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Global Solutions for a Nonlocal Problem with Logarithmic Source Term

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Abstract. The current paper discusses the global existence and asymptotic behavior of solutions of the following new nonlocal problem

$$u_{tt} - M\left(\int_\Omega |\nabla u|^2\, dx\right) \triangle u + \delta u_t = |u|^{\rho-2} u \log|u|, \quad \text{in } \Omega\times]0,\infty[,$$

where

$$M(s) = \begin{cases} a - bs, & \text{for } s \in [0, \frac{a}{b}[, \\ 0, & \text{for } s \in [\frac{a}{b}, +\infty[. \end{cases}$$

If the initial data are appropriately small, we derive existence of global strong solutions and the exponential decay of the energy.

1. Introduction

In this research, we study the following nonlocal problem

(1.1)

$$\begin{aligned} &u_{tt} - M\left(\int_\Omega |\nabla u|^2\, dx\right) \triangle u + \delta u_t = |u|^{\rho-2} u \log|u|, \quad \text{in } \Omega\times]0,\infty[, \\ &u = 0, \quad \text{on } \Gamma\times]0,\infty[, \\ &u(x,0) = u^0(x), u_t(x,0) = u^1(x), \quad x \in \Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ,

$$M(s) = \begin{cases} a - bs, & \text{for } s \in [0, \frac{a}{b}[, \\ 0, & \text{for } s \in [\frac{a}{b}, +\infty[, \end{cases} \tag{1.2}$$

$a, b > 0, \rho > 2$ and $\delta > 0$.

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When $M(s) = a + bs,\ s \geq 0, a \geq 0,\ b \geq 0,\ a + b > 0,\ \delta = 0 = \mu$ and Ω is a finite open interval, equation (1.1) was introduced by Kirchhoff [17] as a model of the transversal oscillation of a string and is called the wave equation of Kirchhoff type after his name. See also Lions [18]. In general, for any nonnegative function M, the equation (1.1) is said a degenerate equation when $M(s)$ has zeros and a nondegenerate one when $M(s) \geq m_0 > 0, \forall s \geq 0$. Equations with logarithmic nonlinearity have a lot of applications in the fields of geophysics, quantum mechanics, inflationary cosmology and so on, see [3, 4, 9, 14]. A number of results on the solutions to problem (1.1) with polynomial nonlinearities instead of the logarithmic source have been established by many researches through various approaches and assumptive conditions; see [1, 2, 5, 12, 20, 22, 25] and references therein. Concerning nonlinear wave equations with logarithmic nonlinearities, there is many literature; see in this regard [7, 8, 26] and references therein.

In [27] Yin et al. investigated the existence and multiplicity of non-trivial solution for the new nonlocal problem

$$\begin{aligned} -\left(a - b\int_\Omega |\nabla u|^2\,dx\right)\triangle u &= |u|^{\rho-2}u, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \Gamma. \end{aligned} \tag{1.3}$$

Also, see [16, 23, 28] for generalizations of (1.3).

At this point, we would like to mention that when $a = 0 = \delta$ the equation (1.1) becomes the quasilinear non well posed problem

$$\begin{aligned} &u_{tt} - \left(\int_\Omega |\nabla u|^2\,dx\right)\triangle u + \delta u_t = |u|^{\rho-2}u\log|u|, \quad \text{in } \Omega\times]0,\infty[, \\ &u = 0, \quad \text{on } \Gamma\times]0,\infty[, \\ &u(x,0) = u^0(x), u_t(x,0) = u^1(x), \quad x \in \Omega, \end{aligned}$$

which can be seen as a boundary value problem for the potential equation as in [19]. The example of Hadamard [15] is the case of the potential equation in $\mathbb{R}^2$. This question has some interest in the study of the optimal control for singular distributed system and it seems to be essentially open.

Inspired by the aforementioned studies, we are concerned with the global solvability of (1.1) with the nonlocal operator given in (1.3). More precisely, under appropriate assumptions imposed on the initial data and the logarithmic term, we shall establish global existence of solutions by using of Tartar method [24] combined with suitable a priori estimates including $|\triangle u(t)|$ and $|\nabla_t u(t)|$ in addition to the usual energy estimate. In this sense, it is worth noting that to handle the logarithmic term we do not use the famous logarithmic Sobolev inequality, but we work with

algebraic inequalities related to the logarithmic function. To the best of our knowledge, this is the first work that take into account existence and decay estimate of global strong solutions for the logarithmic Kircchoff equation.

