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# Integral Inequalities Using Generalized Convexity Property Pertaining to Fractional Integrals and Their Applications

Muhammad Sadaqat Talha[1], Artion Kashuri[2] and Soubhagya Kumar Sahoo[3*]

Abstract. In this study, we established the Hermite-Hadamard type, Simpson type, Ostrowski type and midpoint type integral inequalities for the s-convex functions in the second sense via Katugampola fractional integrals. By using Katugampola fractional integral operators, we obtained several new identities and presented new results for the s-convex function in the second sense. We made connections of our results with various results recognized in the literature. Finally, applications to special means are examined to verify the efficiency of the established results.

## 1. Introduction

In recent years, convex functions have been used in various branches of mathematics, especially in mathematical economics, analysis and geometry, among others. In modern analysis, a large area is now dedicated to the applications of convex functions. Jensen's fundamental work in the years 1905 and 1906 served as the starting point for the systematic study of convex functions. The roots of these functions can be traced back to J. Hadamard and C. Hermite in 1889 and 1893, respectively. However, in their work, these roots were not clearly specified.

The general theory of convex functions serves as a powerful source of tools for studying problems in analysis. In 1987, a book titled "Convex Functions: Inequalities, Beograd" (written in Serbo-Croatian) by Pečarić was entirely devoted to inequalities for convex functions. This

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current volume covers a comprehensive and complete treatment of convex functions, including some specific applications in probability, statistics and connected partial orders.

Integral inequalities are one of the major sections of mathematical inequalities. They have been extensively developed over the last few decades due to the significant applications of integral inequalities in many areas of research. The extensive involvement of integral inequalities in various branches of mathematics attracts mathematicians and researchers to work on them. Since the pronouncement of Ostrowski's inequality, Hardy's inequality and Hermite-Hadamard's inequality, many different researchers have been interested in the subject of integral inequalities. Many generalizations of these inequalities have been proved for several functions such as differentiable functions, convex functions and s-convex functions (see [22, 23]). Gozpinar et al. [18] worked on the Hermite-Hadamard inequalities for twice differentiable s-convex functions. Almutairi et al. [4], Delavar et al. [14] and Mehreen et al. [26] proved some fractional versions of Hermite-Hadamard type inequalities.

The effect of inequalities consisting of convex functions is as profound and magical as in various branches of mathematics, solving many problems at a considerable rate. When dealing in higher dimensions, many physicists have used inequalities of higher-order convex functions. Many new inequalities of different categories involving convex functions were obtained and implemented in other fields. One can find more information in [22, 27]. We now recall some significant definitions and useful results on convex functions (see [28]). Convex functions play a significant role in many different fields of science and is usually defined in the following way:

**Definition 1.1** ([28])**.** Suppose $\Psi : I = [\hbar_1, \hbar_2] \subseteq \Re \to \Re$, is a convex function if

$$\Psi(\hbar_1\varpi + (1-\varpi)\hbar_2) \leq \varpi\Psi(\hbar_1) + (1-\varpi)\Psi(\hbar_2),$$

holds for all $\hbar_1, \hbar_2 \in I$ and $\varpi \in [0,1]$. And $\Psi$ is concave if $(-\Psi)$ is convex.

Hudzik et al. [19] examined the s-convex functions in the second sense. This class of functions is given below:

**Definition 1.2** ([19])**.** Suppose $\Psi : [0,\infty) \to \Re$, is an s-convex function in the second sense if

$$\Psi(\hbar_1\varpi + (1-\varpi)\hbar_2) \leq \varpi^s\Psi(\hbar_1) + (1-\varpi)^s\Psi(\hbar_2),$$

holds for all $\hbar_1, \hbar_2 \in [0,\infty)$ and $\varpi \in [0,1]$. This is usually denoted by $K_s^2$. We can see that s-convexity reduces to ordinary convexity of the functions for $s = 1$, defined on $[0,\infty)$.

Hermite-Hadamard inequality is the most significant inequality that has attracted many mathematicians in this field. In [15], the well-known Hermite-Hadamard inequality is given as:

**Theorem 1.3.** *If $\Psi$ is a convex function on $[\hbar_1, \hbar_2]$ with $\hbar_1 < \hbar_2$, then*

$$\Psi\left(\frac{\hbar_1+\hbar_2}{2}\right) \leq \frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2}\Psi(\varpi)d\varpi \leq \frac{\Psi(\hbar_1)+\Psi(\hbar_2)}{2},$$

*holds.*

Dragomir et al. [16], have proved an important variant of Hadamard's inequality for s-convex functions in the second sense.

**Theorem 1.4** ([16])**.** *Let $\Psi : [0,\infty) \longrightarrow [0,\infty)$ is an s-convex function in the second sense and let $\hbar_1, \hbar_2 \in [0,\infty)$, $\hbar_1 < \hbar_2$, where $s \in (0,1)$. If $\Psi \in L[\hbar_1, \hbar_2]$, then*

$$2^{s-1}\Psi\left(\frac{\hbar_1+\hbar_2}{2}\right) \leq \frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2}\Psi(\varpi)d\varpi \leq \frac{\Psi(\hbar_1)+\Psi(\hbar_2)}{s+1},$$

*holds.*

Simpson's inequalities are well-known results in the literature given as:

**Theorem 1.5** ([33])**.** *Let $\Psi : [\hbar_1, \hbar_2] \longrightarrow \Re$ be a four times continuously differentiable mapping on $(\hbar_1, \hbar_2)$ and $\left|\left|\Psi^{(4)}\right|\right|_\infty = \sup_{\varpi\in(\hbar_1,\hbar_2)} < \infty$, then*

$$\left|\frac{1}{3}\left(\frac{\Psi(\hbar_1)+\Psi(\hbar_2)}{2}+2\Psi\left(\frac{\hbar_1+\hbar_2}{2}\right)\right)-\frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2}\Psi(\varpi)d\varpi\right| \leq \frac{1}{2880}\left\|\Psi^{(4)}\right\|_\infty(\hbar_2-\hbar_1)^2,$$

*holds.*

We can see the generalizations, counterparts and refinements for Simpson's inequality in [6, 34].

The Ostrowski's inequalities are well-known results in the literature given as:

**Theorem 1.6** ([17])**.** *Let $\Psi : I \subset \Re \longrightarrow \Re$ be a differentiable mapping on $I^o$ and let $\hbar_1, \hbar_2 \in (0,1)$ with $\hbar_1 < \hbar_2$. If $|\Psi'(\varpi)| \leq M$ and $\varpi \in [\hbar_1, \hbar_2]$, then*

$$\left|\Psi(\varpi)-\frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2}\Psi(\varpi)d\varpi\right| \leq \frac{M}{\hbar_2-\hbar_1}\left[\frac{(\varpi-\hbar_1)^2+(\hbar_2-\varpi)^2}{2}\right],$$

*holds.*

Alomari et al. [5] have shown an important variant of Ostrowski's inequality for s-convex functions in the second sense.

**Theorem 1.7** ([5]). *Suppose that $\Psi : I \subset [0,\infty) \longrightarrow \Re$ be a differentiable mapping on $I^o$, such that $\Psi \in L_1[\hbar_1,\hbar_2]$, where $\hbar_1,\hbar_2 \in I$ with $\hbar_1 < \hbar_2$. For some fixed $s \in (0,1]$, if $|\Psi'|$ is s-convex in second sense on $[\hbar_1,\hbar_2]$, $|\Psi'(\varpi)| \leq M$ and $\varpi \in [\hbar_1,\hbar_2]$, then*

$$\left|\Psi(\varpi) - \frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2}\Psi(\varpi)d\varpi\right| \leq \frac{M}{\hbar_2-\hbar_1}\left[\frac{(\varpi-\hbar_1)^2+(\hbar_2-\varpi)^2}{s+1}\right],$$

*holds.*

We use the given definitions and concepts for our results.

(1) The Beta function [31]

$$\begin{aligned}\beta(\hbar_1,\hbar_2) &= \frac{\Gamma(\hbar_1)\Gamma(\hbar_2)}{\Gamma(\hbar_1+\hbar_2)}\\ &= \int_0^1 \ell^{\hbar_1-1}(1-\ell)^{\hbar_2-1}\,d\ell, \quad \hbar_1,\hbar_2>0.\end{aligned}$$

(2) The Hypergeometric function [25]

$$\begin{aligned}&{}_2F_1(\hbar_1,\hbar_2;\hbar_3;z)\\ &= \frac{1}{\beta(\hbar_2,\hbar_3-\hbar_2)}\int_0^1 \ell^{\hbar_2-1}(1-\ell)^{\hbar_3-\hbar_2-1}(1-z\ell)\,d\ell,\end{aligned}$$

that $\hbar_3 > \hbar_2 > 0, |z| < 1$.

**Lemma 1.8** ([29]). *For $0 < \Upsilon \leq 1$ and $0 \leq \hbar_1 < \hbar_2$, we have*

$$\left|\hbar_1^{\Upsilon} - \hbar_2^{\Upsilon}\right| \leq (\hbar_2-\hbar_1)^{\Upsilon}.$$

Some necessary and important mathematical preliminaries and definitions of fractional calculus theory are given below:

**Definition 1.9** ([25]). Let $\Psi \in L_1[\hbar_1,\hbar_2]$. The Riemann-Liouville integrals $J^{\Upsilon}_{\hbar_1^+}\Psi(\varpi)$ and $J^{\Upsilon}_{\hbar_2^-}\Psi(\varpi)$ of order $\Upsilon > 0$ are defined by:

$$J^{\Upsilon}_{\hbar_1^+}\Psi(\varpi) = \frac{1}{\Gamma(\Upsilon)}\int_{\hbar_1}^{\varpi}(\varpi-\ell)^{\Upsilon-1}\Psi(\ell)d\ell, \quad \varpi > \hbar_1$$

and

$$J^{\Upsilon}_{\hbar_2^-}\Psi(\varpi) = \frac{1}{\Gamma(\Upsilon)}\int_{\varpi}^{\hbar_2}(\ell-\varpi)^{\Upsilon-1}\Psi(\ell)d\ell, \quad \varpi < \hbar_2$$

respectively, where with $\Gamma(\Upsilon) = \int_0^{\infty} e^{-\Upsilon}\varpi^{\Upsilon-1}d\varpi$. Here $I^0_{\hbar_1^+}\Psi(\varpi) = I^0_{\hbar_2^-}\Psi(\varpi) = \Psi(\varpi)$.

This fractional integral becomes a classical integral for $\Upsilon = 1$. We cannot underestimate the noble impact of fractional calculus in both applied and pure states of science. To obtain new refinements of well-known results, many researchers have intensively used the techniques of fractional calculus, as seen in [10, 12, 13, 30, 37] and the references cited therein. Sarikaya et al. [32] have proven a new version of Hermite-Hadamard's inequality in Riemann-Liouville fractional integral form as follows:

**Theorem 1.10** ([32]). *Let $\Psi : [\hbar_1, \hbar_2] \to \Re$ be a positive function with $0 \leq \hbar_1 < \hbar_2$ and $\Psi \in L_1[\hbar_1, \hbar_2]$. If $\Psi$ is a convex function on $[\hbar_1, \hbar_2]$, then*

$$\Psi\left(\frac{\hbar_1+\hbar_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(\hbar_2-\hbar_1)^{\alpha}}\left[ J^{\alpha}_{\hbar_1^{-}+}\Psi(\hbar_2) + \ J^{\alpha}_{\hbar_2^{-}}\Psi(\hbar_1)\right] \leq \frac{\Psi(\hbar_1)+\Psi(\hbar_2)}{2},$$

*holds for the Riemann-Liouville fractional integrals, where $\Gamma(\alpha)$ is Euler gamma function.*

For further results related to convex function, integral inequalities and Hermite-Hadamard type inequalities involving fractional integrals, see [1, 2, 35].

Set et al. [36] gave a new generalization of the above inequality for s-convex functions as:

**Theorem 1.11** ([36]). *Let $\Psi : [\hbar_1, \hbar_2] \to \Re$ be a positive function with $0 \leq \hbar_1 < \hbar_2$ and $\Psi \in L_1[\hbar_1, \hbar_2]$. If $\Psi$ is an s-convex mapping in the second sense on $[\hbar_1, \hbar_2]$, then*

$$\begin{aligned} 2^{s-1}\Psi\left(\frac{\hbar_1+\hbar_2}{2}\right) &\leq \frac{\Gamma(\Upsilon+1)}{(\hbar_2-\hbar_1)^{\Upsilon}}\left[\frac{J^{\Upsilon}_{\hbar_1^{-}+}\Psi(\hbar_2) + \ J^{\Upsilon}_{\hbar_2^{-}}\Psi(\hbar_1)}{2}\right] \\ &\leq \frac{\Psi(\hbar_1)+\Psi(\hbar_2)}{2}\left[\frac{1}{(\Upsilon+s)} + \beta(\Upsilon, s+1)\right], \end{aligned}$$

*holds for the Riemann-Liouville fractional integrals with $\Upsilon > 0$, and for all $s \in (0,1)$, where $\Gamma(\Upsilon)$ and $\beta$ are Euler gamma and Euler beta functions, respectively.*

In [21], Isćan obtained inequalities for differentiable convex mapping given as:

**Lemma 1.12.** *If $\Psi : I \subseteq \Re \to \Re$ is a differentiable function on $I^o$, such that $\Psi' \in L[\hbar_1, \hbar_2]$, for all $\hbar_1, \hbar_2 \in I$ with $\hbar_1 < \hbar_2$ and $\lambda, \tau \in [0,1]$, then*

$$\begin{aligned} &(1-\tau)\left(\lambda\Psi(\hbar_1) + (1-\lambda)\Psi(\hbar_2)\right) + \tau\Psi\left((1-\lambda)\hbar_1 + \lambda\hbar_2\right) \\ &\quad + \frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2}\Psi(\varpi)d\varpi \end{aligned}$$

$$= (\hbar_2 - \hbar_1) \left[ -\lambda^2 \int_0^1 (\ell - \tau) \Psi' (\ell \hbar_1 + (1 - \ell) [(1 - \lambda) \hbar_1 + \lambda \hbar_2]) \, d\ell \right.$$
$$\left. + (1 - \lambda)^2 \int_0^1 (\ell - \tau) \Psi' (\ell \hbar_2 + (1 - \ell) [(1 - \lambda) \hbar_1 + \lambda \hbar_2]) \, d\ell \right],$$

*holds.*

We pointed out the main inequalities proved in [21], as:

