{1-x}e^{-10x}$. Simple calculus yields:

$$\begin{aligned}\left\|\psi\right\|_{L^1_{\alpha-1}} &= \int_0^1 \frac{1}{40}e^{-10x}dx \\ &= 2.49991 \times 10^{-3},\end{aligned}$$

and

$$\begin{aligned}\sigma &= \sum_{j=1}^{3} 2^{j+1}\left(\frac{1}{j+1}\right)^2 \\ &= \frac{26}{9},\end{aligned}$$

and

$$\begin{aligned}L &= \max\left\{\frac{\alpha+1}{\Gamma(\alpha)}, \frac{2}{|2-\sigma|}\left(\frac{2}{\Gamma(\alpha)} + \sum_{j=1}^{3} 2^{j+1}\frac{\alpha^2 c_j + 2}{\Gamma(\alpha+1)}\right)\right\} \\ &= 132.8666\ . \\ &= \max\left\{\frac{5}{\sqrt{\pi}}, \frac{471}{2\sqrt{\pi}}\right\} \\ &= \frac{471}{2\sqrt{\pi}},\end{aligned}$$

so,

$$\begin{aligned}\frac{1}{2L} &= \frac{\sqrt{\pi}}{471} \\ &= 3.7631 \times 10^{-3},\end{aligned}$$

thus,

$$\left\|\psi\right\|_{L^1_{\alpha-1}} < \frac{1}{2L}.$$

Since all the conditions of the theorem are satisfied, we can conclude that problem (1.1) has at least one solution in X.

The figure below display plots of the function w(x) with varying values of N to improve precision:

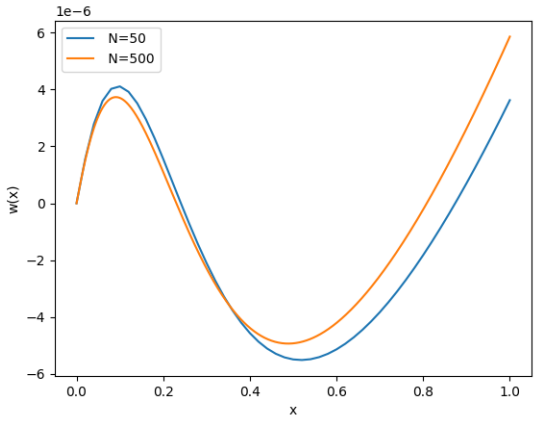


FIGURE 1. Plot of w(x) with 2 values of N.

When zooming in, we get the figure below:

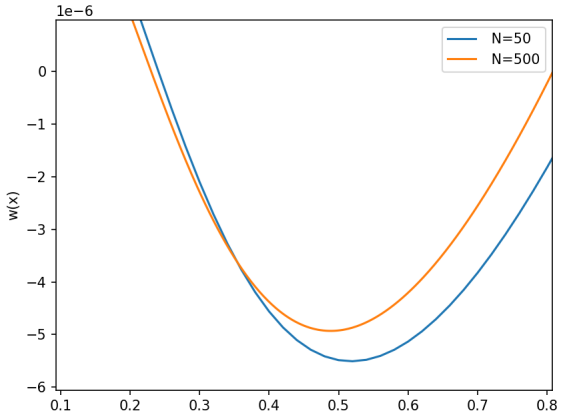


FIGURE 2. A zoomed-in version of the plot of w(x).

In the figure below, we plot $w(x)$ for different values of α and $N = 10000$, we can see that the sign of the function changes when alpha changes.

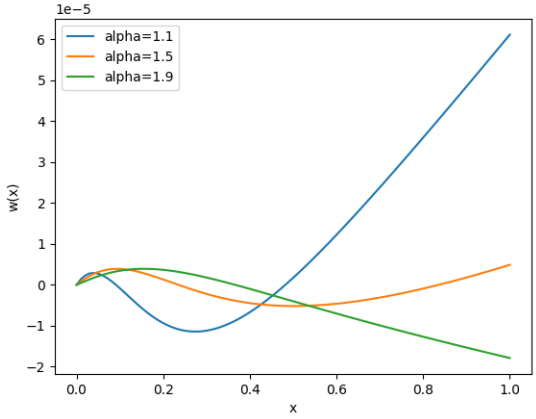


FIGURE 3. Plot of w(x) with 3 values of α.

