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# Existence and Multiplicity Results for a Class of Kirchhoff-Type Equations

Mohammad Reza Heidari Tavani

Abstract. This paper deals with the existence and multiplicity of non-negative generalized solutions for a class of second-order Kirchhoff-type problems. It is important to mention that no asymptotic condition is required in the nonlinear term either at zero or at infinity. Our approach is based on the variational methods and critical point theory.

## 1. Introduction

Consider the following second-order Kirchhoff-type problem

$$\begin{cases} T(u) = \lambda f(t, u(t)), & t \in (a, b), \\ u(a) = u(b) = 0, \end{cases} \tag{1.1}$$

where $\lambda$ is a positive parameter and

$$T(u) = K\left(\int_a^b e^{-G(t)}\left(|u'(t)|^2 + h(t)|u(t)|^2\right) dt\right) \times \left(-u''(t) + g(t)u'(t) + h(t)u(t)\right),$$

in which $K : [0, +\infty[ \to \mathbb{R}$ is a continuous function such that there exist positive numbers $k_1$ and $k_2$ with $k_1 \leq K(t) \leq k_2$ for all $t \geq 0$, $g, h \in L^\infty([a, b])$ such that ess $\inf_{t\in[a,b]} h(t) > -\left(\frac{\pi}{b-a}\right)^2$. Also, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a non-negative $L^1$-Carathéodory function. We consider $G(t) = \int_a^t g(\xi)d\xi$ for all $t \in [a, b]$.

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The problem (1.1) is related to the stationary analogue of the equation stationary problem

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate $x$ and the time $t$, $E$ the Young modulus, $\rho$ the mass density, $h$ the cross-section area, $L$ the length and $\rho_0$ the initial axial tension, proposed by Kirchhoff [17] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. It can be definitely said that Kirchhoff type equations are part of the most important applied problems of science and engineering, which are presented in different order. To see more interesting results, it is recommended to read, for example, references [4, 10, 18, 24]. Second-order Kirchhoff-type problems are significant in describing a large class of elastic deflections. Therefore, many authors have studied the existence and multiplicity of solutions for second-order Kirchhoff type problems, we refer the reader to [3, 7, 9, 11–16, 19–21, 23, 25].

If $K$ is considered as a constant function 1 in problem (1.1), then it appears as follows:

$$\begin{cases} -u''(t) + g(t)u'(t) + h(t)u(t) = \lambda f(t,u(t)), \quad t \in (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{1.3}$$

which is discussed in references [1, 2]. In these papers, the authors investigated the existence and multiplicity of solutions for the problem (1.3) with a Sturm-Liouville second-order differential equation and Dirichlet conditions. From an engineering point of view, it should be noted that this type of problems describes, for example, physical and chemical events, as well as Boyd equation about eddies in the atmosphere, Laplace tidal wave equation, and Meissner equation which arises in a model of a one-dimensional crystal. For example in [5], the authors investigated the following Sturm–Liouville differential equation

$$-(t\,u'(t))' - t^3 u(t) = \lambda f(t,u(t)), \quad t \in (0,b], \text{with } 0 < b < 1, \tag{1.4}$$

related to gas dynamics in a fuel cell that converts chemical energy into electrical energy. It should be noted that fuel cells are a relatively new method of energy production that is an excellent and clean alternative to existing fossil fuel power plants. In fact, fuel cells combine hydrogen and oxygen and produce only waste in the form of water and heat.

The presence of the term $K\left(\int_a^b e^{-G(t)}(|u'(t)|^2 + h(t)|u(t)|^2)dt\right)$ implies that the equation (1.1) is a non-local type and this fact will lead

to serious challenges and will increase the complexity of these types of problems.

For an overview of the second-order Kirchhoff type problems discussed by techniques of variational methods in critical point theory, we refer the reader to reference [23]. In [23], on a bounded domain $\Omega \subset \mathbb{R}^n$, the author has considered a non-local problem of the type

$$\begin{cases} -K\left(\int_\Omega |\nabla u(x)|^2 dx\right)\Delta u = \lambda f(x,u) + \mu\, g(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

and using variational methods, investigated the existence of at least three weak solutions. In the present paper, using two critical point theorems obtained in [6, 22] we establish the existence and multiplicity of non-negative generalized solutions for the problem (1.1).