**Theorem 1.13.** *Let $\Psi : I \subseteq (0, \infty) \to \Re$ be a differentiable mapping on $I^o$, such that $\Psi' \in L[\hbar_1, \hbar_2]$,where $\hbar_1, \hbar_2 \in I^o$ with $\hbar_1 < \hbar_2$ and $\lambda, \tau \in [0, 1]$, If $\Psi'$ is s-convex on $[\hbar_1, \hbar_2]$, and $q \geq 1$, then*

$$\left| (1 - \tau) (\lambda \Psi(\hbar_1) + (1 - \lambda) \Psi(\hbar_2)) + \tau \Psi ((1 - \lambda) \hbar_1 + \lambda \hbar_2) \right.$$
$$\left. + \frac{1}{\hbar_2 - \hbar_1} \int_{\hbar_1}^{\hbar_2} \Psi(\varpi) d\varpi \right|$$
$$\leq (\hbar_2 - \hbar_1) B_3^{1 - \frac{1}{q}} (\tau) \left[ \lambda^2 [|\Psi(\hbar_1)|^q B_1 (\tau, s) + |\Psi(C)|^q B_2 (\tau, s)]^{\frac{1}{q}} \right.$$
$$\left. + (1 - \lambda)^2 [|\Psi(\hbar_2)|^q B_1 (\tau, s) + |\Psi(C)|^q B_2 (\tau, s)]^{\frac{1}{q}} \right],$$

*where*

$$B_1 (\tau, s) = \frac{2 \tau^{s+2}}{(s + 1) (s + 2)} - \frac{\tau}{s + 1} + \frac{1}{s + 2},$$
$$B_2 (\tau, s) = \frac{2 (1 - \tau)^{s+2}}{(s + 1) (s + 2)} - \frac{1 - \tau}{s + 1} + \frac{1}{s + 2},$$
$$B_3 (\tau) = \tau^2 - \tau + \frac{1}{2},$$

*and $C = (1 - \lambda) \hbar_1 + \lambda \hbar_2$.*

**Theorem 1.14.** *Let $\Psi : I \subseteq (0, \infty) \to \Re$ be a differentiable mapping on $I^o$, such that $\Psi' \in L[\hbar_1, \hbar_2]$, where $\hbar_1, \hbar_2 \in I^o$ with $\hbar_1 < \hbar_2$ and $\lambda, \tau \in [0, 1]$. If $\Psi'$ is s-convex on $[\hbar_1, \hbar_2]$, and $q > 1$, then*

$$\left| (1 - \tau) (\lambda \Psi(\hbar_1) + (1 - \lambda) \Psi(\hbar_2)) + \tau \Psi ((1 - \lambda) \hbar_1 + \lambda \hbar_2) \right.$$
$$\left. + \frac{1}{\hbar_2 - \hbar_1} \int_{\hbar_1}^{\hbar_2} \Psi(\varpi) d\varpi \right|$$
$$\leq (\hbar_2 - \hbar_1) \left( \frac{\tau^{p+1} + (1 - \tau)^{p+1}}{p + 1} \right)^{\frac{1}{p}} \left[ \lambda^2 \left( \frac{|\Psi(\hbar_1)|^q + |\Psi(C)|^q}{s + 1} \right)^{\frac{1}{q}} \right.$$

$$+(1-\lambda)^2\left(\frac{|\Psi(\hbar_2)|^q+|\Psi(C)|^q}{s+1}\right)^{\frac{1}{q}}\Bigg],$$

*where* $C=(1-\lambda)\,\hbar_1+\lambda\hbar_2$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

We come by the definition of semi convex-functions given by Alberti et al. [3].

**Definition 1.15** ([3])**.** Let the space $X_c^p(\hbar_1,\hbar_2)$ $(c\in\mathbb{R})$, $1\le p\le\infty$ denotes the space of all complex-valued Lebesgue measurable functions $\Psi$ on $(\hbar_1,\hbar_2)$, for which $\|\Psi\|_{X_c^p}<\infty$, where the norm $\|.\|_{X_c^p}$ is defined by

$$\|\Psi\|_{X_c^p}=\left(\int_{\hbar_1}^{\hbar_2}\mid\varpi^c\Psi(\varpi)\mid^p\frac{d\varpi}{\varpi}\right)^{\frac{1}{p}},\quad(1\le p<\infty),$$

and for $p=\infty$

$$\|\Psi\|_{X_c^\infty}=ess\sup_{\hbar_1\le\varpi\le\hbar_2}\left|\varpi^c\Psi(\varpi)\right|.$$

Over the last few decades, fractional calculus has emerged as an useful and effective mathematical tool, which has been demonstrated applications in different fields of science and engineering. Many researchers are now focusing on newly introduced fractional operators. Katugampola [24], introduced a new fractional integral operator, that generalizes the Hadamard fractional and Riemann-Liouville integrals into a single form as:

**Definition 1.16.** Let $[\hbar_1,\hbar_2]\subset\Re$ be a finite interval. Then the left and right side Katugampola fractional integrals of order $(\Upsilon>0)$ of $\Psi\in X_c^p(\hbar_1,\hbar_2)$ is given by:

$${}^{\rho}I_{\hbar_1^+}^{\Upsilon}\{\Psi(\varpi)\}=\frac{\eth^{1-\Upsilon}}{\Gamma(\Upsilon)}\int_{\hbar_1}^{\varpi}\frac{\ell^{\eth-1}}{(\varpi^{\eth}-\ell^{\eth})^{1-\Upsilon}}\Psi(\ell)d\ell,$$

and

$${}^{\eth}I_{\hbar_2^-}^{\Upsilon}\{\Psi(\varpi)\}=\frac{\eth^{1-\Upsilon}}{\Gamma(\Upsilon)}\int_{\varpi}^{\hbar_2}\frac{\ell^{\eth-1}}{(\ell^{\eth}-\varpi^{\eth})^{1-\Upsilon}}\Psi(\ell)d\ell,$$

with $\hbar_1<\varpi<\hbar_2$ and $\eth>0$, if the integral exists.

**Theorem 1.17** ([24])**.** *Let* $\Upsilon>0$ *and* $\eth>0$, *then for* $\varpi>\hbar_1$,

(1) $\lim\limits_{\eth\to1}{}^{\eth}I_{\hbar_1^+}^{\Upsilon}\Psi(\varpi)=J_{\hbar_1^+}^{\Upsilon}\Psi(\varpi)$,

(2) $\lim\limits_{\eth\to0^+}{}^{\eth}I_{\hbar_1^-}^{\Upsilon}\Psi(\varpi)=H_{\hbar_1^+}^{\Upsilon}\Psi(\varpi)$.

*For right-sided operators, related results also hold.*

The main aim of this paper is to present the generalization of different type of integral inequalities for the s-convex function in the second sense by using Katugampola fractional integrals. Some new identities with the first order derivatives are derived. In this article, our scheme and approach is used to get the main results and may generate new results in the fields of integral inequalities.

## 2. Generalized Integral Inequalities For s-Convex Function Via Katugampola Fractional Integrals

By using s-convex functions, Hermite-Hadamard type inequalities can be expressed in Katugampola fractional integral form as given by Yaldiz et al. [38] as follows:

**Theorem 2.1** ([38]). *Let $\Upsilon > 0$ and $\eth > 0$. Let $\Psi : [\hbar_1^\eth, \hbar_2^\eth] \to \Re$ be a function, such that $\Psi \in X_c^p(\hbar_1^\eth, \hbar_2^\eth)$, for all $\hbar_1^\eth, \hbar_2^\eth \in I$ and $\hbar_1 < \hbar_2$. If $\Psi$ is an s-convex function on $[\hbar_1, \hbar_2]$, then*

$$\begin{aligned}
&\frac{2^s}{\eth}\Psi\left(\frac{\hbar_1^\eth + \hbar_2^\eth}{2}\right) \\
&\quad \leq \frac{2^\Upsilon \eth^{\Upsilon-1}\Gamma(\Upsilon+1)}{\left(\hbar_2^\eth - \hbar_1^\eth\right)^\Upsilon}\left[{}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1^\eth+\hbar_2^\eth}{2}\right)^+}\Psi(\hbar_2^\eth) + {}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1^\eth+\hbar_2^\eth}{2}\right)^-}\Psi(\hbar_1^\eth)\right] \\
&\quad \leq 2^{-s}\left[\Psi\left(\hbar_1^\eth\right) + \Psi\left(\hbar_2^\eth\right)\right]\left[\frac{1}{\eth\left(\Upsilon+s\right)} + \frac{2^{(\Upsilon+s)}\beta_{\frac{1}{2}}\left(\Upsilon, s+1\right)}{\eth}\right],
\end{aligned}$$

*where*

$$\beta_{\frac{1}{2}}\left(\Upsilon, s+1\right) = \int_0^{\frac{1}{2}} \ell^{\Upsilon-1}\left(1-\ell\right)^s d\ell,$$

*holds for Katugampola fractional integrals, where $\Gamma(\Upsilon)$ is Euler gamma function.*

Let $\Psi : I \subseteq (0,\infty) \to \Re$ be a differentiable function on $(\hbar_1, \hbar_2)$, throughout this section we will take

$$\begin{aligned}
B_\Psi\left(\varpi^\eth, \mu^\eth; \Upsilon, \hbar_1^\eth, \hbar_2{}^\eth\right) &= \left(1-\mu^\eth\right)\left[\frac{\left(\varpi^\eth - \hbar_1^\eth\right)^\Upsilon + \left(\hbar_2^\eth - \varpi^\eth\right)^\Upsilon}{\eth\left(\hbar_2^\eth - \hbar_1^\eth\right)}\right]\Psi\left(\varpi^\eth\right) \\
&\quad + \mu^\eth\left[\frac{\left(\varpi^\eth - \hbar_1^\eth\right)^\Upsilon \Psi\left(\hbar_1^\eth\right) + \left(\hbar_2^\eth - \varpi^\eth\right)^\Upsilon \Psi\left(\hbar_2^\eth\right)}{\eth\left(\hbar_2^\eth - \hbar_1^\eth\right)}\right] \\
&\quad - \frac{\Gamma(\Upsilon+1)}{\hbar_2^\eth - \hbar_1^\eth}\left[{}^{\eth}I^{\Upsilon}_{\varpi^-}\Psi(\hbar_1) + {}^{\eth}I^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right],
\end{aligned}$$

where $\hbar_1^\eth, \hbar_2^\eth \in I$ with $\hbar_1 < \hbar_2$, $\Upsilon > 0$, $\mu \in [0,1]$, $\eth > 0$.

Now, we start with a new integral identity as follows:

**Lemma 2.2.** *Let $\Upsilon > 0$ and $\eth > 0$. Let $\Psi : I \subseteq (0,\infty) \to \Re$ be a differentiable function on $(\hbar_1, \hbar_2)$, such that $\Psi \in X_c^p(\hbar_1^\eth, \hbar_2^\eth)$, for all $\hbar_1^\eth, \hbar_2^\eth \in I$ and $\hbar_1 < \hbar_2$.*

(2.1)

$$
\begin{aligned}
&B_\Psi\left(\varpi^\eth, \mu^\eth; \Upsilon, \hbar_1^\eth, \hbar_2{}^\eth\right) \\
&= \frac{\left(\varpi^\eth - \hbar_1^\eth\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^\eth - \hbar_1^\eth\right)} \int_0^1 \left(\ell^{\eth\Upsilon} - \mu^\eth\right) \ell^{\eth-1} \Psi'\left(\ell^\eth \varpi^\eth + (1-\ell^\eth)\hbar_1^\eth\right) d\ell \\
&\quad + \frac{\left(\hbar_2^\eth - \varpi^\eth\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^\eth - \hbar_1^\eth\right)} \int_0^1 \left(\mu^\eth - \ell^{\eth\Upsilon}\right) \ell^{\eth-1} \Psi'\left(\ell^\eth \varpi^\eth + (1-\ell^\eth)\hbar_2^\eth\right) d\ell,
\end{aligned}
$$

*holds.*

*Proof.* By using integration by parts and changing the variable, for $\varpi^\eth \neq \hbar_1^\eth$, we can state

$$
\begin{aligned}
&\int_0^1 \left(\ell^{\eth\Upsilon} - \mu^\eth\right) \ell^{\eth-1} \Psi'\left(\ell^\eth \varpi^\eth + (1-\ell^\eth)\hbar_1^\eth\right) d\ell \\
&= \left(\ell^{\eth\Upsilon} - \mu^\eth\right) \Psi\left(\frac{\ell^\eth \varpi^\eth + (1-\ell^\eth)\hbar_1^\eth}{\eth\left(\varpi^\eth - \hbar_1^\eth\right)}\right)\Bigg|_0^1 \\
&\quad - \Upsilon \int_0^1 \left(\ell^\eth\right)^{\Upsilon-1} \ell^{\eth-1} \Psi'\left(\frac{\ell^\eth \varpi^\eth + (1-\ell^\eth)\hbar_1^\eth}{\varpi^\eth - \hbar_1^\eth}\right) d\ell \\
&= \frac{1}{\eth\left(\varpi^\eth - \hbar_1^\eth\right)} \left[\left(1-\mu^\eth\right)\Psi\left(\varpi^\eth\right) + \mu^\eth \Psi\left(\hbar_1^\eth\right)\right] \\
&\quad - \frac{\Upsilon}{\left(\varpi^\eth - \hbar_1^\eth\right)^{\Upsilon+1}} \int_{\hbar_1^\eth}^{\varpi^\eth} \frac{z^{\eth-1}}{\left(z^\eth - \hbar_1^\eth\right)^{1-\Upsilon}} \Psi\left(z^\eth\right) dz \\
&= \frac{1}{\eth\left(\varpi^\eth - \hbar_1^\eth\right)} \left[\left(1-\mu^\eth\right)\Psi\left(\varpi^\eth\right) + \mu^\eth \Psi\left(\hbar_1^\eth\right)\right] \\
&\quad - \frac{\Gamma(\Upsilon+1)}{\eth^{1-\Upsilon}\left(\varpi^\eth - \hbar_1^\eth\right)^{\Upsilon+1}} \, {}^\eth I_{\varpi-}^\Upsilon \Psi(\hbar_1).
\end{aligned} \tag{2.2}
$$

For $\varpi^\eth \neq \hbar_2^\eth$, we have

$$
\begin{aligned}
&\int_0^1 \left(\mu^\eth - \ell^{\eth\Upsilon}\right) \ell^{\eth-1} \Psi'\left(\ell^\eth \varpi^\eth + (1-\ell^\eth)\hbar_2^\eth\right) d\ell \\
&= \left(\mu^\eth - \ell^{\eth\Upsilon}\right) \Psi\left(\frac{\ell^\eth \varpi^\eth + (1-\ell^\eth)\hbar_2^\eth}{\eth\left(\varpi^\eth - \hbar_2^\eth\right)}\right)\Bigg|_0^1
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
&+\Upsilon\int_0^1\left(\ell^{\eth}\right)^{\Upsilon-1}\ell^{\eth-1}\Psi'\left(\frac{\ell^{\eth}\varpi^{\eth}+(1-\ell^{\eth})\hbar_2^{\eth}}{\varpi^{\eth}-\hbar_2^{\eth}}\right)d\ell\\
&=\frac{1}{\eth\left(\hbar_2^{\eth}-\varpi^{\eth}\right)}\left[\left(1-\mu^{\eth}\right)\Psi\left(\varpi^{\eth}\right)+\mu^{\eth}\Psi\left(\hbar_2^{\eth}\right)\right]\\
&-\frac{\Upsilon}{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}\int_{\hbar_1^{\eth}}^{\varpi^{\eth}}\frac{z^{\eth-1}}{\left(\hbar_2^{\eth}-z^{\eth}\right)^{1-\Upsilon}}\Psi\left(z^{\eth}\right)dz\\
&=\frac{1}{\eth\left(\hbar_2^{\eth}-\varpi^{\eth}\right)}\left[\left(1-\mu^{\eth}\right)\Psi\left(\varpi^{\eth}\right)+\mu^{\eth}\Psi\left(\hbar_2^{\eth}\right)\right]\\
&-\frac{\Gamma(\Upsilon+1)}{\eth^{1-\Upsilon}\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}\,{}^{\eth}I_{\varpi+}^{\Upsilon}\Psi(\hbar_2).
\end{aligned}
$$

Multiplying both sides of (2.2) and (2.3) by $\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}$ and $\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}$ respectively and adding the resulting identities, we get our desired result.

