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On Deferred Statistical Summability in Intuitionistic Fuzzy n-Normed Linear Space

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Abstract. In this paper, we define strong deferred summability and deferred statistical summability in intuitionistic fuzzy n-normed linear spaces (briefly called IF-n-NLS) and study some of their properties. We also define deferred statistical Cauchy sequence deferred statistical completeness and characterize deferred statistical summability in such spaces. Finally, we will compare deferred statistical summability for different pairs of sequences $\alpha(\ell)$, $\gamma(\ell)$,$u(\ell)$ and $v(\ell)$ satisfying $\alpha(\ell) \leq u(\ell) < v(\ell) \leq \gamma(\ell)$.

1. Introduction

In 1965, Zadeh [50] introduced the idea of fuzzy logic to reduce the uncertainty in many mathematical problems. The idea was found very useful, especially in studying population dynamics [6], chaos control [14], computer programming [17], non-linear dynamical systems [20] and many more. With the development of fuzzy sets, many mathematical analysis ideas have grown. One among these is the idea of fuzzy norm initially introduced by Katsaras [23] as a generalization of the crisp norm for those problems whose norms are not predictable exactly due to tremendous uncertainty. The concept was redefined by Felbin [13] in 1992 using the idea of fuzzy real numbers. He has shown that a unique fuzzy norm exists up to fuzzy equivalence within finite dimensional fuzzy normed linear space. Using the idea of n-norm of Gunawan and Mashadi [18], Vijayabalaji and Narayanan [38] introduced the idea

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of fuzzy n-norm. They obtained some results on best approximation sets in α-n-normed space. For some more works in this direction, we refer to the reader [4, 7, 15, 16, 24, 28, 31].

Atanassov [2, 3] felt that the idea of a fuzzy set was not sufficient to resolve many problems, such as complex uncertainty, ambiguity and hesitation in decision-making processes. In view of given this, he generalized fuzzy sets and introduced intuitionistic fuzzy sets (IFS). Saadati and Park [41] used intuitionistic fuzzy sets to generalize the fuzzy norm. They introduced the intuitionistic fuzzy norm (IFN) by adjoining the non-membership function ψ and membership function φ. Subsequently, some extension of IFNLS have appeared in the literature. For instance, Mursaleen and Lohani [36] defined IF-2-NLS and studied some topological properties in these spaces. Vijayabalaji et al. [49] defined IF-n-NLS and studied convergent and Cauchy sequences in these spaces. In recent years, many summability methods such as statistical convergence [22], lacunary statistical convergence [37], ideal convergence [26], λ statistical convergence [32] and deferred statistical convergence [19, 27, 29] have also been considered in IFNLS. Subsequently, these summability methods were generalized in the context of IF-n-NLS (see [9, 45, 46]). We refer to the reader [1, 5, 11, 25, 29, 39, 40, 47] for systematic development on deferred statistical convergence. In the present paper, we will define strong deferred summability and deferred statistical summability in IF-n-NLS and study some of their properties in IF-n-NLS.

2. Preliminaries

This section will explore some of the basic definitions and results needed the sequel. We began with the concept of statistical convergence, independently introduced by Fast [12] and Steinhaus [48] with the help of natural density δ of subsets of $\mathbb{N}$, the set of positive integers. For $\mathbb{K} \subseteq \mathbb{N}$, the natural density δ of $\mathbb{K}$ is defined by

$$\delta(\mathbb{K}) = \lim_{\ell \to \infty} \frac{1}{\ell} |k \leq \ell : k \in \mathbb{K}|$$

provided the limit exists. Here vertical bars denote the cardinality of the enclosed set.

A sequence $p = (p_k)$ is said to be statistically convergent to p_0 if for each $\varepsilon > 0$,

$$\lim_{\ell \to \infty} \frac{1}{\ell} |\{k \leq \ell : |p_k - p_0| \geq \varepsilon\}| = 0$$

i.e., $\delta(\{k \in \mathbb{N} : |p_k - p_0| \geq \varepsilon\}) = 0$ and is denoted by $S - \lim p_k = p_0$.

Statistical convergence has wide applications in measure theory [30], trigo-nometric series [51], approximation theory [10] and many more. It was also studied from sequence spaces point of view and linked with

summability theory by Schoenberg [43], Salat [42], Connor [8] and also see ([21, 33–35]).

Definition 2.1 ([44])**.** Let $\mathcal{I}$ =[0, 1]. A binary operation $\circledast : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ is t-norm if for all $\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d} \in \mathcal{I}$ we have

(1) $\circledast$ is continuous commutative and associative,
(2) $\mathfrak{a} = \mathfrak{a} \circledast 1$,
(3) $\mathfrak{a} \circledast \mathfrak{b} \leq \mathfrak{c} \circledast \mathfrak{d}$ whenever $\mathfrak{a} \leq \mathfrak{c}$ and $\mathfrak{b} \leq \mathfrak{d}$.

Definition 2.2 ([44])**.** Let $\mathcal{I}$ =[0, 1]. A binary operation $\odot : mathcalI \times I \to I$ is t-norm if for all $\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d} \in \mathcal{I}$ we have

(1) $\odot$ is continuous commutative and associative,
(2) $\mathfrak{a} = \mathfrak{a} \odot 0$,
(3) $\mathfrak{a} \odot \mathfrak{b} \leq \mathfrak{c} \odot \mathfrak{d}$ whenever $\mathfrak{a} \leq \mathfrak{c}$ and $\mathfrak{b} \leq \mathfrak{d}$.

Using t-norm and t-conorm, Saadati and Park [41] defined IFNLS as follows:

Definition 2.3. The five-tuple structure $(\mathcal{P}, \varphi, \psi, \circledast,\ \odot)$ be an IFNLS, where $\mathcal{P}$ is a linear space over a field F. φ, ψ are fuzzy sets on $\mathcal{P} \times (0, \infty)$ and denotes the degree of membership and non-membership on $\mathcal{P} \times (0, 1)$ which fulfils the following conditions for every $p,\ t \in \mathcal{P}$ and $\lambda_1,\ \lambda_2 > 0$

(1) $\varphi(p, \lambda) + \psi(p, \lambda) \leq 1$,
(2) $\varphi(p, \lambda) > 0$,
(3) $\varphi(p, \lambda) = 1 \quad \Leftrightarrow \quad p = 0$,
(4) $\varphi(cp, \lambda) = \varphi\left(p, \frac{\lambda}{|c|}\right)$, if $c \neq 0,\ c \in F$,
(5) $\varphi(p, \lambda_1) \circledast \varphi(t, \lambda_2) \leq \varphi(p + t, \lambda_1 + \lambda_2)$,
(6) $\varphi(p, .) : (0, \infty) \to [0, 1]$ is continuous,
(7) $\lim\limits_{\lambda \to \infty} \varphi(p, \lambda) = 1,\ \lim\limits_{\lambda \to 0} \varphi(p, \lambda) = 0$,
(8) $\psi(p, \lambda) < 1$,
(9) $\psi(p, \lambda) = 0 \quad \Leftrightarrow \quad p = 0$,
(10) $\psi(cp, \lambda) = \psi\left(p, \frac{\lambda}{|c|}\right)$ if $c \neq 0,\ c \in F$,
(11) $\psi(p, \lambda_1) \ \odot \ \psi(t, \lambda_2) \geq \psi(p + t, \lambda_1 + \lambda_2)$,
(12) $\psi(p, .) : (0, \infty) \to [0, 1]$ is continuous,
(13) $\lim\limits_{\lambda \to \infty} \psi(p, \lambda) = 0\ ,\ \lim\limits_{\lambda \to 0} \psi(p, \lambda) = 1$.

Example 2.4. Let $(\mathcal{P}, \| \bullet \|)$ be an NLS. Define $\mathfrak{a} \circledast \mathfrak{b} = \mathfrak{a}\mathfrak{b}$, $\mathfrak{a} \odot \mathfrak{b} = \min\{\mathfrak{a} + \mathfrak{b}, 1\}\ \forall\ \mathfrak{a}, \mathfrak{b} \in [0, 1]$. Consider $\varphi(p, \lambda) = \frac{\lambda}{\lambda + \|p\|}$ and $\psi(p, \lambda) = \frac{\|p\|}{\lambda + \|p\|}$ for every $\lambda > 0$ and $p \in \mathcal{P}$. Then $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IFNLS.

By using the idea of n-norm [18], Vijayabalaji et al. [49] generalized the idea of IFNLS and introduced IF-n-NLS. He also studied convergent and Cauchy sequences in these spaces.