## 2. Preliminaries

Let $X := H_0^1([a,b]) = W_0^{1,2}([a,b])$ be the Sobolev space endowed with the norm

$$\|u\|_X = \left(\int_a^b e^{-G(t)}|u'(t)|^2 dt + \int_a^b e^{-G(t)}h(t)|u(t)|^2 dt\right)^{1/2},$$

which is equivalent to the usual norm

$$\|u\| = \left(\int_a^b |u'(t)|^2 dt\right)^{\frac{1}{2}}, \tag{2.1}$$

(see [1, Proposition 2.2]).

**Proposition 2.1** ([1], Proposition 2.1)**.** *For all $u \in X$, one has*

- $\max\limits_{t\in[a,b]} |u(t)| \le \frac{(b-a)^{\frac{1}{2}}}{2}\| u \|$,
- $\|u\|_{L^2[a,b]} \le \frac{b-a}{\pi}\|u\|$.

**Proposition 2.2** ([1], Proposition 2.2)**.** *For any $u \in X$, one has*

$$m\,\|u\| \le \|u\|_X \le M\,\|u\| \tag{2.2}$$

*where $m, M$, with $M \ge m > 0$, are given by*

$$m = \begin{cases} \left(\min\limits_{t\in[a,b]} e^{-G(t)}\right)^{\frac{1}{2}}, & \text{if } \operatorname{ess\,inf}\limits_{t\in[a,b]} h(t) \ge 0, \\ \left[\min\limits_{t\in[a,b]} e^{-G(t)}\left(1 + \operatorname{ess\,inf}\limits_{t\in[a,b]} h(t)\left(\frac{b-a}{\pi}\right)^2\right)\right]^{\frac{1}{2}}, & \text{if } \operatorname{ess\,inf}\limits_{t\in[a,b]} h(t) < 0, \end{cases}$$

*and*

$$M = \begin{cases} \left[\max\limits_{t\in[a,b]} e^{-G(t)}\left(1+\operatorname{ess\,sup}\limits_{t\in[a,b]} h(t)\left(\frac{b-a}{\pi}\right)^2\right)\right]^{\frac{1}{2}}, & \text{if } \operatorname{ess\,sup}\limits_{t\in[a,b]} h(t)\geq 0, \\ \left(\max\limits_{t\in[a,b]} e^{-G(t)}\right)^{\frac{1}{2}}, & \text{if } \operatorname{ess\,sup}\limits_{t\in[a,b]} h(t)< 0, \end{cases}$$

**Remark 2.3** ([1], Remark 2.3)**.** For any $u \in X$, one has

$$\max_{t\in[a,b]} |u(t)| \leq \frac{(b-a)^{\frac{1}{2}}}{2m}\|u\|_X, \tag{2.3}$$

where m is given in Proposition 2.2.

Put

$$F(t,x) = \int_0^x f(t,\xi)d\xi, \quad \text{for all} \quad (t,x)\in[a,b]\times\mathbb{R},$$

and

$$\widetilde{K}(t) = \int_0^t K(\xi)d\xi, \quad \text{for all} \quad t>0.$$

**Remark 2.4.** Since $f : [a,b]\times\mathbb{R}\to\mathbb{R}$ is a non-negative function, $F(t,x)\geq 0$ for all $(t,x)\in[a,b]\times\mathbb{R}^+$.