For $\varpi^{\eth}=\hbar_1^{\eth}$ and $\varpi^{\eth}=\hbar_2^{\eth}$, the identities

$$
\begin{aligned}
&B_{\Psi}\left(\varpi^{\eth},\mu^{\eth};\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\\
&\quad=\eth^{1-\Upsilon}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}\int_0^1\left(\mu^{\eth}-\ell^{\eth\Upsilon}\right)\ell^{\eth-1}\Psi'\left(\ell^{\eth}\hbar_1^{\eth}+(1-\ell^{\eth})\hbar_2^{\eth}\right)d\ell,
\end{aligned}
$$

and

$$
\begin{aligned}
&B_{\Psi}\left(\varpi^{\eth},\mu^{\eth};\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\\
&\quad=\eth^{1-\Upsilon}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}\int_0^1\left(\ell^{\eth\Upsilon}-\mu^{\eth}\right)\ell^{\eth-1}\Psi'\left(\ell^{\eth}\hbar_2^{\eth}+(1-\ell^{\eth})\hbar_1^{\eth}\right)d\ell,
\end{aligned}
$$

can be proved easily by changing the variables and implementing integration by parts to the integrals in the right side. □

**Remark 2.3.** By taking $\eth=1$ in Lemma 2.2, we get the Lemma 2.1 in [20].

**Remark 2.4.** By taking $\eth=\Upsilon=1,\mu=1-\tau$ and $\varpi=(1-\lambda)\hbar_1+\lambda\hbar_2$ in Lemma 2.2, the equality (2.1) becomes the Lemma 1.12.

**Theorem 2.5.** *Let $\Upsilon>0$ and $\eth>0$. Let $\Psi:I\subseteq(0,\infty)\to\Re$ be a differentiable function, such that $\Psi\in X_c^p(\hbar_1^{\eth},\hbar_2^{\eth})$, for all $\hbar_1^{\eth},\hbar_2^{\eth}\in I$, $\mu\in[0,1]$ and $\hbar_1<\hbar_2$. If $|\Psi'|$ is an s-convex function on $[\hbar_1,\hbar_2]$, then*

$$
\begin{aligned}
&\left|B_{\Psi}\left(\varpi^{\eth},\mu^{\eth};\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right|\\
&\quad\leq\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left|\Psi'\left(\varpi^{\eth}\right)\right|T_1\left(\Upsilon;\mu^{\eth},s\right)+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|T_2\left(\Upsilon;\mu^{\eth},s\right)\right]
\end{aligned}
$$

$$+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left|\Psi'\left(\varpi^{\eth}\right)\right|T_1\left(\Upsilon;\mu^{\eth},s\right)+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|T_2\left(\Upsilon;\mu^{\eth},s\right)\right],$$

*where*

$$T_1\left(\Upsilon;\mu^{\eth},s\right)=\frac{2\Upsilon\mu^{\frac{\eth(\Upsilon+s+1)}{\Upsilon}}+s+1}{\eth\left(s+1\right)\left(\Upsilon+s+1\right)}-\frac{\mu^{\eth}}{\eth\left(s+1\right)},$$

*and*

$$T_2\left(\Upsilon;\mu^{\eth},s\right)=\frac{\mu^{\eth}}{\eth}\left[\frac{1-2\left(1-\mu^{\frac{\eth}{\Upsilon}}\right)^{(s+1)}}{s+1}\right]+\beta\left(\eth\left(\Upsilon+1\right),\eth\left(s+1\right)\right)$$
$$-2\beta\left(\mu^{\frac{\eth}{\Upsilon}},\eth\left(\Upsilon+1\right),\eth\left(s+1\right)\right).$$

*Also*

$$\beta\left(\Upsilon,\hbar_1,\hbar_2\right)=\int_0^{\Upsilon}\ell^{\hbar_1-1}\left(1-\ell\right)^{\hbar_2-1}d\ell,\quad 0<\Upsilon\leq 1,\,\hbar_1,\hbar_2>0,$$

*is incomplete Beta function.*

*Proof.* From Lemma 2.2 and by using the property of modulus and s-convexity of $\left|\Psi'\right|$, we obtain

(2.4)

$$\begin{aligned}
&\left|B_{\Psi}\left(\varpi^{\eth},\mu^{\eth};\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right|\\
&\leq\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\int_0^1\left|\ell^{\eth\Upsilon}-\mu^{\eth}\right|\left|\ell^{\eth-1}\right|\left|\Psi'\left(\ell^{\eth}\varpi^{\eth}+(1-\ell^{\eth})\hbar_1^{\eth}\right)\right|d\ell\\
&\quad+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\int_0^1\left|\mu^{\eth}-\ell^{\eth\Upsilon}\right|\left|\ell^{\eth-1}\right|\left|\Psi'\left(\ell^{\eth}\varpi^{\eth}+(1-\ell^{\eth})\hbar_2^{\eth}\right)\right|d\ell\\
&\leq\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\int_0^1\left|\ell^{\eth\Upsilon}-\mu^{\eth}\right|\left|\ell^{\eth-1}\right|\left[\ell^{\eth s}\left|\Psi'\left(\varpi^{\eth}\right)\right|+(1-\ell^{\eth})^s\left|\Psi'\left(\hbar_1^{\eth}\right)\right|\right]d\ell\\
&\quad+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\int_0^1\left|\mu^{\eth}-\ell^{\eth\Upsilon}\right|\left|\ell^{\eth-1}\right|\left[\ell^{\eth s}\left|\Psi'\left(\varpi^{\eth}\right)\right|+(1-\ell^{\eth})^s\left|\Psi'\left(\hbar_2^{\eth}\right)\right|\right]d\ell\\
&\leq\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left|\Psi'\left(\varpi^{\eth}\right)\right|T_1\left(\Upsilon;\mu^{\eth},s\right)+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|T_2\left(\Upsilon;\mu^{\eth},s\right)\right]\\
&\quad+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left|\Psi'\left(\varpi^{\eth}\right)\right|T_1\left(\Upsilon;\mu^{\eth},s\right)+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|T_2\left(\Upsilon;\mu^{\eth},s\right)\right].
\end{aligned}$$

Calculating $T_1\left(\Upsilon;\mu^{\eth},s\right)$ and $T_2\left(\Upsilon;\mu^{\eth},s\right)$, we have

$$\int_0^1\left|\ell^{\eth\Upsilon}-\mu^{\eth}\right|\left|\ell^{\eth-1}\right|\ell^{\eth s}d\ell \tag{2.5}$$

$$= \int_0^{\mu^{\frac{\eth}{\Upsilon}}} \left(\mu^{\eth} - \ell^{\eth\Upsilon}\right) \ell^{\eth-1}\ell^{\eth s} d\ell + \int_{\mu^{\frac{\eth}{\Upsilon}}}^1 \left(\ell^{\eth\Upsilon} - \mu^{\eth}\right) \ell^{\eth-1}\ell^{\eth s} d\ell$$

$$= \frac{2\Upsilon\mu^{\frac{\eth(\Upsilon+s+1)}{\Upsilon}} + s + 1}{\eth\,(s+1)\,(\Upsilon+s+1)} - \frac{\mu^{\eth}}{\eth\,(s+1)}.$$

Similarly,

(2.6)

$$\int_0^1 \left|\ell^{\eth\Upsilon} - \mu^{\eth}\right| \left|\ell^{\eth-1}\right| (1-\ell^{\eth})^s d\ell$$

$$= \int_0^{\mu^{\frac{\eth}{\Upsilon}}} \left(\mu^{\eth} - \ell^{\eth\Upsilon}\right) \ell^{\eth-1}(1-\ell^{\eth})^s d\ell + \int_{\mu^{\frac{\eth}{\Upsilon}}}^1 \left(\ell^{\eth\Upsilon} - \mu^{\eth}\right) \ell^{\eth-1}(1-\ell^{\eth})^s d\ell$$

$$= \mu^{\eth} \int_0^{\mu^{\frac{\eth}{\Upsilon}}} \ell^{\eth-1}(1-\ell^{\eth})^s d\ell - \int_0^{\mu^{\frac{\eth}{\Upsilon}}} \ell^{\eth\Upsilon}\ell^{\eth-1}(1-\ell^{\eth})^s d\ell$$

$$+ \int_{\mu^{\frac{\eth}{\Upsilon}}}^1 \ell^{\eth\Upsilon}\ell^{\eth-1}(1-\ell^{\eth})^s d\ell - \mu^{\eth} \int_{\mu^{\frac{\eth}{\Upsilon}}}^1 \ell^{\eth-1}(1-\ell^{\eth})^s d\ell$$

$$= \frac{\mu^{\eth}}{\eth}\left[\frac{1-2\left(1-\mu^{\frac{\eth}{\Upsilon}}\right)^{(s+1)}}{s+1}\right] + \int_0^1 \ell^{\eth\Upsilon}\ell^{\eth-1}(1-\ell^{\eth})^s d\ell$$

$$- 2\int_0^{\mu^{\frac{\eth}{\Upsilon}}} \ell^{\eth\Upsilon}\ell^{\eth-1}(1-\ell^{\eth})^s d\ell$$

$$= \frac{\mu^{\eth}}{\eth}\left[\frac{1-2\left(1-\mu^{\frac{\eth}{\Upsilon}}\right)^{(s+1)}}{s+1}\right] + \beta\left(\eth\,(\Upsilon+1), \eth\,(s+1)\right)$$

$$- 2\beta\left(\mu^{\frac{\eth}{\Upsilon}}, \eth\,(\Upsilon+1), \eth\,(s+1)\right).$$

If we put (2.5) and (2.6) in the inequality (2.4), we get our desired result. The proof is completed. □

**Corollary 2.6.** *If we put* $\eth = 1$ *in Theorem 2.5, we get the inequality for Riemann-Liouville fractional integrals*

$$\left|(1-\mu)\left[\frac{(\varpi-\hbar_1)^{\Upsilon} + (\hbar_2-\varpi)^{\Upsilon}}{\hbar_2-\hbar_1}\right]\Psi(\varpi)\right.$$

$$\left. + \mu\left[\frac{(\varpi-\hbar_1)^{\Upsilon}\Psi(\hbar_1) + (\hbar_2-\varpi)^{\Upsilon}\Psi(\hbar_2)}{\hbar_2-\hbar_1}\right]\right.$$

$$
\begin{aligned}
&- \frac{\Gamma(\Upsilon+1)}{\hbar_2-\hbar_1}\left[J^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ J^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\Bigg| \\
&\leq \frac{(\varpi-\hbar_1)^{\Upsilon+1}}{(\hbar_2-\hbar_1)}\left[\left|\Psi'(\varpi)\right|T_1(\Upsilon;\mu,s)+\left|\Psi'(\hbar_1)\right|T_2(\Upsilon;\mu,s)\right] \\
&\quad+\frac{(\hbar_2-\varpi)^{\Upsilon+1}}{(\hbar_2-\hbar_1)}\left[\left|\Psi'(\varpi)\right|T_1(\Upsilon;\mu,s)+\left|\Psi'(\hbar_2)\right|T_2(\Upsilon;\mu,s)\right].
\end{aligned}
$$

**Corollary 2.7.** *If we put* $\eth=\Upsilon=1,\mu=1-\tau$ *and* $\varpi=(1-\lambda)\hbar_1+\lambda\hbar_2$ *in Theorem 2.5, we get*

$$
\begin{aligned}
&\Bigg|(1-\tau)\left(\lambda\Psi(\hbar_1)+(1-\lambda)\Psi(\hbar_2)\right)+\tau\Psi\left((1-\lambda)\hbar_1+\lambda\hbar_2\right) \\
&\qquad-\frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2}\Psi(\varpi)d\varpi\Bigg| \\
&\leq(\hbar_2-\hbar_1)\left[\lambda^2\left(\left|\Psi'(\hbar_1)\right|T_1(\tau,s)+\left|\Psi'(C)\right|T_2(\tau,s)\right)\right. \\
&\quad\left.+(1-\lambda)^2\left(\left|\Psi'(\hbar_2)\right|T_1(\tau,s)+\left|\Psi'(C)\right|T_2(\tau,s)\right)\right],
\end{aligned}
$$

*and* $C=(1-\lambda)\hbar_1+\lambda\hbar_2$.