Definition 2.5. The five-tuple structure $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS, where $\mathcal{P}$ is a linear space over a field F. φ, ψ are fuzzy sets on $\mathcal{P}^n \times (0, \infty)$ and denotes the degree of membership and non-membership of $(p_1, p_2, \ldots, p_{n-1}, \ p_n, \lambda) \in \mathcal{P}^n \times (0, 1)$ which fulfils the following conditions for every $(p_1, p_2, \ldots, p_{n-1}, \ p_n) \in \mathcal{P}^n$ and $\lambda_1, \lambda_2 > 0$

(1) $\varphi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) + \psi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) \leq 1$,
(2) $\varphi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) > 0$,
(3) $\varphi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) = 1 \quad \Leftrightarrow \quad p_1, p_2, \ldots, p_{n-1}, p_n$ are linearly dependent,
(4) $\varphi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda)$ is invariant under any permutation of $p_1, p_2, \ldots, p_{n-1}, p_n$,
(5) $\varphi(p_1, p_2, \ldots, p_{n-1}, cp_n, \lambda) = \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_n, \frac{\lambda}{|c|}\right)$ if $c \neq 0$, $c \in F$,
(6) $\varphi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda_1) \circledast \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_n^{'}, \lambda_2\right) \leq \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_n + p_n^{'}, \lambda_1 + \lambda_2\right)$,
(7) $\varphi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) : (0, \infty) \to [0, 1]$ is continuous,
(8) $\lim\limits_{\lambda \to \infty} \varphi(p_1, \ldots, p_n, \lambda) = 1$, $\lim\limits_{\lambda \to 0} \varphi(p_1, \ldots, p_n, \lambda) = 0$,
(9) $\psi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) < 1$,
(10) $\psi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) = 0 \quad \Leftrightarrow \quad p_1, p_2, \ldots, p_{n-1}, p_n$ are linearly dependent,
(11) $\psi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda)$ is invariant under any permutation of $p_1, p_2, \ldots, p_{n-1}, p_n$,
(12) $\psi(p_1, p_2, \ldots, p_{n-1}, cp_n, \lambda) = \psi\left(p_1, p_2, \ldots, p_{n-1}, p_n, \frac{\lambda}{|c|}\right)$ if $c \neq 0$, $c \in F$,
(13) $\psi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda_1) \odot \psi\left(p_1, p_2, \ldots, p_{n-1}, p_n^{'}, \lambda_2\right) \geq \psi\left(p_1, p_2, \ldots, p_{n-1}, p_n + p_n^{'}, \lambda_1 + \lambda_2\right)$,
(14) $\psi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) : (0, \infty) \to [0, 1]$ is continuous,
(15) $\lim\limits_{\lambda \to \infty} \psi(p_1, \ldots, p_n, \lambda) = 0$, $\lim\limits_{\lambda \to 0} \psi(p_1, \ldots, p_n, \lambda) = 1$.

Let $(\mathcal{P}, \|\bullet, \bullet, ..., \bullet\|)$ be n-NLS. Define $\mathfrak{a} \circledast \mathfrak{b} = \mathfrak{a}\mathfrak{b}$, $\mathfrak{a} \odot \mathfrak{b} = \min\{\mathfrak{a} + \mathfrak{b}, 1\}$ $\forall\ \mathfrak{a}, \mathfrak{b} \in [0, 1]$,

$$\varphi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) = \frac{\lambda}{\lambda + \|p_1, p_2, \ldots, p_{n-1}, p_n\|}$$

and

$$\psi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) = \frac{\|p_1, p_2, \ldots, p_{n-1}, p_n\|}{\lambda + \|p_1, p_2, \ldots, p_{n-1}, p_n\|}$$

for every $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$. Then $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS.

Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be convergent sequence to p_0 w.r.t intuitionistic fuzzy n-norm $(\varphi, \psi)^n$ (briefly called IF-n-N) if for every $\varepsilon \in (0,1)$, $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$, $\exists k_0 \in \mathbb{N}$ such that

$$\varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) > 1 - \varepsilon$$

and

$$\psi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon, \quad \forall k \geq k_0.$$

It is denoted by $(\varphi, \psi)^n - \lim p_k = p_0$. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be Cauchy sequence w.r.t IF-n-N $(\varphi, \psi)^n$ if for every $\varepsilon \in (0,1)$, $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$, $\exists\ k_0 \in \mathbb{N}$ such that $\varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \lambda) > 1 - \varepsilon$ and $\psi(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \lambda) < \varepsilon$ for all $k,\ r \geq k_0$.

Definition 2.6 ([46])**.** Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be statistically convergent to $p_0 \in \mathcal{P}$ w.r.t IF-n-N $(\varphi, \psi)^n$ if for every $\varepsilon \in (0,1)$, $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$,

$$\lim_{\ell \to \infty} \frac{1}{\ell} \Big| \big\{ k \leq \ell : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon \\ \text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon \big\} \Big| = 0,$$

equivalently,

$$\delta\Big(\big\{ k \in \mathbb{N} : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon \\ \text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon \big\} \Big) = 0.$$

It is denoted by $[S(\varphi, \psi)^n] - \lim p_k = p_0$.

Recently, S. Melliani et al. [29] defined deferred statistical convergence in IFNLS and obtained some interesting results.
For any two sequences $\alpha = \{\alpha(\ell) : \ell \in \mathbb{N}\}$ and $\gamma = \{\gamma(\ell) : \ell \in \mathbb{N}\}$ of non-negative integers satisfying

$$\alpha(\ell) < \gamma(\ell) \quad \text{and} \quad \gamma(\ell) \to \infty \quad \text{as} \quad \ell \to \infty \tag{2.1}$$

Agnew [1] generalized the Cesàro summability, called deferred Cesàro summability as follows:

A sequence $p = (p_k)$ is called deferred Cesàro summable to p_0 if

$$\lim_{\ell \to \infty} (D_{\alpha,\gamma} p)_\ell = p_0,$$

where

$$(D_{\alpha,\gamma} p)_\ell = \frac{1}{\gamma(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} p_k.$$

kucukaslan and Yilmazturk [25] defined deferred density $\delta_{\alpha,\gamma}$ of subsets of $\mathbb{N}$ and studied deferred statistical summability for real number sequences. This method is called properly deferred if $\alpha(\ell)$ and $\gamma(\ell)$ satisfy the condition $\frac{\alpha(\ell)}{\gamma(\ell)-\alpha(\ell)}$, in addition to equation (2.1).
For $\mathbb{K} \subseteq \mathbb{N}$, the deferred density $\delta_{\alpha,\gamma}$ of $\mathbb{K}$ is defined by

$$\delta_{\alpha,\gamma}(\mathbb{K}) = \lim_{\ell\to\infty} \frac{1}{\gamma(\ell)-\alpha(\ell)} \left|\{\alpha(\ell) < k < \gamma(\ell) : k \in \mathbb{K}\}\right|.$$

Definition 2.7 ([29])**.** Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IFNLS. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be deferred statistical summable to p_0 w.r.t the IFN (φ, ψ) if for every $\varepsilon \in (0,1)$, $\lambda > 0$ we have

$$\lim_{\ell\to\infty} \frac{1}{\gamma(\ell)-\alpha(\ell)} \Big|\big\{\alpha(\ell) < k \le \gamma(\ell) : \varphi(p_k - p_0, \lambda) \le 1-\varepsilon \text{ or } \psi(p_k - p_0, \lambda) \ge \varepsilon\big\}\Big| = 0,$$

equivalently,

$$\delta_{\alpha,\gamma}(\{\alpha(\ell) < k \le \gamma(\ell) : \varphi(p_k - p_0, \lambda) \le 1-\varepsilon \text{ or } \psi(p_k - p_0, \lambda) \ge \varepsilon\}) = 0.$$

It is denoted by $p_k \to p_0\, (D_{\alpha,\gamma}[S(\varphi,\psi)])$.

Definition 2.8 ([29])**.** Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IFNLS. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be deferred statistical Cauchy w.r.t the IFN (φ, ψ) if for every $\varepsilon \in (0,1)$, $\lambda > 0$ and $\exists\ r = r(\varepsilon)$ such that

$$\lim_{\ell\to\infty} \frac{1}{\gamma(\ell)-\alpha(\ell)} \Big|\big\{\alpha(\ell) < k \le \gamma(\ell) : \varphi(p_k - p_r, \lambda) \le 1-\varepsilon \text{ or } \psi(p_k - p_r, \lambda) \ge \varepsilon\big\}\Big| = 0,$$

equivalently,

$$\delta_{\alpha,\gamma}(\{\alpha(\ell) < k \le \gamma(\ell) : \varphi(p_k - p_r, \lambda) \le 1-\varepsilon \text{ or } \psi(p_k - p_r, \lambda) \ge \varepsilon\}) = 0.$$

3. $D_{\alpha,\gamma}[S(\varphi,\psi)^n]$-Summability of Sequence in IF-n-NLS

In this section, we will introduce the notion of deferred statistical summability of sequences in an IF-n-NLS and develop some of its properties. Throughout this work by α and γ we mean two sequences $\alpha = \{\alpha(\ell) : \ell \in \mathbb{N}\}$ and $\gamma = \{\gamma(\ell) : \ell \in \mathbb{N}\}$ of non negative integers satisfying $\alpha(\ell) < \gamma(\ell)$ and $\gamma(\ell) \to \infty$ *as* $\ell \to \infty$. Deferred summability is a generalization of classical Cesàro summability to resolve the problems of ultimate bounds, oscillations and non-preserveness of various convergence for certain sequences of functions. The definition and results in this work will provide a framework to resolve these issues in the sense of generalized summability i.e., statistical summability.