Example 4.2. Consider the problem (1.1) with $\alpha = \frac{3}{2}$, $\lambda_j = 2^{j+1}$, $c_j = \frac{1}{j+1}$ for $j = 1, 2, 3$ and

$$A(x,t) = (1-x)\, e^{-x} \left(\frac{1}{10} + \arctan t \right).$$

We have, $A(x,0) = (1-x)\, e^{-x} \neq 0$ and

$$\begin{aligned} |A(x,t)| &= (1-x)\, e^{-x} \left| \frac{1}{10} + \arctan t \right| \\ &\leq \psi(x)\, \varphi(|t|), \end{aligned}$$

where

$$\psi(x) = \sqrt{1-x} e^{-x}, \qquad \varphi(t) = \frac{1}{10} + \arctan t.$$

Now, through a computation, we get,

$$\begin{aligned} \sigma &= \sum_{j=1}^{3} 2^{j+1} \left(\frac{1}{j+1} \right)^2 = \frac{26}{9}, \\ \kappa &= \frac{5}{\sqrt{\pi}} + \frac{909}{8} \simeq 116.45, \\ \|\psi\|_{L^1_{\alpha-1}} &\simeq 5.267\,7 \times 10^{-3}. \end{aligned}$$

Now, choosing $\eta = 1$, we obtain

$$\left(\kappa + \sum_{j=1}^{k} \frac{2\lambda_j}{|\sigma - 2|} \right) \|\psi\|_{L^1_{\alpha-1}} \varphi(\eta) = 0.836\,93$$

$$\begin{aligned} &< \eta \\ &= 1. \end{aligned}$$

Hence, all the assumptions of the theorem are fulfilled, we conclude that problem (1.1) has at least one solution in X.

Numerical discretization scheme. The spatial domain is $[0,1]$. The mesh is N equal intervals of $h = 1/N$ and $x_l = lh$ for $0 \leqslant l \leqslant N$. From [20] we have :

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{u^{(2)}(\xi)d\xi}{(x-\xi)^{\alpha-1}} \\ &\quad = \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{u^{(2)}(x-\xi)d\xi}{\xi^{\alpha-1}} \\ &\quad = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{l-1} \int_{jh}^{(j+1)h} \frac{u^{(2)}(x-\xi)d\xi}{\xi^{\alpha-1}} \\ &\quad \approx \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{l-1} \frac{u(x_l-(j-1)h) - 2u(x_l-(j)h) + u(x_l-(j+1)h)}{h^2} \\ &\qquad \times \int_{jh}^{(j+1)h} \frac{d\xi}{\xi^{\alpha-1}} \\ &\quad = \frac{h^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{l-1} \left(u_{l-j+1} - 2u_{l-j} + u_{l-j-1}\right) \left[(j+1)^{2-\alpha} - j^{2-\alpha}\right]. \end{aligned}$$

where $u(x_l - (j)h) = u_{l-j}$. Using Python we were able to plot the solution of the examples above.

5. Conclusion

By applying the Krasnoselskii fixed point theorem and the Leray-Schauder alternative, we have established the existence of solutions for the Riesz-Caputo fractional boundary value problem with integral boundary conditions. Moreover, numerical examples have been used to validate and illustrate the results. Further research can be pursued by exploring various types of fractional derivatives and examining their influence on the solutions of boundary value problems. Furthermore, investigating the stability and convergence properties of the obtained solutions would contribute to a better understanding of the behavior of these fractional differential equations.

Acknowledgment. The author thanks the referees for their valuable comments and suggestions that improved this paper.