**Corollary 2.8.** *If we put* $s=1$ *in Theorem 2.5, we get*

$$
\begin{aligned}
&\Bigg|\left(1-\mu^{\eth}\right)\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right]\Psi\left(\varpi^{\eth}\right) \\
&\quad+\mu^{\eth}\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_1^{\eth}\right)+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_2^{\eth}\right)}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right] \\
&\quad-\frac{\Gamma(\Upsilon+1)}{\hbar_2^{\eth}-\hbar_1^{\eth}}\left[\ {}^{\eth}I^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ {}^{\eth}I^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\Bigg| \\
&\leq\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left|\Psi'(\varpi)\right|T_1\left(\Upsilon;\mu^{\eth},1\right)+\left|\Psi'(\hbar_1)\right|T_2\left(\Upsilon;\mu^{\eth},1\right)\right] \\
&\quad+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left|\Psi'(\varpi)\right|T_1\left(\Upsilon;\mu^{\eth},1\right)+\left|\Psi'(\hbar_2)\right|T_2\left(\Upsilon;\mu^{\eth},1\right)\right].
\end{aligned}
$$

**Corollary 2.9.** *If we put* $\eth=s=1$ *in Theorem 2.5, we get*

$$
\begin{aligned}
&\Bigg|(1-\mu)\left[\frac{(\varpi-\hbar_1)^{\Upsilon}+(\hbar_2-\varpi)^{\Upsilon}}{\hbar_2-\hbar_1}\right]\Psi(\varpi) \\
&\quad+\mu\left[\frac{(\varpi-\hbar_1)^{\Upsilon}\Psi(\hbar_1)+(\hbar_2-\varpi)^{\Upsilon}\Psi(\hbar_2)}{\hbar_2-\hbar_1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\Gamma(\Upsilon+1)}{\hbar_2-\hbar_1}\left[J^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ J^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\Bigg| \\
&\leq \frac{(\varpi-\hbar_1)^{\Upsilon+1}}{(\hbar_2-\hbar_1)}\left[\left|\Psi'(\varpi)\right| T_1\left(\Upsilon;\mu^{\eth},1\right)+\left|\Psi'(\hbar_1)\right| T_2\left(\Upsilon;\mu^{\eth},1\right)\right] \\
&\quad+\frac{(\hbar_2-\varpi)^{\Upsilon+1}}{(\hbar_2-\hbar_1)}\left[\left|\Psi'(\varpi)\right| T_1\left(\Upsilon;\mu^{\eth},1\right)+\left|\Psi'(\hbar_2)\right| T_2\left(\Upsilon;\mu^{\eth},1\right)\right].
\end{aligned}
$$

**Corollary 2.10.** *If we put $\varpi^{\eth}=\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}$ and $\mu^{\eth}=\frac{1}{3}$ in Theorem 2.5, we get the Simpson type inequality for Katugampola fractional integrals*

$$
\begin{aligned}
&\left|\frac{2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon-1}}\left(B_{\Psi}\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2},\frac{1}{3};\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right)\right| \\
&=\left|\frac{1}{6}\left[\Psi\left(\hbar_1^{\eth}\right)+4\Psi\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)+\Psi\left(\hbar_2^{\eth}\right)\right]\right. \\
&\quad\left.-\frac{\Gamma(\Upsilon+1)2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}}\left[{}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^-}\Psi(\hbar_1)+\ {}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^+}\Psi(\hbar_2)\right]\right| \\
&\leq\frac{\hbar_2^{\eth}-\hbar_1^{\eth}}{4\eth^{\Upsilon-1}}\left[\ \left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right| T_1\left(\Upsilon;\frac{1}{3},s\right)+\left|\Psi'\left(\hbar_1^{\eth}\right)\right| T_2\left(\Upsilon;\frac{1}{3},s\right)\right. \\
&\quad\left.+\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right| T_1\left(\Upsilon;\frac{1}{3},s\right)+\left|\Psi'\left(\hbar_2^{\eth}\right)\right| T_2\left(\Upsilon;\frac{1}{3},s\right)\right].
\end{aligned}
$$

**Corollary 2.11.** *If we put $\eth=1$ in Corollary 2.10, we get the Simpson type inequality for Riemann-Liouville fractional integrals*

$$
\begin{aligned}
&\left|\frac{2^{\Upsilon-1}}{(\hbar_2-\hbar_1)^{\Upsilon-1}}\left(B_{\Psi}\left(\frac{\hbar_1+\hbar_2}{2},\frac{1}{3};\Upsilon,\hbar_1,\hbar_2\right)\right)\right| \\
&=\left|\frac{1}{6}\left[\Psi(\hbar_1)+4\Psi\left(\frac{\hbar_1+\hbar_2}{2}\right)+\Psi(\hbar_2)\right]\right. \\
&\quad\left.-\frac{\Gamma(\Upsilon+1)2^{\Upsilon-1}}{(\hbar_2-\hbar_1)^{\Upsilon}}\left[J^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^-}\Psi(\hbar_1)+\ J^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^+}\Psi(\hbar_2)\right]\right| \\
&\leq\frac{\hbar_2-\hbar_1}{4}\left[\ \left|\Psi'\left(\frac{\hbar_1+\hbar_2}{2}\right)\right| T_1\left(\Upsilon;\frac{1}{3},s\right)+\left|\Psi'(\hbar_1)\right| T_2\left(\Upsilon;\frac{1}{3},s\right)\right. \\
&\quad\left.+\left|\Psi'\left(\frac{\hbar_1+\hbar_2}{2}\right)\right| T_1\left(\Upsilon;\frac{1}{3},s\right)+\left|\Psi'(\hbar_2)\right| T_2\left(\Upsilon;\frac{1}{3},s\right)\right].
\end{aligned}
$$

**Corollary 2.12.** *If we put* $\varpi^{\eth}=\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}$ *and* $\mu^{\eth}=0$ *in Theorem 2.5, we get the midpoint inequality for Katugampola fractional integrals*

$$\begin{aligned}
&\left|\frac{2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon-1}}\left(B_{\Psi}\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2},0;\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right)\right| \\
&\quad=\left|\Psi\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)-\frac{\Gamma(\Upsilon+1)2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}}\left[{}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^{-}}\Psi(\hbar_1)+{}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^{+}}\Psi(\hbar_2)\right]\right| \\
&\quad\leq\frac{\hbar_2^{\eth}-\hbar_1^{\eth}}{4\eth^{\Upsilon-1}}\left[\frac{\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|}{\eth\left(\Upsilon+s+1\right)}+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|\beta\left(\eth\left(\Upsilon+1\right),\eth\left(s+1\right)\right)\right. \\
&\qquad\left.+\frac{\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|}{\eth\left(\Upsilon+s+1\right)}+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|\beta\left(\eth\left(\Upsilon+1\right),\eth\left(s+1\right)\right)\right].
\end{aligned}$$

**Corollary 2.13.** *If we put* $\eth=1$ *in Corollary 2.12, we get the midpoint inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left|\frac{2^{\Upsilon-1}}{(\hbar_2-\hbar_1)^{\Upsilon-1}}\left(B_{\Psi}\left(\frac{\hbar_1+\hbar_2}{2},0;\Upsilon,\hbar_1,\hbar_2\right)\right)\right| \\
&\quad=\left|\Psi\left(\frac{\hbar_1+\hbar_2}{2}\right)-\frac{\Gamma(\Upsilon+1)2^{\Upsilon-1}}{(\hbar_2-\hbar_1)^{\Upsilon}}\left[J^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^{-}}\Psi(\hbar_1)+J^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^{+}}\Psi(\hbar_2)\right]\right| \\
&\quad\leq\frac{\hbar_2-\hbar_1}{4}\left[\frac{\left|\Psi'\left(\frac{\hbar_1+\hbar_2}{2}\right)\right|}{\Upsilon+s+1}+\left|\Psi'\left(\hbar_1\right)\right|\beta\left(\Upsilon+1,s+1\right)\right. \\
&\qquad\left.+\frac{\left|\Psi'\left(\frac{\hbar_1+\hbar_2}{2}\right)\right|}{\Upsilon+s+1}+\left|\Psi'\left(\hbar_2\right)\right|\beta\left(\Upsilon+1,s+1\right)\right].
\end{aligned}$$

**Corollary 2.14.** *If we put* $\mu^{\eth}=1$ *in Theorem 2.5, we get the trapezoid inequality for Katugampola fractional integrals*

$$\begin{aligned}
&\left|B_{\Psi}\left(\varpi^{\eth},1;\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right| \\
&\quad=\left|\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_1^{\eth}\right)+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_2^{\eth}\right)}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right. \\
&\qquad\left.-\frac{\Gamma(\Upsilon+1)}{\hbar_2^{\eth}-\hbar_1^{\eth}}\left[{}^{\eth}I^{\Upsilon}_{\varpi^{-}}\Psi(\hbar_1)+{}^{\eth}I^{\Upsilon}_{\varpi^{+}}\Psi(\hbar_2)\right]\right| \\
&\quad\leq\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\frac{\Upsilon\left|\Psi'\left(\varpi^{\eth}\right)\right|}{\eth\left(s+1\right)\left(\Upsilon+s+1\right)}\right. \\
&\qquad\left.+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|\left(\frac{1}{\eth\left(s+1\right)}-\beta\left(\eth\left(\Upsilon+1\right),\eth\left(s+1\right)\right)\right)\right]
\end{aligned}$$

$$+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\frac{\Upsilon\left|\Psi'\left(\varpi^{\eth}\right)\right|}{\eth\left(s+1\right)\left(\Upsilon+s+1\right)}\right.$$
$$\left.+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|\left(\frac{1}{\eth\left(s+1\right)}-\beta\left(\eth\left(\Upsilon+1\right),\eth\left(s+1\right)\right)\right)\right].$$

**Corollary 2.15.** *If we put* $\eth=1$ *in Corollary 2.14, we get the trapezoid inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}\left|B_{\Psi}\left(\varpi,1;\Upsilon,\hbar_1,\hbar_2\right)\right| &= \left|\frac{\left(\varpi-\hbar_1\right)^{\Upsilon}\Psi\left(\hbar_1\right)+\left(\hbar_2-\varpi\right)^{\Upsilon}\Psi\left(\hbar_2\right)}{\hbar_2-\hbar_1}\right.\\ &\quad\left.-\frac{\Gamma(\Upsilon+1)}{\hbar_2-\hbar_1}\left[\ J_{\varpi^-}^{\Upsilon}\Psi(\hbar_1)+\ J_{\varpi^+}^{\Upsilon}\Psi(\hbar_2)\right]\right|\\ &\leq \frac{\left(\varpi-\hbar_1\right)^{\Upsilon+1}}{\hbar_2-\hbar_1}\left[\frac{\Upsilon\left|\Psi'\left(\varpi\right)\right|}{\left(s+1\right)\left(\Upsilon+s+1\right)}\right.\\ &\quad\left.+\left|\Psi'\left(\hbar_1\right)\right|\left(\frac{1}{s+1}-\beta\left(\Upsilon+1,s+1\right)\right)\right]\\ &\quad+\frac{\left(\hbar_2-\varpi\right)^{\Upsilon+1}}{\left(\hbar_2-\hbar_1\right)}\left[\frac{\Upsilon\left|\Psi'\left(\varpi\right)\right|}{\left(s+1\right)\left(\Upsilon+s+1\right)}\right.\\ &\quad\left.+\left|\Psi'\left(\hbar_2\right)\right|\left(\frac{1}{s+1}-\beta\left(\Upsilon+1,s+1\right)\right)\right].\end{aligned}$$

**Corollary 2.16.** *Under the assumptions of Theorem 2.5, if* $\left|\Psi'\left(\varpi^{\eth}\right)\right|\leq M$ *for all* $\varpi^{\eth}\in\left[\hbar_1^{\eth},\hbar_2^{\eth}\right]$ *and* $\mu^{\eth}=0$*, then we get the Ostrowski type inequality for Katugampola fractional integral*

$$\begin{aligned}&\left|\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right]\Psi\left(\varpi^{\eth}\right)\right.\\ &\qquad\left.-\frac{\Gamma(\Upsilon+1)}{\hbar_2^{\eth}-\hbar_1^{\eth}}\left[\ {}^{\eth}I_{\varpi^-}^{\Upsilon}\Psi(\hbar_1)+\ {}^{\eth}I_{\varpi^+}^{\Upsilon}\Psi(\hbar_2)\right]\right|\\ &\quad\leq M\left[\frac{1}{\eth\left(\Upsilon+s+1\right)}+\beta\left(\eth\left(\Upsilon+1\right),\eth\left(s+1\right)\right)\right]\\ &\qquad\times\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right].\end{aligned}$$

**Corollary 2.17.** *If we put* $\eth=1$ *in Corollary 2.16, we get the Ostrowski type inequality for Riemann-Liouville fractional integrals*

$$\left|\left[\frac{\left(\varpi-\hbar_1\right)^{\Upsilon}+\left(\hbar_2-\varpi\right)^{\Upsilon}}{\hbar_2-\hbar_1}\right]\Psi\left(\varpi\right)-\frac{\Gamma(\Upsilon+1)}{\hbar_2-\hbar_1}\left[\ J_{\varpi^-}^{\Upsilon}\Psi(\hbar_1)+\ J_{\varpi^+}^{\Upsilon}\Psi(\hbar_2)\right]\right|$$

$$\leq M\left[\frac{1}{\Upsilon+s+1}+\beta\left(\Upsilon+1,s+1\right)\right]\left[\frac{\left(\varpi-\hbar_1\right)^{\Upsilon+1}+\left(\hbar_2-\varpi\right)^{\Upsilon+1}}{\hbar_2-\hbar_1}\right].$$

**Theorem 2.18.** *Let $\Upsilon>0$ and $\eth>0$. Let $\Psi: I\subseteq(0,\infty)\to\Re$ be a differentiable function, such that $\Psi\in X_c^p(\hbar_1^\eth,\hbar_2^\eth)$, for all $\hbar_1^\eth,\hbar_2^\eth\in I$ and $\hbar_1<\hbar_2$. If $|\Psi'|^q$ is an s-convex function on $[\hbar_1,\hbar_2]$ for some fixed $q\geq 1$, then for $\varpi\in[\hbar_1,\hbar_2]$, and $\mu\in[0,1]$, the following inequality for fractional integrals holds*