Definition 3.1. Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be strong deferred summable or $D_{\alpha,\gamma}$-summable to p_0 w.r.t the IF-n-norm $(\varphi, \psi)^n$ if for every $\varepsilon \in (0,1), \lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$, $\exists\ k_0 \in \mathbb{N}$ such that

$$\frac{1}{\gamma(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) > 1 - \varepsilon$$

and

$$\frac{1}{\gamma(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < \varepsilon$$

holds for all $k \geq k_0$.

In this case, we write $D_{\alpha,\gamma}[(\varphi, \psi)^n] - \lim p_k = p_0$, or simply $p_k \to p_0 \quad (D_{\alpha,\gamma}[(\varphi, \psi)^n])$.
Let $D_{\alpha,\gamma}[(\varphi, \psi)^n] = \{p = (p_k) : p_k \to p_0 \quad (D_{\alpha,\gamma}[(\varphi, \psi)^n])\}$.

Definition 3.2. Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be deferred statistical summable to p_0 w.r.t the IF-n-norm $(\varphi, \psi)^n$ if for every $\varepsilon \in (0,1)$, $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$, we have

$$\lim_{\ell \to \infty} \frac{1}{\gamma(\ell) - \alpha(\ell)} \Big| \big\{ \alpha(\ell) < k \leq \gamma(\ell) : \varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon \text{ or } \psi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon \big\} \Big| = 0,$$

or

$$\delta_{\alpha,\gamma}\Big(\big\{ \alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon \text{ or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon \big\} \Big) = 0.$$

In this case, we write $D_{\alpha,\gamma}[S(\varphi, \psi)^n] - \lim p_k = p_0$, or simply $p_k \to p_0\, (D_{\alpha,\gamma}[S(\varphi, \psi)^n])$.

Let $D_{\alpha,\gamma}[S(\varphi, \psi)^n] = \{p = (p_k) : p_k \to p_0\, (D_{\alpha,\gamma}[S(\varphi, \psi)^n])\}$.

Remark 3.3.

(i) If $\gamma(\ell) = \ell$ and $\alpha(\ell) = 0$ then Definition (3.2) corresponds precisely to statistical convergence in IF-n-NLS presented in [46].

(ii) If $\gamma(\ell) = \theta_\ell$ and $\alpha(\ell) = \theta_{\ell-1}$ where (θ_ℓ) is a sequence of non-negative integers with $\theta_\ell - \theta_{\ell-1} \to \infty$ as $\ell \to \infty$, then Definition (3.2) corresponds precisely to lacunary statistical convergence in IF-n-NLS presented in [45].

(iii) If $\gamma(\ell) = \lambda_\ell$ and $\alpha(\ell) = 0$ where (λ_ℓ) is a strictly increasing sequence of positive numbers tending to ∞ then Definition (3.2) corresponds precisely to λ-statistical convergence in IF-n-NLS presented in [9].

Definition (3.2) immediately gives the following Lemma.

Lemma 3.4. *Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS. Then for every $\varepsilon \in (0,1), \lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$, the following are equivalent:*

(i) $D_{\alpha,\gamma}[S(\varphi,\psi)^n] - \lim p_k = p_0$,

(ii)
$$\delta_{\alpha,\gamma}\Big(\big\{\alpha(\ell) < k \le \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \le 1 - \varepsilon \big\}\Big) = 0,$$
$$\delta_{\alpha,\gamma}\Big(\big\{\alpha(\ell) < k \le \gamma(\ell) : \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \ge \varepsilon\big\}\Big) = 0,$$

(iii)
$$\delta_{\alpha,\gamma}\Big(\big\{\alpha(\ell) < k \le \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) > 1 - \varepsilon \text{ and } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < \varepsilon\big\}\Big) = 1,$$

(iv)
$$\delta_{\alpha,\gamma}\Big(\big\{\alpha(\ell) < k \le \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) > 1 - \varepsilon\big\}\Big) = \delta_{\alpha,\gamma}\Big(\big\{\alpha(\ell) < k \le \gamma(\ell) : \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < \varepsilon\big\}\Big) = 1$$

(v)
$$D_{\alpha,\gamma}S - \lim \varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) = 1$$
and
$$D_{\alpha,\gamma}S - \lim \psi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) = 0.$$

Theorem 3.5. *Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS and $p = (p_k)$ be any sequence in $\mathcal{P}$ with $(\varphi,\psi)^n - \lim p_k = p_0$ then*

(i) $D_{\alpha,\gamma}[(\varphi,\psi)^n] - \lim p_k = p_0$.
(ii) $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right] - \lim p_k = p_0$.

Proof. (i) For every $\varepsilon \in (0,1)$, $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$, $\exists$ $k_0 \in \mathbb{N}$ such that $\varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) > 1 - \varepsilon$ and $\psi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon \ \forall \ k \geq k_0$. Assuming the above inequalities hold for values between $\alpha(\ell) + 1$ and $\gamma(\ell)$, the subsequent inequality is derived

$$\sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) > (1-\varepsilon)(\gamma(\ell) - \alpha(\ell))$$

and

$$\sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < \varepsilon\left(\gamma(\ell) - \alpha(\ell)\right),$$

i.e.,

$$\frac{1}{\gamma(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) > (1-\varepsilon)$$

and

$$\frac{1}{\gamma(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < \varepsilon,$$

and therefore we have $D_{\alpha,\gamma}[(\varphi,\psi)^n] - \lim p_k = p_0$.

(ii) Since $(\varphi,\psi)^n - \lim p_k = p_0$. So clearly, we have the containment

$$\Big\{\alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon \text{ or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\Big\} \subseteq \Big\{1,\ 2,\ 3,\ \ldots, k_0 - 1\Big\},$$

gives immediately

$$\begin{aligned}\delta_{\alpha,\gamma}\Big(\Big\{\alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon \text{ or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\Big\}\Big) \\ \leq \delta_{\alpha,\gamma}(\{1, 2, 3, \ldots, k_0 - 1\}) \\ = 0.\end{aligned}$$

Hence,

$$D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right] - \lim p_k = p_0. \qquad \square$$

In general, the converse of Theorem (3.5) does not hold.

Example 3.6. Let $(\mathbb{R}^n, \|\bullet, \bullet, \ldots, \bullet\|)$ be an n-NLS and $\mathfrak{a} \circledast \mathfrak{b} = \mathfrak{a}\mathfrak{b}$, $\mathfrak{a} \odot \mathfrak{b} = \min\{\mathfrak{a} + \mathfrak{b}, 1\}\ \forall\ \mathfrak{a}, \mathfrak{b} \in [0,1]$. For all $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1}, p_n \in \mathbb{R}^n$, define

$$\varphi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) = \frac{\lambda}{\lambda + \|p_1, p_2, \ldots, p_{n-1}, p_n\|}$$

and

$$\psi(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) = \frac{\|p_1, p_2, \ldots, p_{n-1}, p_n\|}{\lambda + \|p_1, p_2, \ldots, p_{n-1}, p_n\|},$$

then it is easy to check $(\mathbb{R}^n, \varphi, \psi, \circledast, \odot)$ is an IF-n-NLS. Now consider the sequence $p = (p_k)$ where

$$p_k = \begin{cases} \left(k^2, 0, 0, \cdots, 0\right) \in {}^n; & \left[\sqrt{\gamma(\ell)}\right] - m_0 < k < \left[\sqrt{\gamma(\ell)}\right] \\ (0, 0, 0, \cdots, 0) \in {}^n; & \text{otherwise} \end{cases}$$

where $\gamma(\ell)$ is a monotone increasing sequence of positive integers and $m_0 (\neq 0) \in \mathbb{N}$ is a fixed number. Now, for every $\varepsilon \in (0,1)$, $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1}, p_n \in \mathbb{R}^n$, consider the following set

$$\begin{aligned} A_n(\varepsilon, \lambda) &= \big\{\alpha(\ell) < k \leq \gamma(\ell) : \varphi(p_1, p_2, \ldots, p_{n-1}, p_k - 0, \lambda) \leq 1 - \varepsilon \\ &\qquad \text{or } \psi(p_1, p_2, \ldots, p_{n-1}, p_k - 0, \lambda) \geq \varepsilon\big\} \\ &= \left\{\alpha(\ell) < k \leq \gamma(\ell) : \frac{\lambda}{\lambda + \|p_1, p_2, \ldots, p_{n-1}, p_k\|} \leq 1 - \varepsilon \right. \\ &\qquad \left. \text{or } \frac{\|p_1, p_2, \ldots, p_{n-1}, p_k\|}{\lambda + \|p_1, p_2, \ldots, p_{n-1}, p_k\|} \geq \varepsilon \right\} \\ &= \left\{\alpha(\ell) < k \leq \gamma(\ell) : \|p_1, p_2, \ldots, p_{n-1}, p_k\| \geq \frac{\varepsilon\lambda}{1 - \varepsilon} > 0 \right\} \\ &\subseteq \left\{\alpha(\ell) < k \leq \gamma(\ell) : p_k = \left(k^2, 0, 0, \cdots, 0\right) \in \mathbb{R}^n \right\} \\ &= \left\{\alpha(\ell) < k \leq \gamma(\ell) : \left[\left|\sqrt{\gamma(\ell)}\right|\right] - m_0 < k < \left[\left|\sqrt{\gamma(\ell)}\right|\right] \right\} \\ &\leq \lim_{\ell \to \infty} \frac{1}{\gamma(\ell) - \alpha(\ell)} m_0 \to 0 \end{aligned}$$