References

1. B. Ahmad, S. K. Ntouyas and A. Alsaedi, *Fractional order differential systems involving right Caputo and left Riemann-Liouville fractional derivatives with nonlocal coupled conditions*. Bound. Value Probl. Berlin, 2019, 109.
2. R. L. Bagley and P. J. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, J. Rheol. 27, 1983, pp. 201-210.
3. D. Baleanu, K. Diethelm, E. Scalas and J. Trujillo, *Fractional Calculus: Models and Numerical Methods. 3, Series on Complexity, Nonlinearity and Chaos*, World Scientific Publishing, Hackensack, (2012).
4. A. Carpinteri, P. Cornetti and A. Sapora, *Nonlocal elasticity: an approach based on fractional calculus*. Meccanica. 49,(2014), pp. 2551-2569.
5. F. Chen, D. Baleanu and GC. Wu, *Existence results of fractional differential equations with Riesz–Caputo derivative*. Eur. Phys. J. Spec. Top. Berlin. 226,(2017), pp. 3411-3425.
6. C.Y. Gu, J. Zhang and G.C. Wu, *Positive solutions of fractional differential equations with the Riesz space derivative*. Appl. Math. Lett. Oxford, 95 (2019), pp. 59-64.
7. M. Feng, X. Zhang and W. Ge, *New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions*. Bound. Value Probl. Berlin., 2011, pp 20.
8. A. Guezane-Lakoud and R. Khaldi, *Solutions for a nonlinear fractional Euler–Lagrange type equation*. SeMA J. Madrid, 76 (2019), pp. 195-202.
9. A. Guezane-Lakoud, R. Khaldi and A. Kılıçman, *Existence of solutions for a mixed fractional boundary value problem*. Adv. Difference Equ. Berlin, 2017, pp 164.
10. A. Guezane-Lakoud and R. Rodríguez-López, *Positive solutions to mixed fractional p-Laplacian boundary value problems*. J. Appl. Anal. Berlin, 29 (2023), pp. 49-58.
11. R. Hilfer, Ed., *Applications of Fractional Calculus in Physics*, World Scientific, River Edge, NJ, USA, 2000.
12. S. Jiang, J. Zhang, Q. Zhang, Z. Zhang, *Fast Evaluation of the Caputo Fractional Derivative and its Applications to Fractional Diffusion Equations*. Commun. Comput. Phys., Hong Kong, 21 (2017), pp. 650-678.
13. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam (2006).

14. S. Li, Z. Zhang, W. Jiang, *Positive solutions for integral boundary value problems of fractional differential equations with delay.* Adv. Difference Equ. Berlin, 2020, 256.
15. I. Merzoug, A. Guezane-Lakoud, R. Khaldi, *Existence of solutions for a nonlinear fractional p-Laplacian boundary value problem.* Rend. Circ. Mat. Palermo, II. Ser 69 (2020), pp. 1099-1106.
16. R. Khaldi, A. Guezane-Lakoud, *On generalized nonlinear Euler-Bernoulli Beam type equations*, Acta Univ. Sapientiae, Mathematica, Cluj-Napoca, 10 (2018), pp. 90-100.
17. R. Khaldi, A. Guezane-Lakoud, *Higher order fractional boundary value problems for mixed type derivatives*, Journal of Nonlinear Functional Analysis, Vol. 2017 , Article ID 30, pp. 1-9.
18. R. Khaldi., A. Guezane-Lakoud, *On a generalized Lyapunov inequality for a mixed fractional boundary value problem*, AIMS Mathematics, 2019, 4(3), pp. 506-515.
19. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications.* Switzerland: Gordon and Breach Science Publishers, 1993.
20. Q. Yang, F. Liu, I. Turner, *Numerical methods for fractional partial differential equations with Riesz space fractional derivatives*, Appl. Math. Model. Amsterdam, 34 (2010), pp. 200-218.

---

[1] Department of Mathematics, Faculty of Science, University Badji Mokhtar Annaba, Laboratory of Advanced Materials, 23000 , Annaba, Algeria.

Email address: takieddine.zeghida@univ-annaba.dz

[2] Department of Mathematics, Faculty of Science, University Badji Mokhtar Annaba, 23000 , Annaba, Algeria.

Email address: rkhadi@yahoo.fr

[3] Department of Mathematics, Faculty of Science, University Badji Mokhtar Annaba, 23000 , Annaba, Algeria.

Email address: a_guezane@yahoo.fr