This article is organized as follows. In Section 2, we prepare some lemmas needed for our arguments and state the local existence theorem. In Section 3, we derive the global solution and its exponential decay.

2. Preliminaries

Throughout this paper, the functions are all real valued and the notations are as usual, in particular we shall denote by $\|.\|_p, (p \geq 1)$ the ususal L^p-norm, $(u, v) = \int_\Omega uv\, dx$ the inner product and by $\|.\|$ the norm in $H_0^1(\Omega)$. Positive constants will be denote by C and will change from line to line.

Lemma 2.1 (Sobolev- Poincaré [13]). *Let q be a number with $2 \leq q < \infty$ $(n = 1, 2)$ or $2 \leq q \leq \frac{2N}{N-2}$ $n \geq 3$), then there is a positive constant $C_* = C(\Omega, q)$ such that*

$$\|u\|_q \leq C_* \|\nabla u\|_2, \quad \text{for } u \in H_0^1(\Omega),$$

Thus, the norm $\|\nabla u\|_2$ is equivalent to the usual norm in $H_0^1(\Omega)$.

Lemma 2.2 (Gagliardo- Nirenberg [13]). *Let $1 \leq r < q \leq \infty$ and $q \leq p$. Then the inequality*

$$\|u\|_q \leq C \|u\|_{W^{m,q}}^{\theta} \|u\|_r^{1-\theta}, \quad \text{for } u \in W^{m,q}(\Omega) \bigcap L^r(\Omega).$$

holds with some constant $C > 0$ and

$$\theta = \left(\frac{1}{r} - \frac{1}{p}\right) \left(\frac{1}{r} + \frac{m}{N} - \frac{1}{q}\right)^{-1},$$

provided that $0 < \theta \leq 1$ $(0 < \theta < 1 \text{if } p = \infty)$.

Lemma 2.3 (Nakao [21]). *Let $\Phi(t)$ be a nonincreasing and nonnegative function on $[0, T]$, $T > 1$, such that*

$$\Phi(t)^{1+r} \leq k_0(\Phi(t) - \Phi(t+1)), \quad \text{on } [0, T],$$

where k_0 is a positive constant and r a nonnegative constant. Then, we have

(i) *if $r > 0$, then $\Phi(t) \leq \left(\Phi(0)^{-r} + k_0^{-1} r[t-1]^+\right)^{-1/r}$, on [0, T], where $[t-1]^+ = \max\{t-1, 0\}$,*

(ii) *if $r = 0$, then $\Phi(t) \leq \Phi(0) e^{-k_1[t-1]^+}$, on [0, T], where $k_1 = \log\left(\frac{k_0}{k_0 - 1}\right)$.*

We note that problem (1.1) can be considered with the new general nonlocal term $M(s) = a - bs^\gamma,\ \forall\ s \in [0,\ \sqrt[\gamma]{\frac{a}{b}}[, \gamma \geq 1$ and $f \in C^1(\mathbb{R})$

$$\begin{aligned} &u_{tt} - M\left(\int_\Omega |\nabla u|^2\, dx\right) \triangle u + \delta u_t = f(u), \quad \text{in } \Omega\times]0,\infty[,\\ &u = 0, \quad \text{on } \Gamma\times]0,\infty[,\\ &u(x,0) = u^0(x), u_t(x,0) = u^1(x), \quad x \in \Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ.

For sake of completeness, we recall the following local existence result, which may be proved by the Banach contraction mapping principle (See [10, 11]).

Theorem 2.4 (Local existence)**.** *Let $\delta > 0$, let $M : [0,+\infty[\to [0,+\infty[$ be a locally Lipschitz function, $f \in C^1(\mathbb{R})$, and let $(u_0, u_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ with $M(\|u_0\|^2) > 0$.*

Then there exists $T = T(\|\triangle u_0\|_2, \|u_1\|) > 0$ such the problem (2.1) admits a unique local solution u in the class

$$C^0\left([0,T); H_0^1(\Omega) \cap H^2(\Omega)\right) \cap C^1\left([0,T); H_0^1(\Omega)\right) \cap C^2\left([0,T); L^2(\Omega)\right).$$

Moreover, u can be uniquely continued to a maximal solution defined in an interval $[0; T[$, and at least one of the following statements is valid:

(i) $T = +\infty$,
(ii) $\limsup_{t\to T^-} \|u_t(t)\|^2 + \|\triangle u(t)\|_2^2 \to +\infty$,
(iii) $\liminf_{t\to T^-} M(\|u(t)\|^2) \to 0$.