$\delta_{\alpha,\gamma}(A_n(\varepsilon, \lambda)) = 0$ as density can't be negative. Thus, $D_{\alpha,\gamma}[S(\varphi, \psi)^n] - \lim p_k = 0$. But it is clear that the sequence $p = (p_k)$ is not convergent in $(\mathbb{R}^n, \varphi, \psi, \circledast, \odot)$ w.r.t the IF-n-norm $(\varphi, \psi)^n$.

Theorem 3.7. *Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS. For any sequence $p = (p_k)$ in $\mathcal{P}$, if $D_{\alpha,\gamma}[S(\varphi, \psi)^n] - \lim p_k$ exists, then it is unique.*

Proof. Let $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right]-\lim p_k=p_0$. Let $\varepsilon\in(0,1)$ be given. Choose $\eta\in(0,1)$ such that

$$(1-\eta)\circledast(1-\eta)>1-\varepsilon,\quad \eta\ \odot\ \eta<\varepsilon. \tag{3.1}$$

For any $\lambda>0$ and $p_1,p_2,\ldots,p_{n-1}\in\mathcal{P}$, if we define the following sets:

$$K_{\varphi,1}(\eta,\lambda)=\left\{\alpha(\ell)<k\leq\gamma(\ell):\varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\ \frac{\lambda}{2}\right)\leq 1-\eta\right\}, \tag{3.2}$$

$$K_{\varphi,2}(\eta,\lambda)=\left\{\alpha(\ell)<k\leq\gamma(\ell):\varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-q_0,\frac{\lambda}{2}\right)\leq 1-\eta\right\}, \tag{3.3}$$

$$K_{\psi,1}(\eta,\lambda)=\left\{\alpha(\ell)<k\leq\gamma(\ell):\psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\frac{\lambda}{2}\right)\geq\eta\right\}, \tag{3.4}$$

$$K_{\psi,2}(\eta,\lambda)=\left\{\alpha(\ell)<k\leq\gamma(\ell):\psi\left(p_1,p_2,\ldots,p_{n-1},p_k-q_0,\frac{\lambda}{2}\right)\geq\eta\right\}, \tag{3.5}$$

then by Lemma (3.4),

$$\delta_{\alpha,\gamma}\left(K_{\varphi,1}(\eta,\lambda)\right)=\delta_{\alpha,\gamma}\left(K_{\psi,1}(\eta,\lambda)\right)=0$$

as $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right]-\lim p_k=p_0$. Similarly,

$$\delta_{\alpha,\gamma}\left(K_{\varphi,2}(\eta,\lambda)\right)=\delta_{\alpha,\gamma}\left(K_{\psi,2}(\eta,\lambda)\right)=0$$

as $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right]-\lim p_k=q_0$. Let

$$K_{\varphi,\psi}(\eta,\lambda)=\left\{K_{\varphi,1}(\eta,\lambda)\cup\ K_{\varphi,2}(\eta,\lambda)\right\}\cap\left\{K_{\psi,1}(\eta,\lambda)\ \cup\ K_{\psi,2}(\eta,\lambda)\right\},$$

then $\delta_{\alpha,\gamma}\left(K_{\varphi,\psi}(\eta,\lambda)\right)=0$, and therefore $\delta_{\alpha,\gamma}\left(K_{\varphi,\psi}(\eta,\lambda)\right)^{\complement}=1$. Let $k\in\left(K_{\varphi,\psi}(\eta,\lambda)\right)^{\complement}$, then we have $k\in\left(\left\{K_{\varphi,1}(\eta,\lambda)\cup K_{\varphi,2}(\eta,\lambda)\right\}\right)^{\complement}$ or $k\in\left(\left\{K_{\psi,1}(\eta,\lambda)\cup K_{\psi,2}(\eta,\lambda)\right\}\right)^{\complement}$.

Case (i) If $k\in\left(\left\{K_{\varphi,1}(\eta,\lambda)\cup K_{\varphi,2}(\eta,\lambda)\right\}\right)^{\complement}$ then we have

$$\begin{aligned}\varphi\left(p_1,p_2,\ldots,p_{n-1},p_0-q_0,\lambda\right)&\geq\varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\frac{\lambda}{2}\right)\\&\quad\circledast\ \varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-q_0,\frac{\lambda}{2}\right)\\&>(1-\eta)\circledast(1-\eta)\quad\text{by (3.2) and (3.3)}\\&>(1-\varepsilon).\quad\text{by (3.1)}\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary so $\varphi(p_1, p_2, \ldots, p_{n-1}, p_0 - q_0, \lambda) = 1$. This implies that $p_0 - q_0 = 0$ i.e., $p_0 = q_0$.

Case(ii) If $k \in (\{K_{\psi,1}(\eta, \lambda) \cup K_{\psi,2}(\eta, \lambda)\})^{\complement}$ then by similar argument as above

$$\begin{aligned}\psi(p_1, p_2, \ldots, p_{n-1}, p_0 - q_0, \lambda) &\leq \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) \\ &\quad \odot \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - q_0, \frac{\lambda}{2}\right) \\ &< \eta \ \odot \ \eta \quad \text{by (3.4) and (3.5)} \\ &< \varepsilon. \quad \text{by (3.1)}\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary so $\psi(p_1, p_2, \ldots, p_{n-1}, p_0 - q_0, \lambda) = 0$, and therefore, $p_0 = q_0$.

Hence, in all cases, $p_0 = q_0$. □

Theorem 3.8. *Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS. If $p = (p_k)$ in $\mathcal{P}$ then $D_{\alpha,\gamma}[S(\varphi, \psi)^n] - \lim p_k = p_0$ if and only if $\exists$ a set $T = \{t_1 < t_2 < \cdots\} \subset \mathbb{N}$ such that $\delta_{\alpha,\gamma}(T) = 1$ and $(\varphi, \psi)^n - \lim p_{t_j} = p_0$.*

Proof. Necessity: We first, assume that $D_{\alpha,\gamma}[S(\varphi, \psi)^n] - \lim p_k = p_0$. For $\lambda > 0$ and $\eta = 1, 2, \ldots$, we denote,

$$A_{\alpha,\gamma}(\eta, \lambda) = \left\{\alpha(\ell) < k \leq \gamma(\ell) : \varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) > 1 - \frac{1}{\eta}\right.$$
$$\left. \text{and } \psi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \frac{1}{\eta}\right\}$$

and

$$B_{\alpha,\gamma}(\eta, \lambda) = \left\{\alpha(\ell) < k \leq \gamma(\ell) : \varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \frac{1}{\eta}\right.$$
$$\left. \text{or } \psi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \frac{1}{\eta}\right\}.$$

By assumption, we have

$$\delta_{\alpha,\gamma}(B_{\alpha,\gamma}(\eta, \lambda)) = 0 \quad \text{or} \quad \delta_{\alpha,\gamma}(A_{\alpha,\gamma}(\eta, \lambda)) = 1. \tag{3.6}$$

Further, we also have $A_{\alpha,\gamma}(\eta+1, \lambda) \subset A_{\alpha,\gamma}(\eta, \lambda)$. To complete the proof of the first part it is sufficient to show that for $k \in A_{\alpha,\gamma}(\eta, \lambda)$ we have $(\varphi, \psi)^n - \lim p_k = p_0$. Suppose it does not hold $(\varphi, \psi)^n - \lim p_k \neq p_0$ for some $k \in A_{\alpha,\gamma}(\eta, \lambda)$. Then, there is $\varepsilon \in (0, 1)$, $\lambda > 0$ and a positive

integer k_0 such that

$$\varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon$$

or

$$\psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon,$$

holds for all $k \geq k_0$.

$$\varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) > 1 - \varepsilon$$

and

$$\psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < \varepsilon,$$

holds for all $k < k_0$. Then

$$\delta_{\alpha,\gamma}\Big(\big\{\alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) > 1 - \varepsilon$$
$$\text{and } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < \varepsilon\big\}\Big) = 0.$$

Since $\varepsilon > \frac{1}{\eta}$, then $\delta_{\alpha,\gamma}\left(A_{\alpha,\gamma}(\eta,\ \lambda)\right) = 0$. This contradicts equation (3.6). Hence we have $(\varphi, \psi)^n - \lim p_k = p_0$.

