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Generalized Grand Lebesgue Spaces Associated to Banach Function Spaces

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Abstract. In this paper we introduce the class of grand Lebesgue spaces associated to a Banach function space X by replacing the role of the L^1-norm by the norm $\|\cdot\|_X$ in the classical construction of the generalized grand Lebesgue spaces. Also, we study the inclusion property of these spaces.

1. Introduction

Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ with finite Lebesgue measure, and let $1 < p < \infty$. Then, the set of all complex-valued measurable functions f on Ω with

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\int_\Omega |f|^{p-\varepsilon} d\lambda \right)^{\frac{1}{p-\varepsilon}} < \infty,$$

where λ is the normalized Lebesgue measure on Ω, is called *grand Lebesgue space* and is denoted by $L^{p)}(\Omega)$. This class of Banach function spaces were introduced by Iwaniec and Sbordone [9] in the study of integrability properties of the Jacobian determinant. Grand Lebesgue spaces are so useful tool to study partial differential equations and they play a key role in function spaces theory. Refer to [6–8] and references therein, where the generalized grand Lebesgue spaces and variants of them are widely investigated. In particular, in [7] the closure property of these spaces with respect to convolution as multiplicative operation is proved.

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If $\theta \geq 0$, then the generalized grand Lebesgue space $L^{p),\theta}$ is defined as the space of all complex-valued measurable functions f on Ω with

$$\|f\|_{p),\theta} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^{\theta} \int_{\Omega} |f|^{p-\varepsilon} d\lambda\right)^{\frac{1}{p-\varepsilon}} < \infty.$$

In fact $L^{p),0}(\Omega)$ reduces to the Lebesgue space $L^{p}(\Omega)$, and also $L^{p),1}(\Omega)$ is equal to $L^{p)}(\Omega)$. We can write

$$\|f\|_{p),\theta} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left\| |f|^{p-\varepsilon} \right\|_1^{\frac{1}{p-\varepsilon}}.$$

The $L^1(\Omega)$-norm plays a key role in the above definitions. We know that $L^1(\Omega)$ is a special Banach function space. In this paper, by replacing $L^1(\Omega)$ with an arbitrary Banach function space X, we introduce a new class of Banach function spaces related to grand Lebesgue spaces which are a generalization of generalized grand Lebesgue spaces. In fact, we consider an arbitrary Banach function space on a measure space $(\Omega, \mathcal{A}, \mu)$ and replace the L^1-norm by the norm of the fixed Banach function space. We prove that the obtained spaces are Banach function spaces. Also, we study the inclusion of the new spaces.

2. Main Results

In this section, first we recall the definition of a Banach function space as in [3].

Definition 2.1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with a non-negative mesaure μ. Let $\mathcal{L}^0(\Omega)$ be the set of all real-valued measurable functions on Ω, and assume that $\mathcal{L}^0_+(\Omega)$ be the set of all nonnegative functions in $\mathcal{L}^0(\Omega)$. A mapping $\rho : \mathcal{L}^0_+(\Omega) \to [0, +\infty]$ is called a *function norm* if for all $f, g \in \mathcal{L}^0_+(\Omega)$, $(f_n)_n \subseteq \mathcal{L}^0_+(\Omega)$, and all sets $E \in \mathcal{A}$, the following conditions hold:

(i) $\rho(f) = 0$ if and only if $f = 0$ μ-a.e.,
(ii) $\rho(\alpha f) = \alpha\rho(f)$ for all $\alpha \geq 0$,
(iii) $\rho(f + g) \leq \rho(f) + \rho(g)$,
(iv) if $g \leq f$ μ-a.e., then $\rho(g) \leq \rho(f)$,
(v) if $f_n \uparrow f$ μ-a.e., then $\rho(f_n) \uparrow \rho(f)$,
(vi) if $\mu(E) < \infty$, then $\rho(\chi_E) < \infty$,
(vii) if $\mu(E) < \infty$, then there exists a constant $C_E = C_E(\rho, E) > 0$ such that for each $h \in \mathcal{L}^0_+(\Omega)$ we have $\int_E h\, d\mu(x) \leq C_E\, \rho(h)$.