Now, we set

$$B_\rho = \sup_{\substack{u\in H_0^1(\Omega)\\ u\neq 0}} \frac{\|u\|\rho}{\|u\|}, \qquad \gamma_1 = \frac{b}{4a}, \qquad \gamma_2 = \frac{B_\rho^{\rho+1}}{3(\rho+1)a}.$$

Define the function

$$h(\lambda) = \frac{1}{4}\lambda^2 - \gamma_1\lambda^4 - \frac{3}{2}\gamma_2\lambda^{\rho+1},$$

then

$$h'(\lambda) = \lambda\left(\frac{1}{2} - 4\gamma_1\lambda^2 - \frac{3}{2}(\rho+1)\gamma_2\lambda^{\rho-1}\right).$$

So, choosing $\lambda \in \mathbb{R}$, such that

$$0 \leq \lambda^2 \leq \frac{1}{16\gamma_1}, \qquad 0 \leq \lambda^{\rho-1} \leq \frac{1}{6(\rho+1)\gamma_2},$$

we get that this $\lambda' s$ satisfy the inequality

$$\frac{1}{2} - 4\gamma_1\lambda^2 - \frac{3}{2}(\rho+1)\gamma_2\lambda^{\rho-1} \geq 0$$

and $h'(\lambda) \geq 0$ for $0 \leq \lambda \leq \lambda_1$, where

$$\lambda_1 = \min\left\{(16\gamma_1)^{-1/2}, [6(\rho+1)\gamma_2]^{-1/(\rho-1)}\right\}. \tag{2.2}$$

Thus $h(0) = 0$, and $h(\lambda) \geq 0$, $\forall \lambda \in [0, \lambda_1]$.

From this , we get

$$h_0(\lambda) = \frac{1}{2}\lambda^2 - \gamma_1\lambda^4 - \gamma_2\lambda^{\rho+1} \tag{2.3}$$

$$\geq \frac{1}{4}\lambda^2 + \frac{1}{2}\gamma_2\lambda^{\rho+1}, \quad \forall \lambda \in [0, \lambda_1]. \tag{2.4}$$

The energy associated with the problem (1.1) is given by

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)), \quad \text{for } u \in H_0^1(\Omega),$$

where

$$J(u(t)) = \frac{a}{2}\|u(t)\|^2 - \frac{b}{4}\|u(t)\|^4 - \frac{1}{\rho}\int_\Omega |u(t)|^\rho \log|u(t)|\, dx + \frac{1}{\rho^2}\|u(t)\|_\rho^\rho.$$

Multiplying equation (1.1) by $u_t(t)$ and integrating it over Ω, we obtain

$$\frac{d}{dt}E(t) + \delta\|u_t(t)\|_2^2 = 0. \tag{2.5}$$

Therefore, $E(t)$ is a nonincreasing function on t, and

$$E(t) + \delta\int_0^t \|u_t(s)\|_2^2\, ds = E(0). \tag{2.6}$$

3. Global Existence and Exponential Decay

In this section, we state the main results of this paper. Firstly, we give the following two propositions.

Proposition 3.1. *If the local solution $u(t)$ of (1.1) satisfies $0 < \|u(t)\| < \lambda_1$ on $[0, T_0]$, then*

$$J(u(t)) \geq a\left(\frac{1}{4}\|u(t)\|^2 + \frac{\gamma_2}{2}\|u(t)\|^{\rho+1}\right), \tag{3.1}$$

and

$$\|u(t)\| \leq \left[\frac{4}{a}E(t)\right]^{1/2}. \tag{3.2}$$

Proof. It is obvious, from (2.3). In fact

$$\begin{aligned} J(u(t)) &\geq a h_0(\|\nabla u(t)\|_2) \\ &\geq a\left(\frac{1}{4}\|\nabla u(t)\|_2^2 + \frac{\gamma_2}{2}\|\nabla u(t)\|_2^\rho\right). \end{aligned}$$