Sufficiency part: Suppose there exist a set $T = \{t_1 < t_2 < \ldots\} \subset \mathbb{N}$ such that $\delta_{\alpha,\gamma}(T) = 1$ and $(\varphi, \psi)^n - \lim_{j\to\infty} p_{t_j} = p_0$. By given hypothesis, $\exists\ k_0 \in \mathbb{N}$ such that for all $\varepsilon \in (0, 1)$ and $\lambda > 0$,

$$\varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) > 1 - \varepsilon$$

and

$$\psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < \varepsilon,$$

holds for all $k \geq k_0,\ k \in T$. Further,the inclusion

$$\Big\{\alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon$$
$$\text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\Big\}$$
$$\subset \mathbb{N} - \{k_0, k_0 + 1, \ldots\}$$

gives

$$\delta_{\alpha,\gamma}\Big(\Big\{\alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon$$
$$\text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\Big\}\Big)$$
$$\leq \delta_{\alpha,\gamma}\left(\mathbb{N}\right) - \delta_{\alpha,\gamma}\left(\{k_0, k_0 + 1, \cdots\}\right) = 0$$

i.e.,

$$\delta_{\alpha,\gamma}\Big(\Big\{\alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon$$
$$\text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\Big\}\Big) = 0$$

as it can't be negative. Thus, $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right]-\lim p_k=p_0$. □

Throught the proof of Theorem (3.9) and Theorem (3.11), we denote $1-\varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) = \overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)$ for simplicity. If $\varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)>1-\varepsilon$ then we restate it as $\overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)<\varepsilon$.

Theorem 3.9. *Let $(\mathcal{P},\varphi,\psi,\circledast,\odot)$ be an IF-n-NLS and $p=(p_k)$ be in $\mathcal{P}$. Then $D_{\alpha,\gamma}[(\varphi,\psi)^n]-\lim p_k=x_0$ implies $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right]-\lim p_k=p_0$.*

Proof. Let $D_{\alpha,\gamma}[(\varphi,\psi)^n]-\lim p_k=p_0$, then for every $\varepsilon\in(0,1)$, $\lambda>0$ and $p_1,p_2,\ldots,p_{n-1}\in\mathcal{P}$, $\exists\ k_0\in\mathbb{N}$ such that

$$\frac{1}{\gamma(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\gamma(\ell)}\varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)>1-\varepsilon$$

and

$$\frac{1}{\gamma(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\gamma(\ell)}\psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)<\varepsilon$$

holds for all $k\geq k_0$. or equivalently,

$$\frac{1}{\gamma(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\gamma(\ell)}\overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)<\varepsilon \tag{3.7}$$

and

$$\frac{1}{\gamma(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\gamma(\ell)}\psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)<\varepsilon \tag{3.8}$$

holds for all $k\geq k_0$. Now

(3.9)

$$\begin{aligned}
&\frac{1}{\gamma(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\gamma(\ell)}\overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\\
&=\frac{1}{\gamma(\ell)-\alpha(\ell)}\left(\left(\sum_{\substack{k=\alpha(\ell)+1\text{ and}\\ \overline{\varphi}(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon}}^{\gamma(\ell)}\right.\right.\\
&\quad\left.\left.+\sum_{\substack{k=\alpha(\ell)+1\text{ and}\\ \overline{\varphi}(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)<\varepsilon}}^{\gamma(\ell)}\right)\overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\right)
\end{aligned}$$

$$\geq \frac{1}{\gamma(\ell)-\alpha(\ell)} \sum_{\substack{k=\alpha(\ell)+1 \text{ and} \\ \overline{\varphi}(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon}}^{\gamma(\ell)} \overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)$$
$$\geq \frac{\varepsilon}{\gamma(\ell)-\alpha(\ell)} \left|\{\alpha(\ell)<k\leq\gamma(\ell):\overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon\}\right|$$
$$\geq A_\ell$$

where

$$A_\ell = \frac{\varepsilon}{\gamma(\ell)-\alpha(\ell)} \left|\{\alpha(\ell)<k\leq\gamma(\ell):\overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon\}\right|.$$

Now

$$\varepsilon > \frac{1}{\gamma(\ell)-\alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)$$
$$\geq A_\ell \quad \text{(by (3.7) and (3.9))}$$

Since $\varepsilon>0$ is arbitrary thus

$$\lim_{\ell\to\infty} A_\ell = 0.$$

This implies

$$\delta_{\alpha,\gamma}(A_\ell) = 0$$

as deferred density can't be negative.

By similar process we have

$$\frac{1}{\gamma(\ell)-\alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) \geq B_\ell \tag{3.10}$$

where

$$B_\ell = \frac{\varepsilon}{\gamma(\ell)-\alpha(\ell)} \left|\{\alpha(\ell)<k\leq\gamma(\ell):\psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon\}\right|.$$

Also, by similar process

$$\delta_{\alpha,\gamma}(B_\ell) = 0.$$

As we have $\delta_{\alpha,\gamma}(A_\ell)=0,\ \delta_{\alpha,\gamma}(B_\ell)=0$. Hence, $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right]-\lim p_k = p_0$. □

The converse of Theorem (3.9) does not hold, in general. For this consider following example.

Example 3.10. Consider the IF-n-NLS as in Example (3.6). Define a sequence $p = (p_k)$ by

$$p_k = \begin{cases} \left(k^2, 0, 0, \cdots, 0\right) \in {}^n; & \left[\sqrt{\gamma(\ell)}\right] - m_0 < k < \left[\sqrt{\gamma(\ell)}\right] \\ (0, 0, 0, \cdots, 0) \in {}^n; & \text{otherwise} \end{cases}$$

where $\gamma(\ell)$ is a monotone increasing sequence of positive integers and $m_0 \neq 0$ is a fixed natural number. Then we have shown in Example (3.6), $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right] - \lim p_k = 0$. However, the sequence $p = (p_k)$ is not $D_{\alpha,\gamma}[(\varphi,\psi)^n]$-convergent to zero as we have

$$\begin{aligned} &\frac{1}{\gamma(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \\ &= \frac{1}{\gamma(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \frac{\|p_1, p_2, \ldots, p_{n-1}, p_k\|}{\lambda + \|p_1, p_2, \ldots, p_{n-1}, p_k\|} \\ &= \frac{1}{\gamma(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \frac{\left(\left[\left|\sqrt{\gamma(\ell)}\right|\right] - m_0\right)^2}{\lambda + \left(\left[\left|\sqrt{\gamma(\ell)}\right|\right]\right)^2} \\ &\geq \frac{\left(\left[\left|\sqrt{\gamma(\ell)}\right|\right] - m_0\right)^2}{\left(\left[\left|\sqrt{\gamma(\ell)}\right|\right]\right)^2} \to 1 \quad \text{as} \quad \ell \to \infty. \end{aligned}$$

The following result shows that the converse of Theorem (3.9) specifically hold for bounded sequences. l_∞^n represents the collection of bounded sequences in IF-n-NLS.