(2.7)

$$\begin{aligned}
&\left|B_\Psi\left(\varpi^\eth,\mu^\eth;\Upsilon,\hbar_1^\eth,{\hbar_2}^\eth\right)\right| \\
&\quad\leq T_3^{1-\frac{1}{q}}\left(\Upsilon;\mu^\eth\right) \\
&\qquad\times\left\{\frac{\left(\varpi^\eth-\hbar_1^\eth\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^\eth-\hbar_1^\eth\right)}\left[\left|\Psi'\left(\varpi^\eth\right)\right|^q T_1\left(\Upsilon;\mu^\eth,s\right)+\left|\Psi'\left(\hbar_1^\eth\right)\right|^q T_2\left(\Upsilon;\mu^\eth,s\right)\right]^{\frac{1}{q}}\right. \\
&\qquad\left.+\frac{\left(\hbar_2^\eth-\varpi^\eth\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^\eth-\hbar_1^\eth\right)}\left[\left|\Psi'\left(\varpi^\eth\right)\right|^q T_1\left(\Upsilon;\mu^\eth,s\right)+\left|\Psi'\left(\hbar_2^\eth\right)\right|^q T_2\left(\Upsilon;\mu^\eth,s\right)\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

*where*

$$T_3\left(\Upsilon;\mu^\eth\right)=\frac{2\Upsilon\mu^{\frac{\eth(\Upsilon+1)}{\Upsilon}}+1}{\eth\left(\Upsilon+1\right)}-\frac{\mu^\eth}{\eth}.$$

*Proof.* From Lemma 2.2 and by using the property of modulus, the power mean inequality and s-convexity of $|\Psi'|^q$, we obtain

(2.8)

$$\begin{aligned}
&\left|B_\Psi\left(\varpi^\eth,\mu^\eth;\Upsilon,\hbar_1^\eth,{\hbar_2}^\eth\right)\right| \\
&\quad\leq\frac{\left(\varpi^\eth-\hbar_1^\eth\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^\eth-\hbar_1^\eth\right)}\int_0^1\left|\ell^{\eth\Upsilon}-\mu^\eth\right|\left|\ell^{\eth-1}\right|\left|\Psi'\left(\ell^\eth\varpi^\eth+(1-\ell^\eth)\hbar_1^\eth\right)\right|d\ell \\
&\qquad+\frac{\left(\hbar_2^\eth-\varpi^\eth\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^\eth-\hbar_1^\eth\right)}\int_0^1\left|\mu^\eth-\ell^{\eth\Upsilon}\right|\left|\ell^{\eth-1}\right|\left|\Psi'\left(\ell^\eth\varpi^\eth+(1-\ell^\eth)\hbar_2^\eth\right)\right|d\ell \\
&\quad\leq\frac{\left(\varpi^\eth-\hbar_1^\eth\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^\eth-\hbar_1^\eth\right)}\left(\int_0^1\left|\ell^{\eth\Upsilon}-\mu^\eth\right|\left|\ell^{\eth-1}\right|d\ell\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\int_0^1\left|\ell^{\eth\Upsilon}-\mu^\eth\right|\left|\ell^{\eth-1}\right|\left|\Psi'\left(\ell^\eth\varpi^\eth+(1-\ell^\eth)\hbar_1^\eth\right)\right|^q d\ell\right)^{\frac{1}{q}} \\
&\qquad+\frac{\left(\hbar_2^\eth-\varpi^\eth\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^\eth-\hbar_1^\eth\right)}\left(\int_0^1\left|\mu^\eth-\ell^{\eth\Upsilon}\right|\left|\ell^{\eth-1}\right|d\ell\right)^{1-\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_0^1 \left|\mu^{\eth}-\ell^{\eth\Upsilon}\right|\left|\ell^{\eth-1}\right|\left|\Psi'\left(\ell^{\eth}\varpi^{\eth}+(1-\ell^{\eth})\hbar_2^{\eth}\right)\right|^q d\ell\right)^{\frac{1}{q}}\\
&\leq \frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left(\int_0^1 \left|\ell^{\eth\Upsilon}-\mu^{\eth}\right|\left|\ell^{\eth-1}\right| d\ell\right)^{1-\frac{1}{q}}\\
&\times \left(\int_0^1 \left|\ell^{\eth\Upsilon}-\mu^{\eth}\right|\left|\ell^{\eth-1}\right|\left[\ell^{\eth s}\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+(1-\ell^{\eth})^s\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q\right] d\ell\right)^{\frac{1}{q}}\\
&+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left(\int_0^1 \left|\mu^{\eth}-\ell^{\eth\Upsilon}\right|\left|\ell^{\eth-1}\right| d\ell\right)^{1-\frac{1}{q}}\\
&\times \left(\int_0^1 \left|\mu^{\eth}-\ell^{\eth\Upsilon}\right|\left|\ell^{\eth-1}\right|\left[\ell^{\eth s}\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+(1-\ell^{\eth})^s\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q\right] d\ell\right)^{\frac{1}{q}}\\
&\leq \frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}T_3^{1-\frac{1}{q}}\left(\Upsilon;\mu^{\eth}\right)\left(\left|\Psi'\left(\varpi^{\eth}\right)\right|^q T_1\left(\Upsilon;\mu^{\eth},s\right)\right.\\
&\left.+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q T_2\left(\Upsilon;\mu^{\eth},s\right)\right)^{\frac{1}{q}}+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}T_3^{1-\frac{1}{q}}\left(\Upsilon;\mu^{\eth}\right)\\
&\left(\left|\Psi'\left(\varpi^{\eth}\right)\right|^q T_1\left(\Upsilon;\mu^{\eth},s\right)+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q T_2\left(\Upsilon;\mu^{\eth},s\right)\right)^{\frac{1}{q}}.
\end{aligned}
$$

Calculating $T_3\left(\Upsilon;\mu^{\eth},s^{\eth}\right)$, we have

(2.9)

$$
\begin{aligned}
\int_0^1 \left|\ell^{\eth\Upsilon}-\mu^{\eth}\right|\left|\ell^{\eth-1}\right| d\ell &= \int_0^{\mu^{\frac{\eth}{\Upsilon}}}\left(\mu^{\eth}-\ell^{\eth\Upsilon}\right)\ell^{\eth-1}d\ell+\int_{\mu^{\frac{\eth}{\Upsilon}}}^1\left(\ell^{\eth\Upsilon}-\mu^{\eth}\right)\ell^{\eth-1}d\ell\\
&= \frac{2\Upsilon\mu^{\frac{\eth(\Upsilon+1)}{\Upsilon}}+1}{\eth\left(\Upsilon+1\right)}-\frac{\mu^{\eth}}{\eth}.
\end{aligned}
$$

If we put (2.9) in the inequality (2.8), we get our desired result. The proof is completed. □

**Remark 2.19.** By taking $\eth=1$ in Theorem 2.18, we get the Theorem 2.1 in [20].

**Remark 2.20.** By taking $\eth=\Upsilon=1, \mu=1-\tau$ and $\varpi=(1-\lambda)\hbar_1+\lambda\hbar_2$ with $\lambda\in[0,1]$ in Theorem 2.18, the inequality (2.7) becomes Theorem 1.13.

**Corollary 2.21.** *If we put $s=1$ in Theorem 2.18, we get*

$$
\left|\left(1-\mu^{\eth}\right)\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right]\Psi\left(\varpi^{\eth}\right)\right.
$$

$$
\begin{aligned}
&+\mu^{\eth}\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_1^{\eth}\right)+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_2^{\eth}\right)}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right]\\
&-\frac{\Gamma(\Upsilon+1)}{\hbar_2^{\eth}-\hbar_1^{\eth}}\left[\ {}^{\eth}I^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ {}^{\eth}I^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\Bigg|\\
&\leq T_3^{1-\frac{1}{q}}\left(\Upsilon;\mu^{\eth}\right)\left\{\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left|\Psi'\left(\varpi^{\eth}\right)\right|^q T_1\left(\Upsilon;\mu^{\eth},1\right)\right.\right.\\
&+\left.\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q T_2\left(\Upsilon;\mu^{\eth},1\right)\right]^{\frac{1}{q}}\\
&+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left|\Psi'\left(\varpi^{\eth}\right)\right|^q T_1\left(\Upsilon;\mu^{\eth},1\right)\right.\\
&\left.\left.+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q T_2\left(\Upsilon;\mu^{\eth},1\right)\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

**Corollary 2.22.** *If we put $\eth=s=1$ in Theorem 2.18, we get*

$$
\begin{aligned}
&\left|(1-\mu)\left[\frac{(\varpi-\hbar_1)^{\Upsilon}+(\hbar_2-\varpi)^{\Upsilon}}{\hbar_2-\hbar_1}\right]\Psi(\varpi)\right.\\
&\quad+\mu\left[\frac{(\varpi-\hbar_1)^{\Upsilon}\Psi(\hbar_1)+(\hbar_2-\varpi)^{\Upsilon}\Psi(\hbar_2)}{\hbar_2-\hbar_1}\right]\\
&\quad\left.-\frac{\Gamma(\Upsilon+1)}{\hbar_2-\hbar_1}\left[J^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ J^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\right|\\
&\leq T_3^{1-\frac{1}{q}}(\Upsilon;\mu)\left\{\frac{(\varpi-\hbar_1)^{\Upsilon+1}}{(\hbar_2-\hbar_1)}\left[\left|\Psi'(\varpi)\right|^q T_1(\Upsilon;\mu,1)\right.\right.\\
&\quad\left.+\left|\Psi'(\hbar_1)\right|^q T_2(\Upsilon;\mu,1)\right]^{\frac{1}{q}}\\
&\quad\left.+\frac{(\hbar_2-\varpi)^{\Upsilon+1}}{(\hbar_2-\hbar_1)}\left[\left|\Psi'(\varpi)\right|^q T_1(\Upsilon;\mu,1)+\left|\Psi'(\hbar_2)\right|^q T_2(\Upsilon;\mu,1)\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

**Corollary 2.23.** *If we put $\varpi^{\eth}=\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}$ and $\mu^{\eth}=\frac{1}{3}$ in Theorem 2.18, we get the Simpson type inequality for Katugampola fractional integrals*

$$
\begin{aligned}
&\left|\frac{2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon-1}}\left(B_{\Psi}\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2},\frac{1}{3};\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right)\right|\\
&\quad=\left|\frac{1}{6}\left[\Psi\left(\hbar_1^{\eth}\right)+4\Psi\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)+\Psi\left(\hbar_2^{\eth}\right)\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\Gamma(\Upsilon+1)2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}}\left[{}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^-}\Psi(\hbar_1)+{}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^+}\Psi(\hbar_2)\right]\Bigg| \\
&\leq \frac{\hbar_2^{\eth}-\hbar_1^{\eth}}{4\eth^{\Upsilon-1}}T_3^{1-\frac{1}{q}}\left(\Upsilon;\frac{1}{3}\right)\Bigg\{\left[\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|^q T_1\left(\Upsilon;\frac{1}{3},s\right)\right. \\
&\quad\left.+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q T_2\left(\Upsilon;\frac{1}{3},s\right)\right]^{\frac{1}{q}} \\
&\quad+\left[\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|^q T_1\left(\Upsilon;\frac{1}{3},s\right)+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q T_2\left(\Upsilon;\frac{1}{3},s\right)\right]^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

**Remark 2.24.** By taking $\eth = 1$ in Corollary 2.23, we get the Corollary 2.3 in [20].

**Corollary 2.25.** *If we put $\varpi^{\eth} = \frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}$ and $\mu^{\eth} = 0$ in Theorem 2.18, we get the midpoint inequality for Katugampola fractional integrals*

$$
\begin{aligned}
&\left|\frac{2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon-1}}\left(B_{\Psi}\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2},0;\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right)\right| \\
&=\left|\Psi\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)-\frac{\Gamma(\Upsilon+1)2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}}\left[{}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^-}\Psi(\hbar_1)+{}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^+}\Psi(\hbar_2)\right]\right| \\
&\leq \frac{\hbar_2^{\eth}-\hbar_1^{\eth}}{4\eth^{\Upsilon-1}}\left(\frac{1}{\eth(\Upsilon+1)}\right) \\
&\quad\times\left\{\left[\frac{\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|^q}{\eth(\Upsilon+s+1)}+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q\beta\left(\eth(\Upsilon+1),\eth(s+1)\right)\right]^{\frac{1}{q}}\right. \\
&\quad\left.+\left[\frac{\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|^q}{\eth(\Upsilon+s+1)}+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q\beta\left(\eth(\Upsilon+1),\eth(s+1)\right)\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

**Remark 2.26.** By taking $\eth = 1$ in Corollary 2.25, we get the Corollary 2.4 in [20].

**Remark 2.27.** By taking $\eth = \Upsilon = 1$ in Corollary 2.25, we get the Theorem 2.4 in [8].

**Corollary 2.28.** *If we put $\mu^{\eth} = 1$ in Theorem 2.18, we get the trapezoid inequality for Katugampola fractional integrals*

$$
\left|B_{\Psi}\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2},1;\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right|
$$

$$
\begin{aligned}
&= \left| \frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_1^{\eth}\right)+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_2^{\eth}\right)}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)} \right. \\
&\quad \left. -\frac{\Gamma(\Upsilon+1)}{\hbar_2^{\eth}-\hbar_1^{\eth}}\left[ {}^{\eth}I^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+ {}^{\eth}I^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\right| \\
&\leq \left(\frac{1}{\eth\left(\Upsilon+1\right)}\right)^{1-\frac{1}{q}} \left\{ \frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)} \left[ \frac{\Upsilon\left|\Psi'\left(\varpi^{\eth}\right)\right|^q}{\eth\left(s+1\right)\left(\Upsilon+s+1\right)} \right.\right. \\
&\quad \left. + \left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q \left(\frac{1}{\eth\left(s+1\right)}-\beta\left(\eth\left(\Upsilon+1\right),\eth\left(s+1\right)\right)\right)\right]^{\frac{1}{q}} \\
&\quad \times \left[ \frac{\Upsilon\left|\Psi'\left(\varpi^{\eth}\right)\right|^q}{\eth\left(s+1\right)\left(\Upsilon+s+1\right)} \right. \\
&\quad \left.\left. + \left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q \left(\frac{1}{\eth\left(s+1\right)}-\beta\left(\eth\left(\Upsilon+1\right),\eth\left(s+1\right)\right)\right) \right]^{\frac{1}{q}} \right\}.
\end{aligned}
$$