So, (3.1) holds. Also

$$\begin{aligned} E(u(t)) &\geq J(u(t)) \\ &\geq \frac{a}{4}\|\nabla u(t)\|_2^2, \end{aligned}$$

which implies (3.2). □

Proposition 3.2. *Under the assumption of Proposition 3.1, the energy $E(t)$ satisfies*

$$E(t) \leq C_E E(0) e^{-kt}, \tag{3.3}$$

where $k = \ln\left(\frac{k_0}{k_0-1}\right)$, $k_0 = 2\left[1+\delta+\frac{1}{\eta_0}+\frac{2}{a}(\frac{\delta+4}{\eta_0}B_2)^2\right]+1$, $B_{\rho+1}$ *is the best constant in the embedding* $H_0^1(\Omega) \hookrightarrow L^{\rho+1}(\Omega)$, $\eta_0 = 1 - \frac{1}{a}\left[B_{\rho+1}^{\rho+1}\left(\frac{4}{a}E(0)\right)^{(\rho-1)/2} + \frac{4b}{a}E(0)\right]$ *and* $C_E = \max\{1, e^k\}$.

Proof. For a moment, we suppose that $T > 1$. Integrating (2.5) from t to we find

$$\begin{aligned} \delta\int_t^{t+1}\|u_t(s)\|_2^2\,ds &= E(t) - E(t+1) \\ &\equiv \delta F^2(t). \end{aligned}$$

Using the mean value theorem for integrals, there exist two points $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t_i)\|_2 \leq 2F(t), \quad i = 1, 2.$$

Next, multiplying (1.1) by u and integrating over Ω , we obtain

$$\begin{aligned} a\|u(t)\|^2 - b\|u(t)\|^4 - \int_\Omega |u(t)|^\rho \log|u(t)|\,dx \\ = \|u_t(t)\|_2^2 - \delta(u_t(t), u(t)) - \frac{d}{dt}(u_t(t), u(t)). \end{aligned} \tag{3.4}$$

On the other hand, it follows from the Sobolev-Poincaré inequality and (3.2) that

$$\begin{aligned} \left|\int_\Omega |u(t)|^\rho \log|u(t)|\,dx\right| &\leq \|u(t)\|_{\rho+1}^{\rho+1} \\ &\leq B_{\rho+1}^{\rho+1}\|u(t)\|^{\rho+1} \\ &\leq B_{\rho+1}^{\rho+1}\|u(t)\|^{\rho-1}\|u(t)\|^2 \\ &\leq B_{\rho+1}^{\rho+1}\left[\frac{4}{a}E(0)\right]^{(\rho-1)/2}\|u(t)\|^2 \end{aligned}$$

and

$$b\|u(t)\|^4 \leq b\left[\frac{4}{a}E(0)\right]\|u(t)\|^2.$$

Thus , we get

(3.5)

$$b\|u(t)\|^4 + \|u(t)\|_{\rho+1}^{\rho+1} \le \frac{1}{a}\left[B_{\rho+1}^{\rho+1}\left(\frac{4}{a}E(0)\right)^{(\rho-1)/2} + \frac{4b}{a}E(0)\right](a\|u(t)\|^2)$$
$$\equiv (1-\eta_0)(a\|u(t)\|^2), \quad 0<\eta_0<1.$$

Then

$$\eta_0 a\|u(t)\|^2 \le a\|u(t)\|^2 - b\|u(t)\|^4 - \|u(t)\|_{\rho+1}^{\rho+1} \equiv I(t). \tag{3.6}$$

From (2.6) and (3.4), integrating the resultant inequality over $[t_1, t_2]$, we have

(3.7)

$$\begin{aligned}
&\eta_0 a\int_{t_1}^{t_2}\|u(s)\|^2\,ds\\
&\le \int_{t_1}^{t_2} I(s)\,ds\\
&\le \int_{t_1}^{t_2}\|u_t(s)\|_2^2\,ds + \delta\int_{t_1}^{t_2}|(u_t(s),u(s))|\,ds - (u_t(t),u(t))|_{t_1}^{t_2}\\
&\le F^2(t) + \left[\delta\left(\int_t^{t+1}\|u_t(s)\|_2^2\,ds\right)^{1/2} + \sum_{i=1}^{2}\|u(t_i)\|_2\right]\sup_{s\in[t,t+1]}\|u(s)\|_2\\
&\le F^2(t) + (\delta+4)B_2F(t)\left(\frac{4}{a}E(t)\right)^{1/2}.
\end{aligned}$$