Theorem 3.11. *Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS and sequence $p = (p_k)$ in l_∞^n then $D_{\alpha,\gamma}[S(\varphi,\psi)^n] - \lim p_k = p_0$ implies $D_{\alpha,\gamma}\left[(\varphi,\psi)^n\right] - \lim p_k = p_0$.*

Proof. Suppose $p = (p_k) \in l_\infty^n$ and $D_{\alpha,\gamma}[S(\varphi,\psi)^n] - \lim p_k = p_0$. By the assumption on $p = (p_k)$ $\exists$ a positive real number M such that $\varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) > 1 - M$ and $\psi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < M$ or equivently,

$$\overline{\varphi}\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < M$$

nad

$$\psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < M$$

holds for all k. Now

$$\frac{1}{\gamma(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \tag{3.11}$$

$$= \frac{1}{\gamma(\ell)-\alpha(\ell)} \left(\left(\sum_{\substack{k=\alpha(\ell)+1 \text{ and} \\ \varphi(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)\leq 1-\varepsilon}}^{\gamma(\ell)} + \sum_{\substack{k=\alpha(\ell)+1 \text{ and} \\ \varphi(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)>1-\varepsilon}}^{\gamma(\ell)} \right) \varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) \right),$$

rewrite (3.11) as follows

$$\frac{1}{\gamma(\ell)-\alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)$$

$$= \frac{1}{\gamma(\ell)-\alpha(\ell)} \left(\left(\sum_{\substack{k=\alpha(\ell)+1 \text{ and} \\ \overline{\varphi}(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon}}^{\gamma(\ell)} + \sum_{\substack{k=\alpha(\ell)+1 \text{ and} \\ \overline{\varphi}(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)<\varepsilon}}^{\gamma(\ell)} \right) \overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) \right)$$

$$< \frac{1}{\gamma(\ell)-\alpha(\ell)} \left(M \sum_{\substack{k=\alpha(\ell)+1 \text{ and} \\ \overline{\varphi}(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon}}^{\gamma(\ell)} 1 + \varepsilon \sum_{\substack{k=\alpha(\ell)+1 \text{ and} \\ \overline{\varphi}(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)<\varepsilon}}^{\gamma(\ell)} 1 \right)$$

$$< \frac{M}{\gamma(\ell)-\alpha(\ell)} \left| \left\{ \alpha(\ell) < k \leq \gamma(\ell) : \overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) \geq \varepsilon \right\} \right|$$

$$+ \frac{\varepsilon}{\gamma(\ell)-\alpha(\ell)} \left| \left\{ \alpha(\ell) < k \leq \gamma(\ell) : \overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) < \varepsilon \right\} \right|$$

Now, we take limit by considering $D_{\alpha,\gamma}[S(\varphi,\psi)^n] - \lim p_k = p_0$ then we have

$$\frac{1}{\gamma(\ell)-\alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) < M.(0) + \varepsilon.(1)$$

$$\frac{1}{\gamma(\ell)-\alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\gamma(\ell)} \overline{\varphi}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) < \varepsilon \text{ or equivently,}$$

$$\frac{1}{\gamma(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\gamma(\ell)}\varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)>1-\varepsilon \tag{3.12}$$

Similarly, the following inequality

$$\begin{aligned}
&\frac{1}{\gamma(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\gamma(\ell)}\psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\\
&\quad=\frac{1}{\gamma(\ell)-\alpha(\ell)}\left(\left(\left(\sum_{\substack{k=\alpha(\ell)+1 \text{ and}\\ \psi(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon}}^{\gamma(\ell)}\right.\right.\right.\\
&\qquad\left.\left.\left.+\sum_{\substack{k=\alpha(\ell)+1 \text{ and}\\ \psi(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)<\varepsilon}}^{\gamma(\ell)}\right)\psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\right)\\
&\quad<\frac{1}{\gamma(\ell)-\alpha(\ell)}\left(M\sum_{\substack{k=\alpha(\ell)+1 \text{ and}\\ \psi(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon}}^{\gamma(\ell)}1+\varepsilon\sum_{\substack{k=\alpha(\ell)+1 \text{ and}\\ \psi(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)<\varepsilon}}^{\gamma(\ell)}1\right)\\
&\quad<\frac{M}{\gamma(\ell)-\alpha(\ell)}\left|\left\{\alpha(\ell)<k\leq\gamma(\ell):\psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon\right\}\right|\\
&\qquad+\frac{\varepsilon}{\gamma(\ell)-\alpha(\ell)}\left|\left\{\alpha(\ell)<k\leq\gamma(\ell):\psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)<\varepsilon\right\}\right|
\end{aligned}$$

By similar process as above we have

$$\frac{1}{\gamma(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\gamma(\ell)}\psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)<\varepsilon \tag{3.13}$$

So, by considering equation (3.12) and (3.13) the proof is completed. □

To prove the next Theorem (3.13) we give the following Lemma.

Lemma 3.12. *If $a=(a_k)$ and $b=(b_k)$ be two sequences of positive integers such that $\lim_{k\to\infty}a_k=a_0$ and $\lim_{k\to\infty}b_k=\infty$, then $\lim_{k\to\infty}a_{b_k}=a_0$.*

Proof. The proof is simple so omitted. □

Theorem 3.13. *Let $(\mathcal{P},\varphi,\psi,\circledast,\odot)$ be an IF-n-NLS and $p=(p_k)$ in $\mathcal{P}$ and sequence $\left(\frac{\alpha(\ell)}{\gamma(\ell)-\alpha(\ell)}\right)$ is bounded then $[S(\varphi,\psi)^n]-\lim p_k=p_0$ implies $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right]-\lim p_k=p_0$.*

Proof. Suppose $[S(\varphi,\psi)^n]-\lim p_k = p_0$ then every $\varepsilon\in(0,1)$, $\lambda>0$ and $p_1,p_2,\ldots,p_{n-1}\in\mathcal{P}$,

$$\lim_{\ell\to\infty}\frac{1}{\ell}\Big|\{k\le\ell:\varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\le 1-\varepsilon \text{ or } \psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\ge\varepsilon\}\Big|=0.$$

Let

$$a_\ell=\frac{1}{\ell}\Big|\{k\le\ell\ :\ \varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\le 1-\varepsilon \text{ or } \psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\ge\varepsilon\}\Big|$$

and $\gamma(\ell)\to\infty$ as $\ell\to\infty$, so by Lemma (3.12), we have

$$\lim_{\ell\to\infty}\frac{1}{\gamma(\ell)}\Big|\{k\le\gamma(\ell):\varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\le 1-\varepsilon \text{ or } \psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\ge\varepsilon\}\Big|=0 \tag{3.14}$$

Now, from the inclusion

$$\Big\{\alpha(\ell)<k\le\gamma(\ell):\varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\le 1-\varepsilon \text{ or } \psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\ge\varepsilon\Big\}$$
$$\subset\Big\{k\le\gamma(\ell)\ :\ \varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\le 1-\varepsilon \text{ or } \psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\ge\varepsilon\Big\},$$

we have

$$\Big|\big\{\alpha(\ell)<k\le\gamma(\ell):\ \varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\le 1-\varepsilon \text{ or } \psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\ge\varepsilon\big\}\Big|$$
$$\le\Big|\big\{k\le\gamma(\ell):\ \varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\le 1-\varepsilon \text{ or } \psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\ge\varepsilon\big\}\Big|,$$

which immediately implies

$$\frac{1}{\gamma(\ell)-\alpha(\ell)}\Big|\big\{\alpha(\ell)<k\le\gamma(\ell):\varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\le 1-\varepsilon \text{ or } \psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\ge\varepsilon\big\}\Big|$$
$$\le\frac{1}{\gamma(\ell)-\alpha(\ell)}\Big|\big\{k\le\gamma(\ell)\ :\ \varphi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\le 1-\varepsilon \text{ or } \psi\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\ge\varepsilon\ \big\}\Big|$$

$$= (1 + R(\ell)) . \frac{1}{\gamma(\ell)} \Big| \big\{ k \leq \gamma(\ell) \ : \ \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon$$

$$\text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon \big\} \Big|$$

$\to 0$. (As $R(\ell)$ is bounded and (3.14) holds)

where $R(\ell) = \frac{\alpha(\ell)}{\gamma(\ell) - \alpha(\ell)}$. This shows that $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right] - \lim p_k = p_0$. □

4. Deferred Statistical Cauchy Sequence and Deferred Completeness in IF-n-NLS

In this section, we will define and study deferred statistical Cauchy and deferred statistical completeness in an IF-n-NLS .

Definition 4.1. Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be deferred statistical Cauchy w.r.t the IF-n-norm $(\varphi,\psi)^n$ if for every $\varepsilon \in (0,1)$, $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$, $\exists\ r = r(\varepsilon)$ such that

$$\lim_{\ell\to\infty} \frac{1}{\gamma(\ell) - \alpha(\ell)} \Big| \big\{ \alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \ \lambda\right) \leq 1 - \varepsilon \text{ or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \ \lambda\right) \geq \varepsilon \big\} \Big| = 0,$$

or equivalently,

$$\delta_{\alpha,\gamma}\Big(\big\{ \alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \lambda\right) \leq 1 - \varepsilon \text{ or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \lambda\right) \geq \varepsilon \big\} \Big) = 0.$$

Definition 4.2. Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be deferred statistical complete if every deferred statistically Cauchy sequence w.r.t the IF-n-norm $(\varphi,\psi)^n$ is deferred statistical convergent in $\mathcal{P}$.