Any function norm induces a Banach space of measurable functions.

Definition 2.2. Let ρ be a function norm on a measure space $(\Omega, \mathcal{A}, \mu)$. Then, the set of all elements f in $\mathcal{L}^0$ with $\|f\|_{X_\rho} := \rho(|f|) < \infty$ is denoted by X_ρ and is called a *Banach function space*.

For a monograph regarding these spaces see [3]. By [3, Theorem 1.6], $(X_\rho, \|\cdot\|_{X_\rho})$ is a Banach space.

For example, Orlicz spaces, Lebesgue spaces and Lorentz spaces are Banach function spaces.

Recall that a convex function $\Phi : [0,\infty) \to [0,\infty)$ is called a *Young function* if $\Phi(0) = 0$, $\lim_{x\to 0^+} \Phi(x) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$. For example, if $1 \le p < \infty$, then the function $x \mapsto x^p$ on $[0,\infty)$ is a *Young function*.

Definition 2.3. Let X_ρ be a Banach function space on Ω. For each function $f \in \mathcal{L}^0(\Omega)$ the Luxemburg norm with respect to X_ρ is defined by,

$$\|f\|_{X(\rho)^\Phi} := \inf\left\{\alpha > 0 : \Phi(\frac{|f|}{\alpha}) \in X_\rho,\ \left\|\Phi(\frac{|f|}{\alpha})\right\|_{X_\rho} \le 1\right\}.$$

Then, the set of all $f \in \mathcal{L}^0(\Omega)$ with $\|f\|_{X(\rho)^\Phi} < \infty$ is denoted by X_ρ^Φ.

As in [4, Proposition 4.6 and Theorem 4.11], in the case where $(\Omega, \mathcal{A}, \mu)$ is (σ)-finite measure space, $(X_\rho^\Phi, \|\cdot\|_{X(\rho)^\Phi})$ is a Banach function space on Ω. If $X := L^1(\Omega)$ and $\rho_0(f) = \int f d\mu$, the space $X_{\rho_0}^\Phi$ will be the usual Orlicz space $L^\Phi(\Omega)$.

Also we recall that a quasi-normed function space X has the σ-Fatou property if for any positive increasing sequence $(f_n)_{n\in\mathbb{N}}$ in X with $\sup_{n\in\mathbb{N}} \|f_n\|_X < \infty$ and converging pointwise μ-a.e. to a function f, then $f \in X$ and $\|f\|_X = \sup_{n\in\mathbb{N}} \|f_n\|_X$.

In the sequel, we give a huge generalization of generalized grand Lebesgue spaces by Banach function spaces.

Throughout, assume that ρ be a function norm on a measure space $(\Omega, \mathcal{A}, \mu)$.

Definition 2.4. Let $1 < p < \infty$ and $\theta \ge 0$. Then, for each $f \in \mathcal{L}_+^0$ we define

$$\tau_{p),\theta,\mu,\rho}(f) := \sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}} \left[\rho(f^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}} : 0 < \varepsilon \le p-1\right\}.$$

Theorem 2.5. *With the above notations, $\tau_{p),\theta,\mu,\rho}$ is a function norm.*

Proof. (i) The condition (1) is trivial.