Let $\Phi,\Psi : X\to\mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2}\widetilde{K}\left(\|u\|_X{}^2\right), \tag{2.4}$$

and

$$\Psi(u) = \int_a^b e^{-G(t)}F(t,u(t))dt, \tag{2.5}$$

for every $u\in X$. The functional $\Psi$ is well defined and Gâteaux differentiable (see [[1], P. 5576] ) whose differential at the point $u\in X$ is

$$\begin{aligned} \Psi'(u)(v) &= \lim_{s\to 0}\frac{\Psi(u+sv)-\Psi(u)}{s} \\ &= \frac{d}{ds}\Psi(u+sv)\Big|_{s=0} \\ &= \int_a^b e^{-G(t)}f(t,u(t))v(t)dt, \end{aligned}$$

and $\Phi$ is bounded from below and convex such that $\inf_X\Phi = \Phi(0) = 0$. Also $\Phi$ is a continuously Gâteaux differentiable functional and in a

similar way, whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \left( \int_a^b e^{-G(t)} \left( u'(t)v'(t) + h(t)u(t)v(t) \right) dt \right) \times K \left( \|u\|_X{}^2 \right),$$

for every $v \in X$. Also, standard methods show that $\Phi$ is sequentially weakly lower semicontinuous and $\Psi$ is sequentially weakly upper semicontinuous. From (2.2), we observe that for all $u \in X$

$$\frac{k_1\, m^2}{2} \|u\|^2 \leq \Phi(u) \leq \frac{k_2\, M^2}{2} \|u\|^2. \tag{2.6}$$

Now, we recall the types of solutions to problem (1.1).
We say that $u : [a, b] \to \mathbb{R}$ is a classical solution to the problem (1.1) if $u \in C^2([a, b])$ ,$u(a) = u(b) = 0$ and

$$K \left( \int_a^b e^{-G(t)} (|u'(t)|^2 + h(t)|u(t)|^2) dt \right) \left( -u'' + g(t)u' + h(t)u \right) = \lambda f(t, u),$$

for every $t \in [a, b]$. Also a function $u : [a, b] \to \mathbb{R}$ is a generalized solution to the problem (1.1) if $u \in C^1([a, b])$, $u' \in AC([a, b])$, which is the set of all absolutely continuous function, $u(a) = u(b) = 0$ and

$$K \left( \int_a^b e^{-G(t)} (|u'(t)|^2 + h(t)|u(t)|^2) dt \right) \left( -u'' + g(t)u' + h(t)u \right) = \lambda f(t, u),$$

for almost every $t \in [a, b]$. And finally, it is a weak solution to the problem (1.1) if $u \in X$ and

$$\begin{aligned} K \left( \int_a^b e^{-G(t)} \left( |u'(t)|^2 + h(t)|u(t)|^2 \right) dt \right) & \\ \times \int_a^b e^{-G(t)} \left( u'(t)v'(t) + h(t)u(t)v(t) \right) dt & \\ - \lambda \int_a^b e^{-G(t)} f(t, u(t))v(t) dt & \\ = 0, & \end{aligned}$$

for every $v \in X$.

Standard methods show that if $f \in C([a, b] \times \mathbb{R})$ and $g, h \in C([a, b])$, then each generalized solution $u$ of the problem (1.1) is a classical solution and each weak solution $u$ to the problem (1.1) is a generalized one. In other words, if the functions $f$, $g$ and $h$ are considered continuous, all three types of solutions to problem (1.1) are the same.

**Definition 2.5.** Assume $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$, for all $u \in X$, $\lambda > 0$. We say that $u$ is a critical point of $I_\lambda$ when $I'_\lambda(u) = 0_{\{X^*\}}$, that is, $I'_\lambda(u)(v) = \Phi'(u)(v) - \lambda\Psi'(u)(v) = 0$ for all $v \in X$.

We will have the following result by a method similar to the proof of Proposition 2.3 of reference [1].

**Proposition 2.6.** *$u$ is a generalized solution of (1.1) $\Longleftrightarrow$ $u$ is a critical point of $I_\lambda$.*

**Lemma 2.7.** *If $u_0 \not\equiv 0$ is a generalized solution for problem (1.1) then $u_0$ is non-negative.*

*Proof.* Let $E = \{t \in [a,b] : u_0(t) < 0\}$ and arguing by contradiction, assume that $E$ is non-empty. From Proposition 2.6 one has, $I'_\lambda(u_0)(v) = 0$ for all $v \in X$. Choose $v(x) = \bar{u}_0 = \max\{-u_0(t), 0\}$. It is clear that $v \in X$. Therefore, we have

$$\left(\int_{[a,b]} e^{-G(t)} \left(u'(t)\bar{u}_0'(t) + h(t)u(t)\bar{u}_0(t)\right) dt\right) \times K\left(\|u\|_X{}^2\right) = \int_{[a,b]} e^{-G(t)} f(t, u(t))\bar{u}_0(t)dt.$$