Theorem 4.3. *Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS. For any sequence $p = (p_k)$, if $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right] - \lim p_k = p_0$, then $p = (p_k)$ is $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right]$-Cauchy.*

Proof. For $\varepsilon \in (0,1)$, choose $\eta \in (0,1)$ such that (3.1) is satisfied. Since $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right] - \lim p_k = p_0$, so for every $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$ we have $\delta_{\alpha,\gamma}\left(K(\eta,\lambda)\right) = 0$ and $\delta_{\alpha,\gamma}\left(K(\eta,\lambda)\right)^{\complement} = 1$, where

$$K(\eta,\lambda) = \left\{ \alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) \leq 1 - \eta \right.$$

$$\text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) \geq \eta\Bigg\}.$$

Let $r \in (K(\eta,\lambda))^{\complement}$ then

$$\varphi\left(p_1, p_2, \ldots, p_{n-1}, p_r - p_0, \frac{\lambda}{2}\right) > 1-\eta \tag{4.1}$$

and

$$\psi\left(p_1, p_2, \ldots, p_{n-1}, p_r - p_0, \frac{\lambda}{2}\right) < \eta,$$

we define,

$$M(\varepsilon,\lambda) = \Big\{\alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \lambda\right) \leq 1-\varepsilon$$
$$\text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \lambda\right) \geq \varepsilon\Big\},$$

so to prove the result, it is sufficient to show that $M(\varepsilon,\lambda) \subseteq K(\eta,\lambda)$. Let $q \in M(\varepsilon,\lambda)$ then

$$\varphi\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_r, \lambda\right) \leq 1-\varepsilon \tag{4.2}$$

or

$$\psi\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_r, \lambda\right) \geq \varepsilon$$

Case 1: If $\varphi\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_r, \lambda\right) \leq 1-\varepsilon$, then we have

$$\varphi\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) \leq 1-\eta$$

therefore $q \in K(\eta,\lambda)$. As otherwise

$$\varphi\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) > 1-\eta,$$

then by (3.1), (4.1) and (4.2) we have

$$\begin{aligned}
1-\varepsilon &\geq \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_r, \lambda\right) \\
&= \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0 + p_0 - p_r,\ \frac{\lambda}{2} + \frac{\lambda}{2}\right) \\
&\geq \varphi\left(p_1, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) \circledast \varphi\left(p_1, \ldots, p_{n-1}, p_0 - p_r, \frac{\lambda}{2}\right) \\
&> (1-\eta) \circledast (1-\eta) \\
&> 1-\varepsilon. \quad \text{(not possible)}
\end{aligned}$$

Hence $M(\varepsilon,\lambda) \subseteq K(\eta,\lambda)$.
Case 2: If $\psi(p_1, p_2, \ldots, p_{n-1}, p_q - p_r, \lambda) \geq \varepsilon$, then we have

$$\psi\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) \geq \eta$$

therefore $q \in K(\eta, \lambda)$. As otherwise

$$\psi\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) < \eta,$$

then by (3.1), (4.1) and (4.2) we have

$$\begin{aligned}
\varepsilon &\leq \psi\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_r, \lambda\right) \\
&= \psi\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0 + p_0 - p_r, \ \frac{\lambda}{2} + \frac{\lambda}{2}\right) \\
&\leq \psi\left(p_1, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) \odot \psi\left(p_1, \ldots, p_{n-1}, p_0 - p_r, \frac{\lambda}{2}\right) \\
&< \eta \odot \eta \\
&< \varepsilon. \quad \text{(not possible)}
\end{aligned}$$

Hence $M(\varepsilon, \lambda) \subseteq K(\eta, \lambda)$. Thus in all the cases, $M(\varepsilon, \lambda) \subseteq K(\eta, \lambda)$. Since $\delta_{\alpha,\gamma}\left(K(\eta, \lambda)\right) = 0$ so $\delta_{\alpha,\gamma}\left(M(\varepsilon, \lambda)\right) = 0$ and therefore $p = (p_k)$ is $D_{\alpha,\gamma}\left[S(\varphi, \psi)^n\right]$-Cauchy. □

Theorem 4.4. *Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS. If every deferred statistical Cauchy sequence has deferred statistical convergent subsequence then $\mathcal{P}$ is deferred statistical complete.*

Proof. Given that $p = (p_k)$ be a $D_{\alpha,\gamma}\left[S(\varphi, \psi)^n\right]$-Cauchy sequence in $\mathcal{P}$ and (p_{k_j}) be a subsequence of (p_k) such that $p_{k_j} \to p_0$ $(D_{\alpha,\gamma}[S(\varphi, \psi)^n])$. We have to prove that $p_k \to p_0$ $(D_{\alpha,\gamma}\left[S(\varphi, \psi)^n\right])$. Let $\varepsilon \in (0, 1)$, choose $\eta \in (0, 1)$ such that (3.1) holds. Since (p_k) is $D_{\alpha,\gamma}\left[S(\varphi, \psi)^n\right]$-Cauchy so for any $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$ we have $\delta_{\alpha,\gamma}\left(E\right) = 0$, where

$$\begin{aligned}
E = \Big\{&\alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \lambda\right) \leq 1 - \eta \\
&\text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \lambda\right) \geq \eta\Big\}.
\end{aligned}$$

Since $p_{k_j} \to p_0$ $\left(D_{\alpha,\gamma}\left[S(\varphi, \psi)^n\right]\right)$. So we have $\delta_{\alpha,\gamma}\left(F\right) = 0$, where

$$\begin{aligned}
F = \Big\{&\alpha(\ell) < k_j \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_{k_j} - p_0, \lambda\right) \leq 1 - \eta \\
&\text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_{k_j} - p_0, \lambda\right) \geq \eta\Big\}.
\end{aligned}$$

Now define

$$\begin{aligned}
G = \Big\{&\alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \eta \\
&\text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \eta\Big\}.
\end{aligned}$$

Now we claim that $E^{\complement} \cap F^{\complement} \subseteq G^{\complement}$. Let $m \in E^{\complement} \cap F^{\complement}$, then $m \in E^{\complement}$ and $m \in F^{\complement}$.

Now first consider if $m \in E^{\complement}$, then

$$\varphi\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_r, \lambda\right) > 1 - \eta \tag{4.3}$$

and

$$\psi\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_r, \lambda\right) < \eta.$$

Now if $m \in F^{\complement}$, then $m = k_j$ for $j \in \mathbb{N}$ and therefore

$$\varphi\left(p_1, p_2, \ldots, p_{n-1}, p_{k_j} - p_0, \lambda\right) > 1 - \eta \tag{4.4}$$

and

$$\psi\left(p_1, p_2, \ldots, p_{n-1}, p_{k_j} - p_0, \lambda\right) < \eta.$$

Now,

$$\begin{aligned}
&\varphi\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_0, \lambda\right) \\
&\quad = \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_{k_j} + p_{k_j} - p_0,\ \frac{\lambda}{2} + \frac{\lambda}{2}\right) \\
&\quad \geq \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_{k_j}, \frac{\lambda}{2}\right) \circledast\ \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_{k_j} - p_0,\ \frac{\lambda}{2}\right) \\
&\quad > (1-\eta) \circledast (1-\eta) \text{ for } r = k_j \quad \text{(by (4.3) and (4.4))} \\
&\quad > 1 - \varepsilon \quad (\text{ by (3.1) })
\end{aligned}$$

and

$$\begin{aligned}
&\psi\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_0, \lambda\right) \\
&\quad = \psi\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_{k_j} + p_{k_j} - p_0, \frac{\lambda}{2} + \frac{\lambda}{2}\right) \\
&\quad \leq \psi\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_{k_j}, \frac{\lambda}{2}\right) \odot\ \psi\left(p_1, p_2, \ldots, p_{n-1}, p_{k_j} - p_0, \frac{\lambda}{2}\right) \\
&\quad < \eta\ \odot\ \eta \text{ for } r = k_j \quad \text{(by (4.3) and (4.4))} \\
&\quad < \varepsilon \quad (\text{ by (3.1)})
\end{aligned}$$

which implies that $m \in G^{\complement}$, so $E^{\complement} \cap F^{\complement} \subseteq G^{\complement}$ or $G \subseteq E \cup F$. Therefore, $\delta_{\alpha,\gamma}(G) \leq \delta_{\alpha,\gamma}(E \cup F) = 0$. Thus, $p_k \to p_0\,(D_{\alpha,\gamma}\,[S(\varphi,\psi)^n])$. Hence $\mathcal{P}$ is deferred statistically complete. □

By similar process in the proof of Theorem (3.8) and by Theorem (4.3) we have the following Theorem.

Theorem 4.5. *Let $(\mathcal{P}, \varphi, \psi, \circledast, \odot)$ be an IF-n-NLS. If $p = (p_k)$ be any sequence in $\mathcal{P}$. Then the following conditions are equivalent:*

(i) *$p = (p_k)$ is deferred statistical summable w.r.t the IF-n-norm $(\varphi, \psi)^n$.*
(ii) *$p = (p_k)$ is deferred statistical Cauchy w.r.t the IF-n-norm $(\varphi, \psi)^n$.*

(iii) $\exists$ *a set* $T = \{t_1 < t_2 < \cdots\} \subset \mathbb{N}$ *such that* $\delta_{\alpha,\gamma}(T) = 1$ *and the subsequence* $(p_{t_j})_{t_j \in T}$ *is Cauchy sequence w.r.t the IF-n-norm* $(\varphi, \psi)^n$.