(ii) Let $f \in \mathcal{L}_+^0(\Omega)$ and $\alpha \ge 0$. Then, since ρ is a function norm we have

$$\begin{aligned}
\tau_{p),\theta,\mu,\rho}(\alpha f) &= \sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}} \left[\rho((\alpha f)^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}} : 0 < \varepsilon \le p-1\right\} \\
&= \sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}} \left[\alpha^{p-\varepsilon}\,\rho(f^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}} : 0 < \varepsilon \le p-1\right\} \\
&= \alpha \sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}} \left[\rho(f^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}} : 0 < \varepsilon \le p-1\right\}
\end{aligned}$$

$$= \alpha\, \tau_{p),\theta,\mu,\rho}(f).$$

(iii) Let $f, g \in \mathcal{L}^0_+(\Omega)$. Let $0 < \varepsilon \leq p-1$. Hence, $1 \leq p-\varepsilon$, and so the mapping $\Phi : x \mapsto x^{p-\varepsilon}$ on $[0,\infty)$ is a Young function. In this case, we have

$$\begin{aligned}
\|f\|_{X(\rho)^\Phi} &= \inf\left\{k>0:\ \left\|\Phi\left(\frac{f}{k}\right)\right\|_{X(\rho)} \leq 1\right\}\\
&= \inf\left\{k>0:\ \left\|\left(\frac{f}{k}\right)^{p-\varepsilon}\right\|_{X(\rho)} \leq 1\right\}\\
&= \inf\left\{k>0:\ \rho\left(\left(\frac{f}{k}\right)^{p-\varepsilon}\right) \leq 1\right\}\\
&= \inf\left\{k>0:\ \frac{1}{k^{p-\varepsilon}}\,\rho\left(f^{p-\varepsilon}\right) \leq 1\right\}\\
&= \inf\left\{k>0:\ \rho\left(f^{p-\varepsilon}\right) \leq k^{p-\varepsilon}\right\}\\
&= \inf\left\{k>0:\ \rho\left(f^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}} \leq k\right\}\\
&= \left[\rho\left(f^{p-\varepsilon}\right)\right]^{\frac{1}{p-\varepsilon}},
\end{aligned}$$

where $\|\cdot\|_{X(\rho)^\Phi}$ is the Luxemburg norm defined in Definition 2.3. Hence, by [4, Proposition 4.6], we have

$$\left[\rho\left((f+g)^{p-\varepsilon}\right)\right]^{\frac{1}{p-\varepsilon}} \leq \left[\rho\left(f^{p-\varepsilon}\right)\right]^{\frac{1}{p-\varepsilon}} + \left[\rho\left(g^{p-\varepsilon}\right)\right]^{\frac{1}{p-\varepsilon}}.$$

Thus,

$$\begin{aligned}
&\tau_{p),\theta,\mu,\rho}(f+g)\\
&= \sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}}\left[\rho((f+g)^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}} : 0<\varepsilon\leq p-1\right\}\\
&\leq \sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}}\left(\left[\rho\left(f^{p-\varepsilon}\right)\right]^{\frac{1}{p-\varepsilon}} + \left[\rho\left(g^{p-\varepsilon}\right)\right]^{\frac{1}{p-\varepsilon}}\right) : 0<\varepsilon\leq p-1\right\}\\
&\leq \sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}}\left[\rho\left(f^{p-\varepsilon}\right)\right]^{\frac{1}{p-\varepsilon}} : 0<\varepsilon\leq p-1\right\}\\
&\quad + \sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}}\left[\rho\left(g^{p-\varepsilon}\right)\right]^{\frac{1}{p-\varepsilon}} : 0<\varepsilon\leq p-1\right\}\\
&= \tau_{p),\theta,\mu,\rho}(f) + \tau_{p),\theta,\mu,\rho}(g).
\end{aligned}$$

(iv) Let $f, g \in \mathcal{L}^0_+(\Omega)$, and $f \leq g$. Since ρ is a function norm, for each $0 < \varepsilon \leq p-1$ we have $\rho(f^{p-\varepsilon}) \leq \rho(g^{p-\varepsilon})$, and so $[\rho(f^{p-\varepsilon})]^{\frac{1}{p-\varepsilon}} \leq [\rho(f^{p-\varepsilon})]^{\frac{1}{p-\varepsilon}}$. This implies that

$$\tau_{p),\theta,\mu,\rho}(f) = \sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}}\left[\rho(f^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}} : 0<\varepsilon\leq p-1\right\}$$

$$\leq \sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}}\,\left[\rho(g^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}}\,:\,0<\varepsilon\leq p-1\right\}$$
$$=\tau_{p),\theta,\mu,\rho}(g).$$