On the other hand, integrating (2.5) over $[t, t_2]$, noting that $E(t_2) \le 2\int_{t_1}^{t_2} E(s)\,ds$ due to $t_2 - t_1 \ge \frac{1}{2}$, using (3.7) and the Young inequality, we have

$$\begin{aligned}
E(t) =& E(t_2) + \delta\int_t^{t_2}\|u_t(s)\|_2^2\,ds\\
\le& 2\int_{t_1}^{t_2}E(s)\,ds + \delta\int_t^{t+1}\|u_t(s)\|_2^2\,ds\\
\le& (1+\delta)\int_t^{t+1}\|u_t(s)\|_2^2\,ds + a\int_{t_1}^{t_2}\|u(s)\|^2\,ds\\
\le& (1+\delta+\frac{1}{\eta_0})F^2(t) + \frac{1}{2}\left(\frac{\delta+4}{\eta_0}B_2\right)^2\frac{4}{a}F^2(t) + \frac{1}{2}E(t).
\end{aligned}$$

Thus

$$E(t) \le k_0(E(t) - E(t+1)).$$

For $0 \leq t \leq T_0 \leq 1$, since $E(t)$ is bounded, we have

$$E(t) \leq E(0) = e^{kt}E(0)e^{-kt} \leq e^k E(0)e^{-kt}.$$

Therefore, noting (2.6) and applying Lemma 2.3, we obtain (3.3). □

Next, in order to state and prove our main result, we introduce the following constants.

$$\begin{aligned}
&\tau_0 = a - b\|u_0\|^2,\\
&\alpha_0 = 2(\rho-2)(1-\theta_0),\\
&\alpha_1 = 2(\rho-2-\mu_1)(1-\theta_1),\\
&\alpha_2 = 2(\rho-2+\mu_2)(1-\theta_2)\\
&\beta_0 = (\rho-2)\theta_0 + 1,\\
&\beta_1 = (\rho-2-\mu_1)\theta_1 + 1,\\
&\beta_2 = (\rho-2+\mu_2)\theta_2 + 1,\\
&K_0 = C_6 \max\left\{C^{\rho-2}_{3(\rho-2)}, C^{\rho-2-\mu_1}_{3(\rho-2)-3\mu_1}, C^{\rho-2+\mu_2}_{3(\rho-2+\mu_2)}\right\},
\end{aligned}$$

where the $C's$ are positive constants corresponding to the respective Sobolev embeddings, $\theta_i, i = 0, 1, 2$ are the exponents given by Lemma 2.2 and (due to $2 < 3(\rho-2) < \frac{2N}{N-2}$) μ_1, μ_2 are positive constants such that $2 < 3(\rho-2) - 3\mu_1 < \frac{2N}{N-2}$ and $2 < 3(\rho-2+\mu_2) < \frac{2N}{N-2}$, $N = 3$.

The main result of this paper is the following theorem.

Theorem 3.3. *Let $N = 3$ and $\frac{8}{3} < \rho < 4$. Assume further that $\{u_0, u_1\}$ belong to $(H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ with*

$$\|u_0\| < \min\left\{\left(\frac{a}{b}\right)^{1/2}, \lambda_1\right\}, \qquad \left[\frac{4}{a}E(0)\right]^{1/2} < \lambda_1, \tag{3.8}$$

and

$$\left[\frac{4b^2E(0)}{\tau_0 a}(H(0)+\beta_*)\right]^{1/2}\left[1 - d_1\sum_{i=0}^{2}\frac{2}{k\alpha_i}\left[\frac{4E(0)}{a}\right]^{\alpha_i/2}\right]^{-\frac{1}{2(\beta_i^*-1)}} < 1, \tag{3.9}$$

where

$$\begin{aligned}
&\beta_* = \min_{0\leq i\leq 2}\{\beta_i\},\\
&\beta^* = \max_{0\leq i\leq 2}\{\beta_i\},\\
&d_1 = \frac{(\beta^*-1)4K_0(H(0)+\beta_*)^{\beta^*-1}}{\delta\tau_0},
\end{aligned}$$