**Remark 2.29.** By taking $\eth = 1$ in Corollary 2.28, we get the Corollary 2.5 in [20].

**Remark 2.30.** By taking $\eth = \Upsilon = 1$ in Corollary 2.28, we get the Theorem 7 in [9].

**Corollary 2.31.** *Under the assumptions of Theorem 2.5, if* $\left|\Psi'\left(\varpi^{\eth}\right)\right| \leq M$ *for all* $\varpi^{\eth} \in \left[\hbar_1^{\eth}, \hbar_2^{\eth}\right]$ *and* $\mu^{\eth} = 0$, *then we get the Ostrowski type inequality for Katugampola fractional integral*

$$
\begin{aligned}
&\left| \left[ \frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)} \right] \Psi\left(\varpi^{\eth}\right) \right. \\
&\qquad \left. -\frac{\Gamma(\Upsilon+1)}{\hbar_2^{\eth}-\hbar_1^{\eth}}\left[ {}^{\eth}I^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+ {}^{\eth}I^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\right| \\
&\leq M\left(\frac{1}{\eth\left(\Upsilon+1\right)}\right)^{1-\frac{1}{q}} \left[ \frac{1}{\eth\left(\Upsilon+s+1\right)}+\beta\left(\eth\left(\Upsilon+1\right),\eth\left(s+1\right)\right)\right]^{\frac{1}{q}} \\
&\qquad \times \left[ \frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)} \right].
\end{aligned}
$$

**Remark 2.32.** By taking $\eth = 1$ in Corollary 2.31, we get the Corollary 2.6 in [20].

**Remark 2.33.** By taking $\eth = \Upsilon = 1$ in Corollary 2.31, we get the Theorem 4 in [7].

**Theorem 2.34.** *Let* $\Upsilon > 0$ *and* $\eth > 0$. *Let* $\Psi : I \subseteq (0,\infty) \to \Re$ *be a differentiable function, such that* $\Psi \in X_c^p(\hbar_1^{\eth}, \hbar_2^{\eth})$, *for all* $\hbar_1^{\eth}, \hbar_2^{\eth} \in I$

*and $\hbar_1 < \hbar_2$. If $|\Psi'|^q$ is an s-convex function on $[\hbar_1, \hbar_2]$ for some fixed $q > 1, \frac{1}{p} + \frac{1}{q} = 1$, then for $\varpi \in [\hbar_1, \hbar_2]$, and $\mu \in [0,1]$, the following inequality for fractional integrals holds*

(2.10)

$$\begin{aligned}
&\left|B_\Psi\left(\varpi^\eth, \mu^\eth; \Upsilon, \hbar_1^\eth, {\hbar_2}^\eth\right)\right| \\
&\quad \leq T_4^{\frac{1}{p}}\left(\Upsilon; \mu^\eth, p\right)\left\{\frac{\left(\varpi^\eth - \hbar_1^\eth\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^\eth - \hbar_1^\eth\right)}\left(\frac{\left|\Psi'\left(\varpi^\eth\right)\right|^q + \left|\Psi'\left(\hbar_1^\eth\right)\right|^q}{\eth\left(s+1\right)}\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \frac{\left(\hbar_2^\eth - \varpi^\eth\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^\eth - \hbar_1^\eth\right)}\left(\frac{\left|\Psi'\left(\varpi^\eth\right)\right|^q + \left|\Psi'\left(\hbar_2^\eth\right)\right|^q}{\eth\left(s+1\right)}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

*where*

$$\begin{aligned}
&T_4\left(\Upsilon; \mu^\eth, p\right) \\
&= \begin{cases}
\frac{1}{\eth\Upsilon(p+1)} & \mu = 0 \\
\left\{\frac{\mu^{\frac{\eth(\Upsilon p+1)}{\Upsilon}}}{\Upsilon}\beta\left(\frac{\eth}{\Upsilon}, \eth p + 1\right)\right. & \\
\qquad \left. + \frac{\left(1-\mu^\eth\right)^{p+1}}{\eth(\Upsilon p+1)}\ {}_2F_1\left(1 - \frac{1}{\Upsilon}, 1; \Upsilon p + 2; 1 - \mu^\eth\right)\right\} & 0 < \mu < 1 \\
\frac{1}{\Upsilon}\beta\left(\eth\left(p+1\right), \frac{1}{\eth\Upsilon}\right) & \mu = 0.
\end{cases}
\end{aligned}$$

*Proof.* From Lemma 2.2 and by using the property of modulus, the Hölder's inequality and s-convexity of $|\Psi'|^q$, we obtain

(2.11)

$$\begin{aligned}
&\left|B_\Psi\left(\varpi^\eth, \mu^\eth; \Upsilon, \hbar_1^\eth, {\hbar_2}^\eth\right)\right| \\
&\quad \leq \frac{\left(\varpi^\eth - \hbar_1^\eth\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^\eth - \hbar_1^\eth\right)}\left(\int_0^1 \left|\ell^{\eth\Upsilon} - \mu^\eth\right|^p \left|\ell^{\eth-1}\right| d\ell\right)^{\frac{1}{p}} \\
&\qquad \times \left(\int_0^1 \left|\ell^{\eth-1}\right| \left|\Psi'\left(\ell^\eth\varpi^\eth + (1-\ell^\eth)\hbar_1^\eth\right)\right|^q d\ell\right)^{\frac{1}{q}} \\
&\qquad + \frac{\left(\hbar_2^\eth - \varpi^\eth\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^\eth - \hbar_1^\eth\right)}\left(\int_0^1 \left|\mu^\eth - \ell^{\eth\Upsilon}\right|^p \left|\ell^{\eth-1}\right| d\ell\right)^{\frac{1}{p}} \\
&\qquad \times \left(\int_0^1 \left|\ell^{\eth-1}\right| \left|\Psi'\left(\ell^\eth\varpi^\eth + (1-\ell^\eth)\hbar_2^\eth\right)\right|^q d\ell\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left(\int_0^1\left|\ell^{\eth\Upsilon}-\mu^{\eth}\right|^p\left|\ell^{\eth-1}\right|d\ell\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1\left|\ell^{\eth-1}\right|\left(\ell^{\eth s}\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+(1-\ell^{\eth})^s\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q\right)d\ell\right)^{\frac{1}{q}}\\
&\quad+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left(\int_0^1\left|\mu^{\eth}-\ell^{\eth\Upsilon}\right|^p\left|\ell^{\eth-1}\right|d\ell\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1\left|\ell^{\eth-1}\right|\left(\ell^{\eth s}\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+(1-\ell^{\eth})^s\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q\right)d\ell\right)^{\frac{1}{q}}\\
&\leq \frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}T_4^{\frac{1}{p}}\left(\Upsilon;\mu^{\eth},p\right)\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q}{\eth\left(s+1\right)}\right)^{\frac{1}{q}}\\
&\quad+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}T_4^{\frac{1}{p}}\left(\Upsilon;\mu^{\eth},p\right)\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q}{\eth\left(s+1\right)}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Calculating $T_4\left(\Upsilon;\mu^{\eth},p\right)$, we have

(2.12)

$$
\begin{aligned}
&\int_0^1\left|\ell^{\eth\Upsilon}-\mu^{\eth}\right|^p\left|\ell^{\eth-1}\right|d\ell\\
&=\int_0^{\mu^{\frac{\eth}{\Upsilon}}}\left(\mu^{\eth}-\ell^{\eth\Upsilon}\right)\ell^{\eth-1}d\ell+\int_{\mu^{\frac{\eth}{\Upsilon}}}^1\left(\ell^{\eth\Upsilon}-\mu^{\eth}\right)\ell^{\eth-1}d\ell\\
&=\begin{cases}
\frac{1}{\eth\Upsilon(p+1)} & \mu=0\\
\left\{\begin{array}{l}\frac{\mu^{\frac{\eth(\Upsilon p+1)}{\Upsilon}}}{\Upsilon}\beta\left(\frac{\eth}{\Upsilon},\eth p+1\right)\\ \quad+\frac{\left(1-\mu^{\eth}\right)^{p+1}}{\eth(\Upsilon p+1)}\ {}_2F_1\left(1-\frac{1}{\Upsilon},1;\Upsilon p+2;1-\mu^{\eth}\right)\end{array}\right\} & 0<\mu<1\\
\frac{1}{\Upsilon}\beta\left(\eth\left(p+1\right),\frac{1}{\eth\Upsilon}\right) & \mu=1.
\end{cases}
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\int_0^1\left|\ell^{\eth-1}\right|\left(\ell^{\eth s}\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+(1-\ell^{\eth})^s\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q\right)d\ell\\
&\quad=\left|\Psi'\left(\varpi^{\eth}\right)\right|^q\int_0^1\ell^{\eth-1}\ell^{\eth s}d\ell+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q\int_0^1\ell^{\eth-1}(1-\ell^{\eth})^s d\ell
\end{aligned}
\tag{2.13}
$$

$$= \frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q + \left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q}{\eth\,(s+1)}.$$

Similarly,

$$\begin{aligned}(2.14)\quad &\int_0^1 \left|\ell^{\eth-1}\right| \left(\ell^{\eth s}\left|\Psi'\left(\varpi^{\eth}\right)\right|^q + (1-\ell^{\eth})^s \left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q\right) d\ell \\ &= \left|\Psi'\left(\varpi^{\eth}\right)\right|^q \int_0^1 \ell^{\eth-1}\ell^{\eth s} d\ell + \left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q \int_0^1 \ell^{\eth-1}(1-\ell^{\eth})^s d\ell \\ &= \frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q + \left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q}{\eth\,(s+1)}.\end{aligned}$$

If we put (2.12), (2.13) and (2.14) in the inequality (2.11), we get our desired result. The proof is completed. □

**Remark 2.35.** By taking $\eth = 1$ in Theorem 2.34, we get the Theorem 2.2 in [20].

**Remark 2.36.** By taking $\eth = \Upsilon = 1, \mu = 1-\tau$ and $\varpi = (1-\lambda)\hbar_1 + \lambda\hbar_2$ with $\lambda \in [0,1]$ in Theorem 2.34, the inequality (2.10) becomes Theorem 1.14.

**Corollary 2.37.** *If we put $s = 1$ in Theorem 2.34, we get*

$$\begin{aligned}&\left|\left(1-\mu^{\eth}\right)\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon} + \left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right]\Psi\left(\varpi^{\eth}\right)\right. \\ &\quad + \mu^{\eth}\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_1^{\eth}\right) + \left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_2^{\eth}\right)}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right] \\ &\quad \left. - \frac{\Gamma(\Upsilon+1)}{\hbar_2^{\eth}-\hbar_1^{\eth}}\left[{}^{\eth}I^{\Upsilon}_{\varpi^-}\Psi(\hbar_1) + {}^{\eth}I^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\right| \\ &\leq T_4^{\frac{1}{p}}\left(\Upsilon;\mu^{\eth},p\right)\left\{\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q + \left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q}{2\eth}\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q + \left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q}{2\eth}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

**Corollary 2.38.** *If we put $\eth = s = 1$ in Theorem 2.34, we get*

$$\begin{aligned}&\left|(1-\mu)\left[\frac{(\varpi-\hbar_1)^{\Upsilon} + (\hbar_2-\varpi)^{\Upsilon}}{\hbar_2-\hbar_1}\right]\Psi(\varpi)\right. \\ &\quad \left. + \mu\left[\frac{(\varpi-\hbar_1)^{\Upsilon}\Psi(\hbar_1) + (\hbar_2-\varpi)^{\Upsilon}\Psi(\hbar_2)}{\hbar_2-\hbar_1}\right]\right.\end{aligned}$$

$$
\begin{aligned}
&-\frac{\Gamma(\Upsilon+1)}{\hbar_2-\hbar_1}\left[J^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ J^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\Bigg| \\
&\leq T_4^{\frac{1}{p}}(\Upsilon;\mu,p)\left\{\frac{(\varpi-\hbar_1)^{\Upsilon+1}}{\hbar_2-\hbar_1}\left(\frac{|\Psi'(\varpi)|^q+|\Psi'(\hbar_1)|^q}{2}\right)^{\frac{1}{q}}\right. \\
&\left.+\ \frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{(\hbar_2-\hbar_1)}\left(\frac{|\Psi'(\varpi)|^q+|\Psi'(\hbar_2)|^q}{2}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

**Corollary 2.39.** *If we put* $\varpi^{\eth}=\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}$ *and* $\mu^{\eth}=\frac{1}{3}$ *in Theorem 2.34, we get the Simpson type inequality for Katugampola fractional integrals*

$$
\begin{aligned}
&\left|\frac{2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon-1}}\left(B_{\Psi}\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2},\frac{1}{3};\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right)\right| \\
&=\left|\frac{1}{6}\left[\Psi\left(\hbar_1^{\eth}\right)+4\Psi\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)+\Psi\left(\hbar_2^{\eth}\right)\right]\right. \\
&\quad\left.-\frac{\Gamma(\Upsilon+1)2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}}\left[{}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^-}\Psi(\hbar_1)+\ {}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^+}\Psi(\hbar_2)\right]\right| \\
&\leq\frac{\hbar_2^{\eth}-\hbar_1^{\eth}}{4\eth^{\Upsilon-1}}T_4^{\frac{1}{p}}\left(\Upsilon;\frac{1}{3},p\right)\left\{\left(\frac{\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|^q+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q}{\eth(s+1)}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|^q+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q}{\eth(s+1)}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

**Remark 2.40.** By taking $\eth=1$ in Corollary 2.39, we get the Corollary 2.9 in [20].

**Corollary 2.41.** *If we put* $\varpi^{\eth}=\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}$ *and* $\mu^{\eth}=0$ *in Theorem 2.34, we get the midpoint inequality for Katugampola fractional integrals*

$$
\begin{aligned}
&\left|\frac{2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon-1}}\left(B_{\Psi}\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2},0;\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right)\right| \\
&=\left|\Psi\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)-\frac{\Gamma(\Upsilon+1)2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}}\right. \\
&\quad\left.\times\left[{}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^-}\Psi(\hbar_1)+\ {}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^+}\Psi(\hbar_2)\right]\right|
\end{aligned}
$$

$$\leq \frac{\hbar_2^{\eth}-\hbar_1^{\eth}}{4\eth^{\Upsilon-1}}\left(\frac{1}{\eth\Upsilon\left(p+1\right)}\right)^{\frac{1}{p}}\left\{\left(\frac{\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|^q+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q}{\eth\left(s+1\right)}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|^q+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q}{\eth\left(s+1\right)}\right)^{\frac{1}{q}}\right\}.$$