(v) Let $f\in\mathcal{L}^0_+(\Omega)$, $(f_n)_n\subseteq\mathcal{L}^0_+(\Omega)$, and $f_n\uparrow f$ μ-a.e. Since ρ is a function norm, by [4, Theorem 4.12], the Luxemburg norm $\|\cdot\|_{X(\rho)^\Phi}$ has the σ-Fatou property. So applying the item (4) we have

$$\begin{aligned}\tau_{p),\theta,\mu,\rho}(f_n)\uparrow\sup_n\tau_{p),\theta,\mu,\rho}(f_n)&=\sup_n\sup_{0<\varepsilon\leq p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\,\left[\rho(f_n^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}}\\&=\sup_{0<\varepsilon\leq p-1}\sup_n\varepsilon^{\frac{\theta}{p-\varepsilon}}\,\left[\rho(f_n^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}}\\&=\sup_{0<\varepsilon\leq p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\,\sup_n\left[\rho(f_n^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}}\\&=\sup_{0<\varepsilon\leq p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\,\left[\rho(f^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}}\\&=\tau_{p),\theta,\mu,\rho}(f).\end{aligned}$$

(vi) Let $E\in\mathcal{A}$ with $\mu(E)<\infty$. Then we have

$$\begin{aligned}\tau_{p),\theta,\mu,\rho}(\chi_E)&=\sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}}\,\left[\rho(\chi_E^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}}\,:\,0<\varepsilon\leq p-1\right\}\\&=\sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}}\,\left[\rho(\chi_E)\right]^{\frac{1}{p-\varepsilon}}\,:\,0<\varepsilon\leq p-1\right\}\\&=(p-1)^\theta\,\left[\rho(\chi_E)\right]^{\frac{1}{p}}<\infty,\end{aligned}$$

because $\rho(\chi_E)<\infty$.

(vii) For the item (7) in Definition 2.1 let $E\in\mathcal{A}$ with $\mu(E)<\infty$. Since ρ is a function norm, there is a constant number $C_E>0$ such that for each $h\in\mathcal{L}^0_+(\Omega)$, $\int_E h\,d\mu(x)\leq C_E\,\rho(h)$. But, by [10, Theorem 6.2 and Theorem 5.1] for each $0<\varepsilon<p-1$ we have

$$\rho(h)\leq 2\,\left[\rho\left(h^{p-\varepsilon}\right)\right]^{\frac{1}{p-\varepsilon}}\,\left[\rho\left(1\right)\right]^{\frac{1}{q}},$$

where $\frac{1}{p-\varepsilon}+\frac{1}{q}=1$. Hence,

$$\begin{aligned}\int_E h\,d\mu(x)&\leq 2\,C_E\,\left[\rho\left(1\right)\right]^{1-\frac{1}{p-\varepsilon}}\,\varepsilon^{-\frac{\theta}{p-\varepsilon}}\,\varepsilon^{\frac{\theta}{p-\varepsilon}}\,\left[\rho\left(h^{p-\varepsilon}\right)\right]^{\frac{1}{p-\varepsilon}}\\&\leq 2\,C_E\,\left[\rho\left(1\right)\right]^{1-\frac{1}{p-\varepsilon}}\,\varepsilon^{-\frac{\theta}{p-\varepsilon}}\,\tau_{p),\theta,\mu,\rho}(h)\\&\leq 2\,C_E\,\left[\rho\left(1\right)\right]^{1-\frac{1}{p}}\,(p-1)^{-\theta}\,\tau_{p),\theta,\mu,\rho}(h).\end{aligned}$$