Since $u_0$ is a generalized solution for problem (1.1) one has

$$\left(\int_E e^{-G(t)} \left(u_0{}'(t)\bar{u}_0'(t) + h(t)u_0(t)\bar{u}_0(t)\right) dt\right) \times K\left(\|u_0\|_X{}^2\right) = \int_E e^{-G(t)} f(t, u_0(t))\bar{u}_0(t)dt,$$

that is

$$(m_E)^2 \int_E |u_0{}'(t)|^2 dt \leq \int_E e^{-G(t)} \left(|u_0{}'(t)|^2 + h(t)|u_0(t)|^2\right) dt \leq 0, \tag{2.7}$$

where

$$m_E = \left(\min\left\{\inf_{t\in E} e^{-G(t)}, \inf_{t\in E} e^{-G(t)} \left(1 + \operatorname*{ess\,inf}_{t\in E} h(t)\left(\frac{|E|}{\pi}\right)^2\right)\right\}\right)^{\frac{1}{2}},$$

and $|E|$ denote the Lebesgue measure of set $E$.

From (2.7) we see that $u_0(t) = 0$ for every $t \in E$, which means that $E = \emptyset$. Therefore $u_0(t) \geq 0$ for every $t \in [a,b]$ and the proof is complete. □

## 3. Existence of One Solution

In this section, our goal is to prove the existence of a non-trivial solution for problem (1.1). Our tool for this research is the following critical point theorem attributed to Ricceri introduced in [22].

**Theorem 3.1.** *Let $X$ be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functional such that $\Phi$ is (strongly) continuous, sequentially weakly lower semicontinuous and coercive. Further, assume that $\Psi$ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$ , put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

*Then, for each $r > \inf_X \Phi$ and each $\lambda \in ]0, \frac{1}{\varphi(r)}[$, the restriction of $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of $I_\lambda$ in $X$.*

Now, we present the main theorem of this section.

**Theorem 3.2.** *Suppose that $f(t, 0) \neq 0$ for every $t \in [a, b]$. Then, for each $r > 0$ and each $\lambda \in ]0, \lambda^*(r)[$, where*

$$\lambda^*(r) = \frac{r}{\max\limits_{t \in [a,b]} e^{-G(t)} \displaystyle\int_a^b \max\limits_{|\xi| \leq \frac{1}{m}\left(\frac{(b-a)r}{2\,k_1}\right)^{\frac{1}{2}}} F(t, \xi)\, dt}, \tag{3.1}$$

*the problem (1.1) admits one non-trivial and non-negative generalized solution $u_\lambda \in X$ and*

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0.$$

*Proof.* We apply Theorem 3.1 with $X = W_0^{1,2}([a, b])$ and $\Phi, \Psi : X \to \mathbb{R}$ as defined in previous section. The functionals $\Phi$ and $\Psi$ satisfy the assumptions of regularity requested in Theorem 3.1. Fix $r > 0$ and $\lambda \in ]0, \lambda^*(r)[$. Bearing Proposition 2.1 and (2.6) in mind,

$$\begin{aligned}
\Phi^{-1}(]-\infty, r[) &= \{u \in X;\ \Phi(u) < r\} \\
&\subseteq \left\{u \in X;\ \frac{k_1\, m^2}{2}\|u\|^2 < r\right\} \\
&\subseteq \left\{u \in X;\ \max_{t \in [a,b]} |u(t)| \leq \frac{1}{m}\left(\frac{(b-a)r}{2\,k_1}\right)^{\frac{1}{2}}\right\},
\end{aligned}$$