and $H(0) = \frac{\|\nabla u_1\|^2}{\tau_0} + \|\triangle u_0\|_2^2$. *Then problem* (1.1) *admits a unique global solution*

$$u \in C([0,+\infty[; H_0^1(\Omega)\cap H^2(\Omega))\cap C^1([0,+\infty[; H_0^1(\Omega))\cap C^2([0,+\infty[; L^2(\Omega)),$$

and the energy satisfies

$$E(t) \le Ce^{-kt}, \quad \text{for } t \ge 0, \tag{3.10}$$

with some constant $C > 0$.

Remark 3.4. It is easy to see that, from (2.2), the condition $\left(\frac{4}{a}E(0)\right)^{1/2} < \lambda_1$ implies

$$\frac{1}{a}\left[B_{\rho+1}^{\rho+1}\left(\frac{4}{a}E(0)\right)^{(\rho-1)/2} + \frac{4b}{a}E(0)\right] < 1. \tag{3.11}$$

In fact, by definition of $B_{\rho+1}$ and λ_1 we have

$$\begin{aligned}\frac{1}{a}B_{\rho+1}^{\rho+1}\left(\frac{4}{a}E(0)\right)^{(\rho-1)/2} &< 3(\rho+1)\gamma_2\lambda_1^{\rho-1}\\ &< \frac{1}{2}\end{aligned}$$

and similarly, we get $\frac{1}{a}(\frac{4b}{a}E(0)) < \frac{1}{4}$. So, (3.11) holds.

Proof. Let $u(t)$ be a unique solution of the problem (1.1) in the sense Theorem 2.4 on $[0, T_0[$, with T_0 the maximal time where the solution exists. Here, we note that by assumption (3.8), we have $\|u(t)\| < \lambda_1$. Suppose that it were not the case. Then there exists $0 < t_1 < T_0$ such that $\|u(t_1)\| = \lambda_1$. Putting

$$t^* = \inf\{t_1 \in]0, T_0[: \|u(t_1)\| = \lambda_1\},$$

we see, by continuity of the norm, that $\|u(t^*)\| = \lambda_1$, so $0 < t^* < T_0$. Let $t \in [0, t^*[$. Then $\|u(t)\| < \lambda_1$. Thus, inequality (3.2) provides

$$\|u(t)\| < \left[\frac{4}{a}E(0)\right]^{1/2}, \quad \forall t \in [0, t^*[.$$

This implies

$$\lambda_1 = \|u(t^*)\| \le \left[\frac{4}{a}E(0)\right]^{1/2}$$

which contradicts (3.8).

Now, we introduce the function

$$N(t) = a - b\|u(t)\|^2, \quad \text{for } t \in [0, T_0[.$$

So, under the assumption (3.8), we have

$$N(0) = a - b\|u_0\|^2 = \tau_0 > 0$$

Then there exists T_0' such that $0 < T_0' \leq T_0$ and

$$N(t) > \tau_0, \quad \text{for } 0 \leq t \leq T_0'.$$

We can define

$$T_1 = \sup\{t \in [0, +\infty[: N(t) > \tau_0, 0 \leq s < t\}.$$

We see that $T_1 > 0$ and $N(t) > \tau_0, \quad \text{for } 0 \leq s < T_1$.

Let us set

$$H(t) = \frac{\|\nabla u_t(t)\|^2}{N(t)} + \|\triangle u(t)\|_2^2, \qquad f(u(t)) = |u(t)|^{\rho-2}u(t)\log|u(t)|.$$

So, for $0 \leq s < T_1$, taking the derivative of $H(t)$ we have

$$\frac{d}{dt}H(t) + 2\delta\frac{\|u_t(t)\|^2}{N(t)}\left(1 + \frac{N'(t)}{N(t)}\right) = \frac{2}{N(t)}\left(\nabla f(u(t)), \nabla u_t(t)\right).$$

Here, we observe that

$$\begin{aligned} \left|\frac{N'(t)}{N(t)}\right| &\leq \left|\frac{b(\nabla u(t), \nabla u_t(t))}{N(t)}\right| \\ &\leq \frac{b\|u(t)\|}{N^{1/2}(t)}H^{1/2}(t) \\ &\leq \left(\frac{4b^2}{\tau_0 a}E(0)\right)^{1/2} H^{1/2}(t). \end{aligned} \tag{3.12}$$