5. Comparison of $D_{\alpha,\gamma}[S(\phi,\psi)^n]$ and $D_{u,v}[S(\phi,\psi)^n]$

In this section, we consider two different pairs of sequences $\alpha(\ell)$,$\gamma(\ell)$ and $u(\ell)$, $v(\ell)$, of positive integers satisfying

$$\alpha(\ell) \leq u(\ell) < v(\ell) \leq \gamma(\ell) \quad \text{for all } \ell \in \mathbb{N}, \tag{5.1}$$

and study the relations among the sets $D_{\alpha,\gamma}[S(\phi,\psi)^n]$ and $D_{u,v}[S(\phi,\psi)^n]$.

The deferred limit of a sequence can be different for the different choice of integers sequence $\alpha(\ell)$, $\gamma(\ell)$ and therefore we can have different sets $D_{\alpha,\gamma}[S(\phi,\psi)^n]$ accordingly. So if we choose different pairs of sequences $\alpha(\ell)$,$\gamma(\ell)$ and $u(\ell)$, $v(\ell)$, satisfying (5.1) then we can have interesting relationships between the sets.

Theorem 5.1. *Let* $u = u(\ell)$ *and* $v = v(\ell)$ *be two positive integers sequences satisfying* $\alpha(\ell) \leq u(\ell) < v(\ell) \leq \gamma(\ell)$ *such that* $\{k : alpha(\ell) < k \leq u(\ell)\}$ *and* $\{k : v(\ell) < k \leq \gamma(\ell)\}$ *are finite sets for all* $\ell \in \mathbb{N}$ *then* $D_{u,v}[S(\varphi,\psi)^n] - \lim p_k = p_0$ *implies that* $D_{\alpha,\gamma}[S(\varphi,\psi)^n] - \lim p_k = p_0$.

Proof. Suppose $D_{u,v}[S(\varphi,\psi)^n] - \lim p_k = p_0$ then for every $\varepsilon \in (0,1)$, $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$ we have

$$\lim_{\ell\to\infty} \frac{1}{v(\ell) - u(\ell)} \Big| \big\{ u(\ell) < k \leq v(\ell) : \varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon \text{ or } \psi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon \big\} \Big| = 0.$$

Now define sets

$$K_{\alpha,\gamma}(\varphi,\psi)^n = \big\{ \alpha(\ell) < k \leq \gamma(\ell) : \varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon \text{ or } \psi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon \big\},$$

$$K_{\alpha,u}(\varphi,\psi)^n = \big\{ \alpha(\ell) < k \leq u(\ell) : \varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon \text{ or } \psi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon \big\},$$

$$K_{u,v}(\varphi,\psi)^n = \big\{ u(\ell) < k \leq v(\ell) : \varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon \text{ or } \psi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon \big\},$$

$$K_{v,\gamma}(\varphi,\psi)^n = \big\{ v(\ell) < k \leq \gamma(\ell) : \varphi(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon$$

$$\text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\Big\}.$$

It is clear that

$$K_{\alpha,\gamma}(\varphi,\psi)^n = K_{\alpha,u}(\varphi,\psi)^n \cup K_{u,v}(\varphi,\psi)^n \cup K_{v,\gamma}(\varphi,\psi)^n.$$

Now

$$\begin{aligned}\frac{\gamma(\ell)-\alpha(\ell)}{\gamma(\ell)-\alpha(\ell)}\left|K_{\alpha,\gamma}(\varphi,\psi)^n\right| \leq \Bigg(&\frac{u(\ell)-\alpha(\ell)}{u(\ell)-\alpha(\ell)}\left|K_{\alpha,u}(\varphi,\psi)^n\right| \\ &+ \frac{v(\ell)-u(\ell)}{v(\ell)-u(\ell)}\left|K_{u,v}(\varphi,\psi)^n\right| \\ &+ \frac{\gamma(\ell)-v(\ell)}{\gamma(\ell)-v(\ell)}\left|K_{v,\gamma}(\varphi,\psi)^n\right|\Bigg),\end{aligned}$$

and therefore

$$\begin{aligned}\frac{1}{\gamma(\ell)-\alpha(\ell)}\left|K_{\alpha,\gamma}(\varphi,\psi)^n\right| \leq \frac{1}{\gamma(\ell)-\alpha(\ell)}\Bigg(&\frac{u(\ell)-\alpha(\ell)}{u(\ell)-\alpha(\ell)}\left|K_{\alpha,u}(\varphi,\psi)^n\right| \\ &+ \frac{v(\ell)-u(\ell)}{v(\ell)-u(\ell)}\left|K_{u,v}(\varphi,\psi)^n\right| \\ &+ \frac{\gamma(\ell)-v(\ell)}{\gamma(\ell)-v(\ell)}\left|K_{v,\gamma}(\varphi,\psi)^n\right|\Bigg).\end{aligned}$$

Taking limit $\ell \to \infty$ and by using the fact that deferred density of a finite set is zero. We have

$$\lim_{\ell\to\infty}\frac{1}{\gamma(\ell)-\alpha(\ell)}\left|K_{\alpha,\gamma}(\varphi,\psi)^n\right| = 0.$$

Hence, $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right] - \lim p_k = p_0$. □

We now give conditions on $\alpha(\ell)$, $u(\ell)$, $v(\ell)$ and $\gamma(\ell)$ such that the converse of the above Theorem (5.1) holds.

Theorem 5.2. *Let $\alpha(\ell)$, $u(\ell)$, $v(\ell)$ and $\gamma(\ell)$ be sequence of natural number satisfying $\alpha(\ell) \leq u(\ell) < v(\ell) \leq \gamma(\ell)$ such that $\lim_{\ell\to\infty} \frac{\gamma(\ell)-\alpha(\ell)}{v(\ell)-u(\ell)} = r > 0$. Then $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right] - \lim p_k = p_0$ implies that $D_{u,v}\left[S(\varphi,\psi)^n\right] - \lim p_k = p_0$.*

Proof. Let us define set $K_{u,v}(\varphi,\psi)^n$ and $K_{\alpha,\gamma}(\varphi,\psi)^n$ as follows

$$\begin{aligned}K_{u,v}(\varphi,\psi)^n = \Big\{&u(\ell) < k \leq v(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1-\varepsilon \\ &\text{or } \psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\Big\}\end{aligned}$$

and

$$K_{\alpha,\gamma}(\varphi,\psi)^n = \Big\{\alpha(\ell) < k \leq \gamma(\ell) : \varphi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1-\varepsilon$$

or $\psi\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\Big\}$

It is clear to see

$$K_{u,v}(\varphi,\psi)^n \subset K_{\alpha,\gamma}(\varphi,\psi)^n,$$

and therefore

$$\left|K_{u,v}(\varphi,\psi)^n\right| \leq \left|K_{\alpha,\gamma}(\varphi,\psi)^n\right|.$$

Thus, we have

(5.2)

$$\frac{1}{u(\ell)-v(\ell)}\left|K_{u,v}(\varphi,\psi)^n\right| \leq \frac{\gamma(\ell)-\alpha(\ell)}{u(\ell)-v(\ell)}.\frac{1}{\gamma(\ell)-\alpha(\ell)}\left|K_{\alpha,\gamma}(\varphi,\psi)^n\right|$$

Since $D_{\alpha,\gamma}\left[S(\varphi,\psi)^n\right] - \lim p_k = p_0$ so we have,

$$\lim_{\ell\to\infty}\frac{1}{u(\ell)-v(\ell)}\left|K_{u,v}(\varphi,\psi)^n\right| = 0$$

Hence, $D_{u,v}\left[S(\varphi,\psi)^n\right] - \lim p_k = p_0$. □

6. Conclusion

The evolution of fuzzy norms led to the introduction of the intuitionistic fuzzy norm, which provides a versatile mathematical framework for handling uncertainty and vagueness. Higher-dimensional spaces are important for capturing the complexity of real-world problems, modeling multi-dimensional data and providing a framework for advanced mathematical and computational techniques in various disciplines. In the present study, we studied deferred summability, deferred statistical summability in IF-n-NLS. Theorems (3.5) establish a link between usual convergence and these summability concepts in this space. The properties and relationships among these concepts are demonstrated in Theorems (3.7), (3.8), (3.9) and (3.11). The condition in Theorem (3.13), explores cases where statistical summability implies deferred statistical summability. Additionally, we studied deferred statistical Cauchy sequences, deferred statistical completeness and their outcomes. Finally, Theorems (5.1) and (5.2) present comparisons to understand the convergence behavior of deferred statistically summability with itself for different pairs of sequences in IF-n-NLS.

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