and since $0 \in \Phi^{-1}(]-\infty, r[)$ it follows that

$$\varphi(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

$$
\begin{aligned}
&\leq \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)}\Psi(v)}{r}\\
&= \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)}\int_a^b e^{-G(t)}F(t,v(t))dt}{r}\\
&\leq \frac{\max\limits_{t\in[a,b]} e^{-G(t)}\int_a^b \max\limits_{|\xi|\leq\frac{1}{m}\left(\frac{(b-a)r}{2\,k_1}\right)^{\frac{1}{2}}} F(t,\xi)\,dt}{r}\\
&= \frac{1}{\lambda^*(r)}\\
&< \frac{1}{\lambda}.
\end{aligned}
$$

In conclusion, $\lambda \in ]0, \lambda^*(r)\,[\subseteq ]0, \frac{1}{\varphi(r)}[$. Since all assumptions of Theorem 3.1 are verified, for each $\lambda \in ]0, \lambda^*(r)[$, the restriction of $I_\lambda$ to $\Phi^{-1}(]-\infty, r[)$ admits a non-zero local minimum $u_\lambda$ in $X$. Now, we show that $\lim\limits_{\lambda\to 0^+}\|u_\lambda\| = 0$.

From $u_\lambda \in \Phi^{-1}(]-\infty, r[)$ and since $\Phi$ is coercive there exists a $C > 0$ such that $\|u_\lambda\| \leq C$. Also compact embedding $X \to C([a,b])$ imply that the functional $\Psi$ has compact derivative and so there exists a positive constant $L$ such that for all $0 < \lambda < \lambda^*(r)$ one has

$$
|\Psi'(u_\lambda)(u_\lambda)| \leq \left\|\Psi'(u_\lambda)\right\|_X \|u_\lambda\| \leq LC. \tag{3.2}
$$

Therefore, from $I'_\lambda(u_\lambda) = 0$ and in particular $I'_\lambda(u_\lambda)(u_\lambda) = \Phi'(u_\lambda)(u_\lambda) - \lambda\Psi'(u_\lambda)(u_\lambda) = 0$ we have $\lim\limits_{\lambda\to 0^+}\Phi'(u_\lambda)(u_\lambda) = 0$.

In other words

$$
\lim_{\lambda\to 0^+}\left(\left(\int_a^b e^{-G(t)}\left(|u'_\lambda(t)|^2 + h(t)|u_\lambda(t)|^2\right)dt\right)\times K\left(\|u_\lambda\|_X{}^2\right)\right) = 0,
$$

which means

$$
\lim_{\lambda\to 0^+}\left(\|u_\lambda\|_X{}^2 \times K\left(\|u_\lambda\|_X{}^2\right)\right) = 0,
$$

so we have in conclusion $\lim\limits_{\lambda\to 0^+}\|u^*_\lambda\| = 0$, and the proof is complete. □

As a consequence of Theorem 3.2, the following corollary is presented.

**Corollary 3.3.** *Suppose that $\Gamma : \mathbb{R} \to \mathbb{R}$ is a non-negative continuous function such that $\Gamma(0) \neq 0$ and $\alpha : [a,b] \to \mathbb{R}$ is a function such that $\alpha \in L^1([0,1])$, $\alpha(t) \geq 0$ for all $t \in [0,1]$, $\alpha \not\equiv 0$. Then for every*

$\lambda \in (0, \varpi)$ *where*

$$\varpi = \frac{m\,(2k_1)^{\frac{1}{2}}}{\max\limits_{t\in[0,1]} e^{-G(t)}\,\|\alpha\|_{L^1([0,1])}\left(\max\limits_{|\xi|\le\frac{1}{m}\left(\frac{1}{2\,k_1}\right)^{\frac{1}{2}}}\Gamma(\xi)\right)},$$

*the problem*

(3.3)
$$\begin{cases} K\left(\displaystyle\int_0^1 e^{-G(t)}(|u'(t)|^2 + h(t)|u(t)|^2)dt\right) \\ \quad \times(-u''(t) + g(t)u'(t) + h(t)u(t)) = \lambda\,\alpha(t)\,\Gamma(u(t)), \quad t \in (0,1), \\ \\ u(0) = u(1) = 0, \end{cases}$$

*admits one non-trivial and non-negative generalized solution in* $X$.