So, $\tau_{p),\theta,\mu,\rho}$ is a function norm. □

Definition 2.6. Let ρ be a function norm on a measure space $(\Omega, \mathcal{A}, \mu)$, $1 < p < \infty$ and $\theta \geq 0$. Then, the set of all $f \in \mathcal{L}^0(\Omega)$ with

$$\|f\|_{p),\theta,\mu,\rho} := \sup\left\{\varepsilon^{\frac{\theta}{p-\varepsilon}}\left[\rho(|f|^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}} : 0 < \varepsilon \leq p-1\right\} < \infty,$$

is denoted by $X_\rho^{p),\theta,\mu}$. Indeed, by the Definition 2.2, $X_\rho^{p),\theta,\mu} = X_{\tau_{p),\theta,\mu,\rho}}$. This space is called generalized grand Lebesgue space related to the Banach function space X_ρ. As usual, two elements $f, g \in X_\rho^{p),\theta,\mu}$ are considered equal if $f = g$ μ-a.e.

The following theorem is a direct conclusion of Theorem 2.5 and [3, Theorem 1.7].

Theorem 2.7. *Let ρ be a function norm on a measure space $(\Omega, \mathcal{A}, \mu)$, $1 < p < \infty$ and $\theta \geq 0$. Then, the $\left(X_\rho^{p),\theta,\mu}, \|\cdot\|_{p),\theta,\mu,\rho}\right)$ is a Banach space. Moreover the following properties hold for all f, g, f_n $(n = 1, 2, \ldots)$, in $L^0(\Omega)$ and all measurable subsets $E \in \mathcal{A}$:*

(i) *(the lattice property) If $|g| < |f|$ μ-a.e. and $f \in X_{\tau_{p),\theta,\mu,\rho}}$, then $g \in X_{\tau_{p),\theta,\mu,\rho}}$ and $\|g\|_{\tau_{p),\theta,\mu,\rho}} \leq \|f\|_{\tau_{p),\theta,\mu,\rho}}$; in particular, a measurable function f belongs to $X_{\tau_{p),\theta,\mu,\rho}}$ if and only if $|f|$ belongs to $X_{\tau_{p),\theta,\mu,\rho}}$, and in that case $\|f\|_{\tau_{p),\theta,\mu,\rho}} = \||f|\|_{\tau_{p),\theta,\mu,\rho}}$.*

(ii) *(the Fatou property) Suppose $f_n \in X_{\tau_{p),\theta,\mu,\rho}}$, $f_n > 0, (n = 1, 2, \ldots)$, and $f_n \uparrow f$ μ-a.e. If $f \in X_{\tau_{p),\theta,\mu,\rho}}$, then $\|f_n\|_{\tau_{p),\theta,\mu,\rho}} \uparrow \|f\|_{\tau_{p),\theta,\mu,\rho}}$, and if $f \notin X_{\tau_{p),\theta,\mu,\rho}}$, then $\|f_n\|_{\tau_{p),\theta,\mu,\rho}} \uparrow \infty$.*

(iii) *(Fatou's lemma) If $f_n \in X_{\tau_{p),\theta,\mu,\rho}}, (n = 1, 2, \ldots)$, $f_n \to f$ μ-a.e., and $\liminf_{n\to\infty} \|f_n\|_{\tau_{p),\theta,\mu,\rho}} < \infty$, then $f \in X_{\tau_{p),\theta,\mu,\rho}}$ and,*

$$\|f\|_{\tau_{p),\theta,\mu,\rho}} \leq \liminf_{n\to\infty} \|f_n\|_{\tau_{p),\theta,\mu,\rho}}.$$

(iv) *Every simple function belongs to $X_{\tau_{p),\theta,\mu,\rho}}$.*

(v) *To each set $E \in \mathcal{A}$ of finite measure there corresponds a constant C_E satisfying $0 < C_E < \infty$ such that*

$$\int_E f d\mu \leq C_E \|f\|_{\tau_{p),\theta,\mu,\rho}},$$

for all $f \in X_{\tau_{p),\theta,\mu,\rho}}$.