**Remark 2.42.** By taking $\eth=1$ in Corollary 2.41, we get the Corollary 2.10 in [20].

**Remark 2.43.** By taking $\eth=\Upsilon=1$ in Corollary 2.41, we get the Theorem 2.3 in [8].

**Corollary 2.44.** *If we put $\mu^{\eth}=1$ in Theorem 2.34, we get the trapezoid inequality for Katugampola fractional integrals*

$$\left|B_{\Psi}\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2},1;\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right|$$

$$=\left|\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_1^{\eth}\right)+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_2^{\eth}\right)}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right.$$

$$\left.-\frac{\Gamma(\Upsilon+1)}{\hbar_2^{\eth}-\hbar_1^{\eth}}\left[\ {}^{\eth}I^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ {}^{\eth}I^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\right|$$

$$\leq\left(\frac{\beta\left(\eth\left(p+1\right),\frac{1}{\eth\Upsilon}\right)}{\Upsilon}\right)^{1-\frac{1}{q}}\left\{\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q}{\eth\left(s+1\right)}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q}{\eth\left(s+1\right)}\right)^{\frac{1}{q}}\right\}.$$

**Remark 2.45.** By taking $\eth=1$ in Corollary 2.44, we get the Corollary 2.11 in [20].

**Remark 2.46.** By taking $\eth=\Upsilon=1$ in Corollary 2.44, we get the Theorem 6 in [9].

**Corollary 2.47.** *Under the assumptions of Theorem 2.34, if $\left|\Psi'\left(\varpi^{\eth}\right)\right|\leq M$ for all $\varpi^{\eth}\in\left[\hbar_1^{\eth},\hbar_2^{\eth}\right]$ and $\mu^{\eth}=0$, then we get the Ostrowski type inequality for Katugampola fractional integral*

$$\left|\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right]\Psi\left(\varpi^{\eth}\right)\right.$$

$$\left.-\frac{\Gamma(\Upsilon+1)}{\hbar_2^{\eth}-\hbar_1^{\eth}}\left[\ {}^{\eth}I^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ {}^{\eth}I^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\right|$$

$$\leq M\left(\frac{1}{\eth\left(\Upsilon+1\right)}\right)^{\frac{1}{p}}\left[\frac{2}{\eth\left(s+1\right)}\right]^{\frac{1}{q}}\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right].$$

**Remark 2.48.** By taking $\eth=1$ in Corollary 2.47, we get the Corollary 2.12 in [20].

**Remark 2.49.** By taking $\eth=\Upsilon=1$ in Corollary 2.47, we get the Theorem 3 in [7].

**Theorem 2.50.** *Let $\Upsilon>0$ and $\eth>0$. Let $\Psi: I\subseteq(0,\infty)\to\Re$ be a differentiable function, such that $\Psi\in X_c^p(\hbar_1^{\eth},\hbar_2^{\eth})$, for all $\hbar_1^{\eth},\hbar_2^{\eth}\in I$ and $\hbar_1<\hbar_2$. If $|\Psi'|^q$ is an s-convex function on $[\hbar_1,\hbar_2]$ for some fixed $q>1,\frac{1}{p}+\frac{1}{q}=1$, then for $\varpi\in[\hbar_1,\hbar_2]$, and $\mu\in[0,1]$, the following inequality for fractional integrals holds*

$$\begin{aligned}
&\left|B_{\Psi}\left(\varpi^{\eth},\mu^{\eth};\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right|\\
&\leq\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left(\frac{T_4\left(\Upsilon;\mu^{\eth},p\right)}{p}\right)+\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q}{q\eth\left(s+1\right)}\right)\right]\\
&\quad+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left(\frac{T_4\left(\Upsilon;\mu^{\eth},p\right)}{p}\right)+\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q}{q\eth\left(s+1\right)}\right)\right].
\end{aligned}$$

*Proof.* From Lemma 2.2 and by using the property of modulus, the Young's inequality i.e.

$$\mathsf{xy}\leq\frac{1}{\mathsf{p}}\mathsf{x}^{\mathsf{p}}+\frac{1}{\mathsf{q}}\mathsf{y}^{\mathsf{q}},$$

and s-convexity of $|\Psi'|^q$, we obtain

(2.15)

$$\begin{aligned}
&\left|B_{\Psi}\left(\varpi^{\eth},\mu^{\eth};\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right|\\
&\leq\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left(\frac{1}{p}\int_0^1\left|\ell^{\eth\Upsilon}-\mu^{\eth}\right|^p\left|\ell^{\eth-1}\right|d\ell\right)\right.\\
&\quad\left.+\left(\frac{1}{q}\int_0^1\left|\ell^{\eth-1}\right|\left|\Psi'\left(\ell^{\eth}\varpi^{\eth}+(1-\ell^{\eth})\hbar_1^{\eth}\right)\right|^q d\ell\right)\right]\\
&\quad+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left(\frac{1}{p}\int_0^1\left|\mu^{\eth}-\ell^{\eth\Upsilon}\right|^p\left|\ell^{\eth-1}\right|d\ell\right)\right.\\
&\quad\left.+\left(\frac{1}{q}\int_0^1\left|\ell^{\eth-1}\right|\left|\Psi'\left(\ell^{\eth}\varpi^{\eth}+(1-\ell^{\eth})\hbar_2^{\eth}\right)\right|^q d\ell\right)\right]\\
&\leq\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left(\frac{1}{p}\int_0^1\left|\ell^{\eth\Upsilon}-\mu^{\eth}\right|^p\left|\ell^{\eth-1}\right|d\ell\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{q}\int_0^1\left|\ell^{\eth-1}\right|\left(\ell^{\eth s}\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+(1-\ell^{\eth})^s\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q\right)d\ell\right)\Bigg]\\
&+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\Bigg[\left(\frac{1}{p}\int_0^1\left|\mu^{\eth}-\ell^{\eth\Upsilon}\right|^p\left|\ell^{\eth-1}\right|d\ell\right)\\
&+\left(\frac{1}{q}\int_0^1\left|\ell^{\eth-1}\right|\left(\ell^{\eth s}\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+(1-\ell^{\eth})^s\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q\right)d\ell\right)\Bigg]\\
&\leq\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left(\frac{T_4\left(\Upsilon;\mu^{\eth},p\right)}{p}\right)+\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q}{q\eth\left(s+1\right)}\right)\right]\\
&+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left(\frac{T_4\left(\Upsilon;\mu^{\eth},p\right)}{p}\right)+\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q}{q\eth\left(s+1\right)}\right)\right].
\end{aligned}
$$

The proof is completed. □

**Corollary 2.51.** *If we put* $\eth=1$ *in Theorem 2.50, we get*

$$
\begin{aligned}
&\Bigg|(1-\mu)\left[\frac{\left(\varpi-\hbar_1\right)^{\Upsilon}+\left(\hbar_2-\varpi\right)^{\Upsilon}}{\hbar_2-\hbar_1}\right]\Psi\left(\varpi\right)\\
&\quad+\mu\left[\frac{\left(\varpi-\hbar_1\right)^{\Upsilon}\Psi\left(\hbar_1\right)+\left(\hbar_2-\varpi\right)^{\Upsilon}\Psi\left(\hbar_2\right)}{\hbar_2-\hbar_1}\right]\\
&\quad-\frac{\Gamma(\Upsilon+1)}{\hbar_2-\hbar_1}\left[J^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ J^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\Bigg|\\
&\leq\frac{\left(\varpi-\hbar_1\right)^{\Upsilon+1}}{\left(\hbar_2-\hbar_1\right)}\left[\left(\frac{T_4\left(\Upsilon;\mu,p\right)}{p}\right)+\left(\frac{\left|\Psi'\left(\varpi\right)\right|^q+\left|\Psi'\left(\hbar_1\right)\right|^q}{q\left(s+1\right)}\right)\right]\\
&\quad+\frac{\left(\hbar_2-\varpi\right)^{\Upsilon+1}}{\left(\hbar_2-\hbar_1\right)}\left[\left(\frac{T_4\left(\Upsilon;\mu,p\right)}{p}\right)+\left(\frac{\left|\Psi'\left(\varpi\right)\right|^q+\left|\Psi'\left(\hbar_2\right)\right|^q}{q\left(s+1\right)}\right)\right].
\end{aligned}
$$

**Corollary 2.52.** *If we put* $\eth=\Upsilon=1,\mu=1-\tau$ *and* $\varpi=(1-\lambda)\hbar_1+\lambda\hbar_2$ *in Theorem 2.50, we get*

$$
\begin{aligned}
&\Bigg|\left(1-\tau\right)\left(\lambda\Psi\left(\hbar_1\right)+(1-\lambda)\Psi\left(\hbar_2\right)\right)\\
&\quad+\tau\Psi\left((1-\lambda)\hbar_1+\lambda\hbar_2\right)-\frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2}\Psi(\varpi)d\varpi\Bigg|\\
&\leq\left(\hbar_2-\hbar_1\right)\left(\frac{\tau^{p+1}+(1-\tau)^{p+1}}{p\left(p+1\right)}\right)\\
&\quad\times\left[\lambda^2\left(\frac{\left|\Psi'\left(C\right)\right|^q+\left|\Psi'\left(\hbar_1\right)\right|^q}{q\left(s+1\right)}\right)+(1-\lambda)^2\left(\frac{\left|\Psi'\left(C\right)\right|^q+\left|\Psi'\left(\hbar_2\right)\right|^q}{q\left(s+1\right)}\right)\right],
\end{aligned}
$$

*where* $C=(1-\lambda)\hbar_1+\lambda\hbar_2$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

**Corollary 2.53.** *If we put* $s=1$ *in Theorem 2.50, we get*

$$\begin{aligned}
&\left|\left(1-\mu^{\eth}\right)\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right]\Psi\left(\varpi^{\eth}\right)\right.\\
&\quad+\mu^{\eth}\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_1^{\eth}\right)+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_2^{\eth}\right)}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right]\\
&\quad\left.-\frac{\Gamma(\Upsilon+1)}{\hbar_2^{\eth}-\hbar_1^{\eth}}\left[\ {}^{\eth}I^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ {}^{\eth}I^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\right|\\
&\leq\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left(\frac{T_4\left(\Upsilon;\mu^{\eth},p\right)}{p}\right)+\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q}{2q\eth}\right)\right]\\
&\quad+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left(\frac{T_4\left(\Upsilon;\mu^{\eth},p\right)}{p}\right)+\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q}{2q\eth}\right)\right].
\end{aligned}$$

**Corollary 2.54.** *If we put* $\eth=s=1$ *in Theorem 2.50, we get*

$$\begin{aligned}
&\left|(1-\mu)\left[\frac{(\varpi-\hbar_1)^{\Upsilon}+(\hbar_2-\varpi)^{\Upsilon}}{\hbar_2-\hbar_1}\right]\Psi(\varpi)\right.\\
&\quad+\mu\left[\frac{(\varpi-\hbar_1)^{\Upsilon}\Psi(\hbar_1)+(\hbar_2-\varpi)^{\Upsilon}\Psi(\hbar_2)}{\hbar_2-\hbar_1}\right]\\
&\quad\left.-\frac{\Gamma(\Upsilon+1)}{\hbar_2-\hbar_1}\left[J^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ J^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\right|\\
&\leq\frac{(\varpi-\hbar_1)^{\Upsilon+1}}{(\hbar_2-\hbar_1)}\left[\left(\frac{T_4\left(\Upsilon;\mu,p\right)}{p}\right)+\left(\frac{|\Psi'(\varpi)|^q+|\Psi'(\hbar_1)|^q}{2q}\right)\right]\\
&\quad+\frac{(\hbar_2-\varpi)^{\Upsilon+1}}{(\hbar_2-\hbar_1)}\left[\left(\frac{T_4\left(\Upsilon;\mu,p\right)}{p}\right)+\left(\frac{|\Psi'(\varpi)|^q+|\Psi'(\hbar_2)|^q}{2q}\right)\right].
\end{aligned}$$

**Corollary 2.55.** *If we put* $\varpi^{\eth}=\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}$ *and* $\mu^{\eth}=\frac{1}{3}$ *in Theorem 2.50, we get the Simpson type inequality for Katugampola fractional integrals*

$$\begin{aligned}
&\left|\frac{2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon-1}}\left(B_{\Psi}\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2},\frac{1}{3};\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right)\right|\\
&=\left|\frac{1}{6}\left[\Psi\left(\hbar_1^{\eth}\right)+4\Psi\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)+\Psi\left(\hbar_2^{\eth}\right)\right]\right.
\end{aligned}$$

$$-\frac{\Gamma(\Upsilon+1)2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}}\left[{}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^-}\Psi(\hbar_1)+{}^{\eth}I^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^+}\Psi(\hbar_2)\right]\Bigg|$$

$$\leq\frac{\hbar_2^{\eth}-\hbar_1^{\eth}}{4\eth^{\Upsilon-1}}\left\{\left(\frac{T_4\left(\Upsilon;\frac{1}{3},p\right)}{p}\right)+\left(\frac{\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|^q+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q}{q\eth\left(s+1\right)}\right)\right.$$

$$\left.+\left(\frac{T_4\left(\Upsilon;\frac{1}{3},p\right)}{p}\right)+\left(\frac{\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|^q+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q}{q\eth\left(s+1\right)}\right)\right\}.$$

**Corollary 2.56.** *If we put $\eth=1$ in Corollary 2.54, we get the Simpson type inequality for Riemann-Liouville fractional integrals*

$$\left|\frac{2^{\Upsilon-1}}{\left(\hbar_2-\hbar_1\right)^{\Upsilon-1}}\left(B_\Psi\left(\frac{\hbar_1+\hbar_2}{2},\frac{1}{3};\Upsilon,\hbar_1,\hbar_2\right)\right)\right|$$

$$=\left|\frac{1}{6}\left[\Psi\left(\hbar_1\right)+4\Psi\left(\frac{\hbar_1+\hbar_2}{2}\right)+\Psi\left(\hbar_2\right)\right]\right.$$

$$\left.-\frac{\Gamma(\Upsilon+1)2^{\Upsilon-1}}{\left(\hbar_2-\hbar_1\right)^{\Upsilon}}\left[J^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^-}\Psi(\hbar_1)+J^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^+}\Psi(\hbar_2)\right]\right|$$

$$\leq\frac{\hbar_2-\hbar_1}{4}\left\{\left(\frac{T_4\left(\Upsilon;\frac{1}{3},p\right)}{p}\right)+\left(\frac{\left|\Psi'\left(\frac{\hbar_1+\hbar_2}{2}\right)\right|^q+\left|\Psi'\left(\hbar_1\right)\right|^q}{q\left(s+1\right)}\right)\right.$$

$$\left.+\left(\frac{T_4\left(\Upsilon;\frac{1}{3},p\right)}{p}\right)+\left(\frac{\left|\Psi'\left(\frac{\hbar_1+\hbar_2}{2}\right)\right|^q+\left|\Psi'\left(\hbar_2\right)\right|^q}{q\left(s+1\right)}\right)\right\}.$$