*Proof.* It is obvious $\varpi > 0$ and so it can be fixed a $\lambda \in (0, \varpi)$. Let $f(t,x) = \alpha(t)\,\Gamma(x)$ for all $(t,x) \in [0,1] \times \mathbb{R}$. We have

$$\begin{aligned} F(t,x) &= \int_0^x f(t,\xi)d\xi \\ &= \int_0^x \alpha(t)\,\Gamma(\xi)d\xi \\ &\le \left(\alpha(t)\max_{|\xi|\le|x|}\Gamma(\xi)\right)|x|, \end{aligned}$$

and so from (3.1) by choosing $r = 1$, $a = 0$ and $b = 1$ we have

$$\begin{aligned} \frac{1}{\lambda^*(1)} &= \max_{t\in[0,1]} e^{-G(t)} \int_0^1 \max_{|\xi|\le\frac{1}{m}\left(\frac{1}{2\,k_1}\right)^{\frac{1}{2}}} F(t,\xi)\,dt \\ &\le \max_{t\in[0,1]} e^{-G(t)}\,\|\alpha\|_{L^1([0,1])}\left(\max_{|\xi|\le\frac{1}{m}\left(\frac{1}{2\,k_1}\right)^{\frac{1}{2}}}\Gamma(\xi)\right)\frac{1}{m}\left(\frac{1}{2\,k_1}\right)^{\frac{1}{2}} \\ &= \frac{1}{\varpi} \\ &< \frac{1}{\lambda}. \end{aligned}$$

Therefore $\lambda \in ]0, \varpi[ \subseteq ]0, \lambda^*(1)[$ and so Theorem 3.2 guarantees the conclusion. □

## 4. Existence of Three Solutions

First, we will introduce a non-standard state of the Palais-Smale condition. This concept is presented in [6] by Bonanno.

**Definition 4.1.** (see [6]) Fix $r \in ]-\infty, +\infty]$. A Gâteaux differentiable function $I : X \to \mathbb{R}$ satisfies the Palais-Smale condition cut off upper at $r$ (in short $(PS)^{[r]}$ -condition) if any sequence $\{u_n\} \subseteq X$ such that

- $\{I(u_n)\}$ is bounded,
- $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
- $\Phi(u_n) < r \quad \forall n \in \mathbb{N}$,

has a convergent subsequence.

**Proposition 4.2.** *Fix $\lambda > 0$. Then $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition for any $r > 0$.*

*Proof.* Fix $r > 0$. Consider sequence $\{u_n\} \subseteq X$ such that $\{I_\lambda(u_n)\}$ is bounded, $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_{X^*} = 0$ and $\Phi(u_n) < r, \quad \forall n \in \mathbb{N}$.

Since $\Phi(u_n) < r$, from (2.6) one has, $\| u_n \| < \frac{1}{m}\sqrt{\frac{2r}{k_1}}$ and so $\{u_n\}$ is bounded in $X$. Therefore $\{u_n\}$ admits a subsequence that we rename again with $\{u_n\}$, which is weakly convergent to $u \in X$ (because $X$ is reflexive). From the compact embedding $X \to C([a,b])$, $\{u_n\}$ is strongly convergent to $u$ in $C([a,b])$. Again, since the embedding $X \to C([a,b])$ is continuous, there is $\sigma > 0$ such that $\max_{t\in[a,b]} |u_n(t)| \leq \sigma$. So since $\{u_n\}$ uniformly convergent to $u$ and $f(t, u_n(t)) \leq \max_{\xi\leq\sigma} f(t,\xi)$, one has

$$\begin{aligned} &\left|\lim_{n\to+\infty}\int_a^b e^{-G(t)} f(t,u_n(t))(u_n(t)-u(t))dt\right| \\ &\quad\leq \max_{t\in[a,b]} e^{-G(t)} \left|\lim_{n\to+\infty}\int_a^b f(t,u_n(t))(u_n(t)-u(t))dt\right| \\ &\quad= 0. \end{aligned} \tag{4.1}$$