(vi) *If $f_n \to f$ in $X_{\tau_{p),\theta,\mu,\rho}}$, then $f_n \to f$ in measure on every set of finite measure; in particular, some subsequence of $(f_n)_{n\in\mathbb{N}}$ converges to f pointwise μ-a.e.*

Remark 2.8. If we define $\rho_0(f) := \int_\Omega f\, d\mu$ for all $f \in \mathcal{L}^0_+(\Omega)$, then we have

$$X_{\rho_0}^{p),\theta,\mu} = L^{p),\theta}.$$

So, the above spaces are generalizations of the known generalized grand Lebesgue spaces.

Corollary 2.9. *If Φ is a Young function, then putting*

$$\|f\|^* := \sup_{0<\varepsilon\leq p-1} \inf \left\{ \varepsilon^{\frac{\theta}{p-\varepsilon}} \alpha^{\frac{1}{p-\varepsilon}} : \alpha > 0, \ \left\| \Phi\left(\frac{|f^{p-\varepsilon}|}{\alpha} \right) \right\|_1 \leq 1 \right\}$$

and

$$L_{\Phi}^{p),\theta,\mu} := \{f \in \mathcal{L}^0(\Omega) : \|f\|^* < \infty\},$$

the pair $(L_{\Phi}^{p),\theta,\mu}, \|\cdot\|^*)$ is a Banach function space.

Proof. If $\rho_0(f) := \int_\Omega f \, d\mu$ and $\rho_1 = \|\cdot\|_{X(\rho_0)^\Phi}$, then

$$\left(L_{\Phi}^{p),\theta,\mu}, \|\cdot\|^*\right) = \left(X_{\rho_1}^{p),\theta,\mu}, \|\cdot\|_{p),\theta,\mu,\rho_1}\right).$$

Therefore, according to Theorem 2.7, the theorem is proved. □

Similar to the Corollary 2.9, we can obtain more Banach function spaces according to Definition 2.6 and Theorem 2.7. There are many areas for research in Banach spaces. For example, it is possible to check the closedness of these spaces with respect to multiplicative operations such as convolution. Also spaceability of this spaces is favorite. For instance, see [1, 2, 11–13]. In this article, the main goal is to introduce the generalized grand Lebesgue space related to a Banach function space X_ρ.

3. Inclusion Property

In this section, we give some necessary or sufficient conditions for the inclusion between the new spaces introduced in Section 2.

Theorem 3.1. *Let ρ_1, ρ_2 be function norms on a finite measure space $(\Omega, \mathcal{A}, \mu)$, $1 < p_1, p_2 < \infty$ and $\theta_1, \theta_2 \geq 0$. Then, the inclusion*

$$X_{\rho_1}^{p_1),\theta_1,\mu} \subseteq X_{\rho_2}^{p_2),\theta_2,\mu} \tag{3.1}$$

implies that there exists a constant $C > 0$ such that

$$\|f\|_{p_2),\theta_2,\mu,\rho_2} \leq C \, \|f\|_{p_1),\theta_1,\mu,\rho_1} \tag{3.2}$$

for all $f \in X_{\rho_1}^{p_1),\theta_1,\mu}$.

Proof. Assume that the inclusion (3.1) holds. We will prove that the inclusion map

$$I : X_{\rho_1}^{p_1),\theta_1,\mu} \to X_{\rho_2}^{p_2),\theta_2,\mu}$$

is continuous. By [5], this implies that, for some constant $C > 0$, the inequality (3.2) is valid. By closed graph theorem, it would be enough to prove that the set

$$G(I) := \left\{(f, I(f)) : f \in X_{\rho_1}^{p_1),\theta_1,\mu}\right\}$$

is closed in $X_{\rho_1}^{p_1),\theta_1,\mu} \times X_{\rho_2}^{p_2),\theta_2,\mu}$. For this, assume that $(f_n) \subseteq X_{\rho_1}^{p_1),\theta_1,\mu}$ such that