Thus, we obtain from Lemmas 2.1 and 2.2 that

$$\begin{aligned} &\left|\frac{2}{N(t)}(\nabla f(u(t)), \nabla u_t(t))\right| \\ &\leq \frac{2}{N(t)}\left[\||u(t)|^{\rho-2}\nabla u(t)\|_2\right. \\ &\quad \left.+(\rho-1)\|(|u(t)|^{\rho-2}\log|u(t)|)\nabla u(t)\|_2\right]\|u_t(t)\| \\ &\leq \frac{2}{N(t)}\left[\|u(t)\|_{3(\rho-2)}^{\rho-2} + \frac{\rho-1}{e\mu_1}\|u(t)\|_{3(\rho-2)-3\mu_1}^{\rho-2-\mu_1}\right. \\ &\quad \left.+\frac{\rho-1}{e\mu_2}\|u(t)\|_{3(\rho-2+\mu_2)}^{\rho-2+\mu_2}\right]\|\nabla u(t)\|_6\|u_t(t)\| \\ &\leq \frac{2K_0}{N(t)^{1/2}}\left[\|u(t)\|^{(\rho-2)(1-\theta_0)}H(t)^{\frac{(\rho-2)\theta_0+1}{2}}\right. \\ &\quad + \|u(t)\|^{(\rho-2-\mu_1)(1-\theta_1)}H(t)^{\frac{(\rho-2-\mu_1)\theta_1+1}{2}} \\ &\quad \left.+ \|u(t)\|^{(\rho-2+\mu_2)(1-\theta_2)}H(t)^{\frac{(\rho-2+\mu_2)\theta_2+1}{2}}\right]\frac{\|u_t(t)\|}{N^{1/2}(t)} \\ &\leq \frac{4K_0}{\delta\tau_0}\left[\|u(t)\|^{\alpha_0}H(t)^{\beta_0} + \|u(t)\|^{\alpha_1}H(t)^{\beta_1}\right. \end{aligned} \tag{3.13}$$

$$+\|u(t)\|^{\alpha_2}H(t)^{\beta_2}\Big] + \delta\frac{\|u_t(t)\|^2}{N(t)}.$$

Hence, it follows from (3.12) and (3.13) that

$$\frac{d}{dt}H(t) + \delta\frac{\|u_t(t)\|^2}{N(t)}\left[1-\left(\frac{4b^2E(0)}{\tau_0 a}\right)^{1/2}H^{1/2}(t)\right] \leq \frac{4K_0}{\delta\tau_0}\sum_{i=0}^{2}\|u(t)\|^{\alpha_i}H(t)^{\beta_i}. \tag{3.14}$$

If

$$1-\left(\frac{4b^2E(0)}{\tau_0 a}\right)^{1/2}H^{1/2}(t) > 0, \quad \forall t\in[0,T_1[\tag{3.15}$$

does not hold, it implies a contradiction. In fact, from (3.9) and the continuity of $H(t)$, we can see that there exists $t^* > 0$ such that

$$1-\left(\frac{4b^2E(0)}{\tau_0 a}\right)^{1/2}H^{1/2}(t^*) = 0, \tag{3.16}$$

$$1-\left(\frac{4b^2E(0)}{\tau_0 a}\right)^{1/2}H^{1/2}(t) > 0, \quad \forall t\in[0,t^*[. \tag{3.17}$$

Now, integrating (3.14) from 0 to t^* and using (3.17) we have

$$H(t^*) \leq H(0) + \frac{4K_0}{\delta\tau_0}\int_0^{t^*}\sum_{i=0}^{2}\|u(s)\|^{\alpha_i}H(s)^{\beta_i}\,ds$$

and, hence by Theorem 2.6 in [6]

$$H(t^*) \leq (H(0)+\beta_*).\left[1-d_1\int_0^{t^*}\sum_{i=0}^{2}\|u(s)\|^{\alpha_i}\,ds\right]^{\frac{1}{1-\beta^*}} \tag{3.18}$$