**Corollary 2.57.** *If we put $\varpi^{\eth}=\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}$ and $\mu^{\eth}=0$ in Theorem 2.50, we get the midpoint inequality for Katugampola fractional integrals*

$$\left|\frac{2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon-1}}\left(B_\Psi\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2},0;\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right)\right|$$

$$=\left|\Psi\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)-\frac{\Gamma(\Upsilon+1)2^{\Upsilon-1}}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}}\left[{}^{\eth}I^{\Upsilon}_{(\frac{\hbar_1+\hbar_2}{2})^-}\Psi(\hbar_1)+{}^{\eth}I^{\Upsilon}_{(\frac{\hbar_1+\hbar_2}{2})^+}\Psi(\hbar_2)\right]\right|$$

$$\leq\frac{\hbar_2^{\eth}-\hbar_1^{\eth}}{4\eth^{\Upsilon-1}}\left\{\left(\frac{1}{\eth\Upsilon p\left(p+1\right)}\right)+\left(\frac{\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|^q+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q}{q\eth\left(s+1\right)}\right)\right.$$

$$\left.+\left(\frac{1}{\eth\Upsilon p\left(p+1\right)}\right)+\left(\frac{\left|\Psi'\left(\frac{\hbar_1^{\eth}+\hbar_2^{\eth}}{2}\right)\right|^q+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q}{q\eth\left(s+1\right)}\right)\right\}.$$

**Corollary 2.58.** *If we put $\eth = 1$ in Corollary 2.57, we get the midpoint inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left|\frac{2^{\Upsilon-1}}{(\hbar_2-\hbar_1)^{\Upsilon-1}}\left(B_\Psi\left(\frac{\hbar_1+\hbar_2}{2},0;\Upsilon,\hbar_1,\hbar_2\right)\right)\right| \\
&\quad=\left|\Psi\left(\frac{\hbar_1+\hbar_2}{2}\right)-\frac{\Gamma(\Upsilon+1)2^{\Upsilon-1}}{(\hbar_2-\hbar_1)^{\Upsilon}}\left[\ J^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^-}\Psi(\hbar_1)+\ J^{\Upsilon}_{\left(\frac{\hbar_1+\hbar_2}{2}\right)^+}\Psi(\hbar_2)\right]\right| \\
&\quad\le\frac{\hbar_2-\hbar_1}{4}\left\{\left(\frac{1}{\Upsilon p\,(p+1)}\right)+\left(\frac{\left|\Psi'\left(\frac{\hbar_1+\hbar_2}{2}\right)\right|^q+|\Psi'(\hbar_1)|^q}{q\,(s+1)}\right)\right. \\
&\qquad\left.+\left(\frac{1}{\Upsilon p\,(p+1)}\right)+\left(\frac{\left|\Psi'\left(\frac{\hbar_1+\hbar_2}{2}\right)\right|^q+|\Psi'(\hbar_2)|^q}{q\,(s+1)}\right)\right\}.
\end{aligned}$$

**Corollary 2.59.** *If we put $\mu^{\eth} = 1$ in Theorem 2.50, we get the trapezoid inequality for Katugampola fractional integrals*

$$\begin{aligned}
&\left|B_\Psi\left(\varpi^{\eth},1;\Upsilon,\hbar_1^{\eth},{\hbar_2}^{\eth}\right)\right| \\
&\quad=\left|\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_1^{\eth}\right)+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}\Psi\left(\hbar_2^{\eth}\right)}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right. \\
&\qquad\left.-\frac{\Gamma(\Upsilon+1)}{\hbar_2^{\eth}-\hbar_1^{\eth}}\left[\ {}^{\eth}I^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ {}^{\eth}I^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\right| \\
&\quad\le\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left(\frac{\beta\left(\eth\,(p+1),\frac{1}{\eth\Upsilon}\right)}{p\Upsilon}\right)+\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+\left|\Psi'\left(\hbar_1^{\eth}\right)\right|^q}{q\eth\,(s+1)}\right)^{\frac{1}{q}}\right] \\
&\qquad+\frac{\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\left(\frac{\beta\left(\eth\,(p+1),\frac{1}{\eth\Upsilon}\right)}{p\Upsilon}\right)+\left(\frac{\left|\Psi'\left(\varpi^{\eth}\right)\right|^q+\left|\Psi'\left(\hbar_2^{\eth}\right)\right|^q}{q\eth\,(s+1)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

**Corollary 2.60.** *If we put $\eth = 1$ in Corollary 2.59, we get the trapezoid inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&|B_\Psi(\varpi,1;\Upsilon,\hbar_1,\hbar_2)| \\
&\quad=\left|\frac{(\varpi-\hbar_1)^{\Upsilon}\Psi(\hbar_1)+(\hbar_2-\varpi)^{\Upsilon}\Psi(\hbar_2)}{\hbar_2-\hbar_1}\right. \\
&\qquad\left.-\frac{\Gamma(\Upsilon+1)}{\hbar_2-\hbar_1}\left[\ J^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ J^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\right| \\
&\quad\le\frac{(\varpi-\hbar_1)^{\Upsilon+1}}{\hbar_2-\hbar_1}\left[\left(\frac{\beta\left(p+1,\frac{1}{\Upsilon}\right)}{p\Upsilon}\right)+\left(\frac{|\Psi'(\varpi)|^q+|\Psi'(\hbar_1)|^q}{q\,(s+1)}\right)^{\frac{1}{q}}\right] \\
&\qquad+\frac{(\hbar_2-\varpi)^{\Upsilon+1}}{(\hbar_2-\hbar_1)}\left[\left(\frac{\beta\left(p+1,\frac{1}{\Upsilon}\right)}{p\Upsilon}\right)+\left(\frac{|\Psi'(\varpi)|^q+|\Psi'(\hbar_2)|^q}{q\,(s+1)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

**Corollary 2.61.** *Under the assumptions of Theorem 2.50, if* $\left|\Psi'\left(\varpi^{\eth}\right)\right| \leq M$ *for all* $\varpi^{\eth} \in \left[\hbar_1^{\eth}, \hbar_2^{\eth}\right]$ *and* $\mu^{\eth} = 0$, *then we get the Ostrowski type inequality for Katugampola fractional integral*

$$\left|\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon}+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon}}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right]\Psi\left(\varpi^{\eth}\right)-\frac{\Gamma(\Upsilon+1)}{\hbar_2^{\eth}-\hbar_1^{\eth}}\left[\ {}^{\eth}I^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ {}^{\eth}I^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\right|$$
$$\leq M\left[\left(\frac{1}{\eth\Upsilon p\,(p+1)}\right)+\left(\frac{2}{q\eth\,(s+1)}\right)\right]\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^{\Upsilon+1}+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^{\Upsilon+1}}{\eth^{\Upsilon-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\right].$$

**Corollary 2.62.** *If we put* $\eth = 1$ *in Corollary 2.61, we get the Ostrowski type inequality for Riemann-Liouville fractional integrals*

$$\left|\left[\frac{(\varpi-\hbar_1)^{\Upsilon}+(\hbar_2-\varpi)^{\Upsilon}}{\hbar_2-\hbar_1}\right]\Psi(\varpi)-\frac{\Gamma(\Upsilon+1)}{\hbar_2-\hbar_1}\left[\ J^{\Upsilon}_{\varpi^-}\Psi(\hbar_1)+\ J^{\Upsilon}_{\varpi^+}\Psi(\hbar_2)\right]\right|$$
$$\leq M\left[\left(\frac{1}{\Upsilon p\,(p+1)}\right)+\left(\frac{2}{q\,(s+1)}\right)\right]\left[\frac{(\varpi-\hbar_1)^{\Upsilon+1}+(\hbar_2-\varpi)^{\Upsilon+1}}{\hbar_2-\hbar_1}\right].$$

## 3. Applications to Some Special Means

In this section, we discuss some applications of Theorem 2.5 to special means to consider a particular example of s-convexity. First of all, we recall some already known notions related to means [11]. For arbitrary positive real numbers $\hbar_1, \hbar_2, \hbar_1 \neq \hbar_2$, we define the following:

(1) The arithematic mean:

$$A(\hbar_1,\hbar_2)=\frac{\hbar_1+\hbar_2}{2}, \qquad \hbar_1,\hbar_2\in\Re.$$

(2) The logarithmic mean:

$$L(\hbar_1,\hbar_2)=\frac{\hbar_2-\hbar_1}{\log\hbar_2-\log\hbar_1}.$$

(3) The generalized logarithmic mean:

$$L_s(\hbar_1,\hbar_2)=\left[\frac{\hbar_2^{s+1}-\hbar_1^{s+1}}{(\hbar_2-\hbar_1)\,(s+1)}\right]^{\frac{1}{s}}, \quad s\in\Re\backslash\{-1,0\},\ \hbar_1,\hbar_2\in\Re,\ \hbar_1<\hbar_2.$$

**Proposition 3.1.** *Let* $\varpi^{\eth} \in \Re^+$ *and* $0 < \hbar_1 < \hbar_2$. *Then for some fixed* $s \in (0,1]$, *we have*

(3.1)

$$\left|\frac{2}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\varpi^{s\eth}\left(1-\mu^{\eth}\right)A\left(\left(\varpi^{\eth}-\hbar_1^{\eth}\right),\left(\hbar_2^{\eth}-\varpi^{\eth}\right)\right)\right.\right.$$
$$\left.\left.+\,\mu^{\eth}A\left(\left(\varpi^{\eth}-\hbar_1^{\eth}\right)\hbar_1^{s\eth},\left(\hbar_2^{\eth}-\varpi^{\eth}\right)\hbar_2^{s\eth}\right)\right]-L^{s+\eth}_{s+\eth-1}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)\right|$$

$$\leq \frac{s\eth}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\Bigg[\ \left(\varpi^{\eth}-\hbar_1^{\eth}\right)^2\left(\varpi^{s\eth-1}T_1\left(1;\mu^{\eth},s\right)+\hbar_1^{s\eth-1}T_2\left(1;\mu^{\eth},s\right)\right)$$
$$+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^2\left(\varpi^{s\eth-1}T_1\left(1;\mu^{\eth},s\right)+\hbar_2^{s\eth-1}T_2\left(\Upsilon;\mu^{\eth},s\right)\right)\Bigg],$$

*where*

$$T_1\left(1;\mu^{\eth},s\right)=\frac{2\mu^{\eth(s+2)}+s+1}{\eth\left(s+1\right)\left(s+2\right)}-\frac{\mu^{\eth}}{\eth\left(s+1\right)},$$

*and*

$$T_2\left(1;\mu^{\eth},s\right)=\frac{\mu^{\eth}}{\eth}\left[\frac{1-2\left(1-\mu^{\eth}\right)^{(s+1)}}{s+1}\right]+\beta\left(2\eth,\eth\left(s+1\right)\right)$$
$$-2\beta\left(\mu^{\eth},2\eth,\eth\left(s+1\right)\right).$$

*Proof.* From Theorem 2.5 with $\Psi(x)=x^s$, $x\in\left[\hbar_1^{\eth},\hbar_2^{\eth}\right]$, $\Upsilon=1$ and $s\in(0,1]$, then we obtain the inequality (3.1). □

**Proposition 3.2.** *Let $\varpi^{\eth}\in\Re^{+}$ and $0<\hbar_1<\hbar_2$. Then for a fixed $s\in(0,1]$, we have*

(3.2)

$$\Bigg|\frac{2}{\eth\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\Bigg[\varpi^{s\eth}\left(1-\mu^{\eth}\right)A\left(\left(\varpi^{\eth}-\hbar_1^{\eth}\right),\left(\hbar_2^{\eth}-\varpi^{\eth}\right)\right)$$
$$+\mu^{\eth}A\left(\left(\varpi^{\eth}-\hbar_1^{\eth}\right)\hbar_1^{s\eth},\left(\hbar_2^{\eth}-\varpi^{\eth}\right)\hbar_2^{s\eth}\right)\Bigg]-L_{s+\eth-1}^{s+\eth}\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)\Bigg|$$
$$\leq\frac{2s\eth T_4\left(1;\mu^{\eth},p\right)}{\left(\hbar_2^{\eth}-\hbar_1^{\eth}\right)}\left[\frac{\left(\varpi^{\eth}-\hbar_1^{\eth}\right)^2A^{q(s\eth-1)}(\varpi,\hbar_1)+\left(\hbar_2^{\eth}-\varpi^{\eth}\right)^2A^{q(s\eth-1)}(\varpi,\hbar_2)}{\eth(s+1)}\right]^{\frac{1}{q}},$$

*where*

$$T_4\left(1;\mu^{\eth},p\right)=\begin{cases}\frac{1}{\eth(p+1)} & \mu=0\\ \Big\{\mu^{\eth(p+1)}\beta\left(\eth,\eth p+1\right) & \\ \quad+\frac{\left(1-\mu^{\eth}\right)^{p+1}}{\eth(p+1)}\ {}_2F_1\left(0,1;p+2;1-\mu^{\eth}\right)\Big\} & 0<\mu<1\\ \beta\left(\eth\left(p+1\right),\frac{1}{\eth}\right) & \mu=1.\end{cases}$$

*Proof.* From Theorem 2.34 with $\Psi(x)=x^s$, $x\in\left[\hbar_1^{\eth},\hbar_2^{\eth}\right]$, $\Upsilon=1$ and $s\in(0,1]$, then we obtain the inequality (3.2). □

## 4. Conclusion

This study focuses on introducing novel Hermite-Hadamard type, Simpson type, Ostrowski type and midpoint type integral inequalities for s-convex functions. A general auxiliary identity related to the first-order derivatives is obtained. In this paper, we have derived a new Hermite-Hadamard type inequality for s-convex functions. To the best of our knowledge, these results are new in the literature, guiding us to develop refinements of different fractional inequalities in terms of special functions. We present a rigorous variety of bounds by employing power mean, Hölder and Young inequalities. Additionally, some achievements have been obtained using Katugampola fractional integral operators. Applications to special means of real numbers are also presented. All the inequalities are closely related and hold importance in the fields of integral inequalities. These identities can be applied to derive different results in mathematical inequalities, fractional calculus, convex analysis and various areas of applied and pure mathematics.

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