$$(f_n, f_n) = (f_n, I(f_n)) \to (f, g)$$

in $X_{\rho_1}^{p_1),\theta_1,\mu} \times X_{\rho_2}^{p_2),\theta_2,\mu}$. So, we have $f_n \to f$ in $X_{\rho_1}^{p_1),\theta_1,\mu}$ and $f_n \to g$ in $X_{\rho_2}^{p_2),\theta_2,\mu}$. So, by Theorem 2.7 and [3, Theorem 1.4], there is a subsequence $(f_{n_j})_j$ such that $f_{n_j} \to f$ a.e. and $f_{n_j} \to g$ a.e. This implies that $f = g$ a.e. and hence $(f, g) \in G(I)$. Therefore, $G(I)$ is closed. □

For the sequel, we need the next fact.

If $p > 0$ and the function $\Phi_{(p)}$ is defined by $\Phi_{(p)}(x) := x^p$ for all $x \geq 0$, then we denote $X_\rho^p := X_\rho^{\Phi_{(p)}}$. In fact, $f \in X_\rho^p$ if and only if we have $[\rho(|f|^p)]^{\frac{1}{p}} < \infty$.

Theorem 3.2. *Let ρ be a function norm on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$ such that*

$$\sup\left\{\|\chi_E\|_{X_\rho} : E \in \mathcal{A} \text{ and } \chi_E \in X_\rho\right\} < \infty, \tag{3.3}$$

and $1 < q < p < \infty$. Then, $X_\rho^p \subseteq X_\rho^q$.

Proof. Assume that $1 < q < p < \infty$. By the relation (3.3), the condition [14, Theorem 2.4(ix)] holds. Hence, by [14, Theorem 2.4 (1)], the proof is complete. □

Theorem 3.3. *Let ρ be a function norm on a measure space $(\Omega, \mathcal{A}, \mu)$, $1 < q < p < \infty$ and $\theta \geq 0$. Let $C > 0$ be a constant such that for each $0 < \varepsilon \leq q - 1$,*

$$\left[\rho(|f|^{q-\varepsilon})\right]^{\frac{1}{q-\varepsilon}} \leq C \left[\rho(|f|^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}},$$

for all $f \in X_\rho^{p),\theta,\mu}$. Then,

$$X_\rho^{p),\theta,\mu} \subseteq X_\rho^{q),\theta,\mu}. \tag{3.4}$$

Proof. Let $1 < q < p < \infty$ and $\theta \geq 0$. Let $f \in X_\rho^{p),\theta,\mu}$. Note that, since $p - q > 0$ and $0 < \frac{1}{(p-\varepsilon)(q-\varepsilon)} \leq 1$, we have

$$\begin{aligned} \varepsilon^{\frac{\theta(p-q)}{(p-\varepsilon)(q-\varepsilon)}} &\leq \max\left\{1, \varepsilon^{\theta(p-q)}\right\} \\ &\leq \max\left\{1, (q-1)^{\theta(p-q)}\right\}, \end{aligned}$$

for all $0 < \varepsilon \leq q-1$. Therefore, setting $M := C \max\left\{1, (q-1)^{\theta(p-q)}\right\}$, we have

$$\begin{aligned}
\tau_{q),\theta,\mu,\rho}(|f|) &= \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \left[\rho(|f|^{q-\varepsilon})\right]^{\frac{1}{q-\varepsilon}} \\
&\leq C \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \left[\rho(|f|^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}} \\
&\leq C \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \left[\rho(|f|^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}} \\
&= C \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta(p-q)}{(p-\varepsilon)(q-\varepsilon)}} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left[\rho(|f|^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}} \\
&\leq M \sup_{0<\varepsilon\leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left[\rho(|f|^{p-\varepsilon})\right]^{\frac{1}{p-\varepsilon}} \\
&= M\, \tau_{p),\theta,\mu,\rho}(|f|) \\
&< \infty,
\end{aligned}$$

because $f \in X_\rho^{p),\theta,\mu}$. Hence, the inclusion (3.4) holds.

□

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