On the other hand, we see from (3.2) and (3.3) that

$$\int_0^{t^*}\|u(s)\|^{\alpha_i}\,ds \leq \frac{2^{\alpha_i}}{a^{\alpha_i/2}}\int_0^{t^*}E^{\alpha_i/2}(s)\,ds \leq \frac{2}{k\alpha_i}\left[\frac{4E(0)}{a}\right]^{\alpha_i/2} \tag{3.19}$$

By (3.18) and (3.19), we infer that

$$H(t^*) \leq (H(0)+\beta_*)\left[1-d_1\sum_{i=0}^{2}\frac{2}{k\alpha_i}\left[\frac{4E(0)}{a}\right]^{\alpha_i/2}\right]^{-\frac{1}{\beta^*-1}}$$

Hence, using (3.9) we get

$$1 - \left(\frac{4b^2E(0)}{\tau_0 a}\right)^{1/2} H^{1/2}(t^*) > 0,$$

which contradicts (3.16). Thus (3.15) is true, and then

$$H(t) < \left[\frac{4b^2}{\tau_0 a}E(0)\right]^{-1}, \quad \forall t \in [0, T_1[. \tag{3.20}$$

Now, if $T_1 < T_0$, then we have $N(T_1) = 0$. Thus, it follows by (3.20) that $\|u_t(T_1)\| = 0$.

Induce a variable $v(t) = u(T_1 - t)$. Hence $v(t)$ satisfies

$$\begin{aligned}
&v_{tt} - M\left(\|\nabla v(t)\|_2^2\right)\triangle v = \delta v_t + f(v), \quad \text{in } \Omega\times]0, T_1],\\
&v = 0, \quad \text{on } \Gamma\times]0, T_1],\\
&v(0) = 0 = v_t(0), \quad \text{in } \Omega,
\end{aligned}$$

where $f(v) = |v|^{\rho-2}v\log|v|$.

Multiplying this equation by v_t as in integrating it over Ω we get

$$\begin{aligned}
\frac{d}{dt}E(v(t)) &= \delta\|v_t(t)\|_2^2\\
&\leq C(\|v_t(t)\|_2^2 + J(v(t))\\
&\leq CE(v(t)), \quad \forall t \in [0, T_1].
\end{aligned} \tag{3.21}$$

Notice that $E(v(0)) = 0$. Then, it follows from (3.21), applying the Gronwall inequality, that

$$\begin{aligned}
\frac{a}{4}\|v(t)\|^2 &\leq E(v(t))\\
&= 0.
\end{aligned}$$

Consequently $\|u(T_1 - t)\| = 0$, for $0 \leq t \leq T_1$. So $u(0) = 0$.

Thus $a - b\|u(0)\|^2 = \tau_0$ implies $a = \tau_0$, which is a contradiction. Therefore $N(t) > 0$, $\forall t \geq 0$ and (3.20) holds for all $t \geq 0$. Moreover, from Proposition 3.2 we obtain the decay estimate (3.10). □

Remark 3.5. It seems to be interesting to study a global solution for Kirchhoff equation with nonlinear source and boundary damping term or with nonlinear boundary damping and source term, i.e.

$$\begin{aligned}
&u_{tt} - M\left(\int_\Omega |\nabla u|^2\,dx\right)\triangle u = |u|^{\rho-2}u\log|u|, \quad \text{in } \Omega\times]0,\infty[,\\
&u = 0, \quad \text{on } \Gamma_0\times]0,\infty[,\\
&\frac{\partial}{\partial\nu}u = g(u_t), \quad \text{on } \Gamma_1\times]0,\infty[,\\
&u(x,0) = u^0(x), u_t(x,0) = u^1(x), \quad x \in \Omega,
\end{aligned}$$

and

$$\begin{aligned}
&u_{tt} - M\left(\int_\Omega |\nabla u|^2\, dx\right)\triangle u = 0, \quad \text{in } \Omega\times]0,\infty[,\\
&u = 0, \quad \text{on } \Gamma_0\times]0,\infty[,\\
&\frac{\partial}{\partial\nu}u = g(u_t) + |u|^{\rho-2}u\log|u|, \quad \text{on } \Gamma_1\times]0,\infty[,\\
&u(x,0) = u^0(x), u_t(x,0) = u^1(x), \quad x\in\Omega,
\end{aligned}$$

with $M(s)$ given in (1.2).

We plan to address these questions in a future research.

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