From $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_{X^*} = 0$, there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0^+$, such that

$$\begin{aligned} &\left|K\left(\|u_n\|_X{}^2\right)\int_a^b e^{-G(t)}\left(u_n{}'(t)v'(t) + h(t)u_n(t)v(t)\right)dt \right. \\ &\qquad \left. -\lambda\int_a^b e^{-G(t)} f(t,u_n(t))v(t)dt|\right| \end{aligned} \tag{4.2}$$

$$\leq \varepsilon_n,$$

for all $n \in \mathbb{N}$ and for all $v \in X$ with $\|v\| \leq 1$. Put $v(x) = \frac{u_n(t)-u(t)}{\|u_n-u\|}$. From (4.2) one has

$$\begin{aligned}
&\left| K\left(\|u_n\|_X{}^2\right) \int_a^b e^{-G(t)} \left( {u_n}'(t)(u_n'(t) - u'(t)) + h(t)u_n(t)(u_n(t) - u(t))\right) dt \right. \\
&\qquad \left. - \lambda \int_a^b e^{-G(t)} f(t, u_n(t))(u_n(t) - u(t))dt \right| \\
&\leq \varepsilon_n \|u_n - u\|,
\end{aligned}$$

that is,

(4.3)

$$\begin{aligned}
&\left| K\left(\|u_n\|_X{}^2\right) \int_a^b e^{-G(t)} \left( {u_n}'(t)(u_n'(t) - u'(t)) + h(t)u_n(t)(u_n(t) - u(t))\right) dt \right| \\
&\qquad \leq \lambda \left| \int_a^b e^{-G(t)} f(t, u_n(t))(u_n(t) - u(t))dt \right| + \varepsilon_n \|u_n - u\|,
\end{aligned}$$

Now, from inequality $|a||b| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$ one has

$$\begin{aligned}
&K\left(\|u_n\|_X{}^2\right) \int_a^b e^{-G(t)} \left( {u_n}'(t)(u_n'(t) - u'(t)) + h(t)u_n(t)(u_n(t) - u(t))\right) dt \\
&\quad \geq k_1 \int_a^b e^{-G(t)} \left( {u_n}'(t)(u_n'(t) - u'(t)) + h(t)u_n(t)(u_n(t) - u(t))\right) dt \\
&\quad \geq k_1 \int_a^b e^{-G(t)} \left( |u_n'(t)|^2 - \frac{1}{2}|u_n'(t)|^2 - \frac{1}{2}|u'(t)|^2 \right. \\
&\qquad \left. + h(t)|u_n(t)|^2 - \frac{1}{2}h(t)|u_n(t)|^2 - \frac{1}{2}h(t)|u(t)|^2 \right) dt \\
&\quad = \frac{k_1}{2} \int_a^b e^{-G(t)} \left[ \left(|u_n'(t)|^2 + h(t)|u_n(t)|^2\right) - \left(|u'(t)|^2 + h(t)|u(t)|^2\right) \right] dt \\
&\quad = \frac{k_1}{2} \left( \|u_n\|_X^2 - \|u\|_X^2 \right).
\end{aligned}$$

Therefore, from (4.1) and (4.3) when $\varepsilon_n \to 0^+$, we have

$$\limsup_{n \to +\infty} \|u_n\|_X \ \leq \ \|u\|_X.$$

Since $X$ is a uniformly convex Banach space, $u_n \to u$ strongly in $X$ ([8]-Proposition 3.30) and the proof is complete. □

Below we mention a critical point theorem in [6], which will be the main tool in proving the important theorems of this paper.

**Theorem 4.3** ([6], Theorem 7.3)**.** *Let $X$ be a real Banach space,and let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functions with $\Phi$ bounded from below and convex such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.*

*Assume that there are two positive constants $r_1, r_2$ and $\overline{u} \in X$, with $2r_1 < \Phi(\overline{u}) < \frac{r_2}{2}$, such that*

- $(b_1)$ $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(\overline{u})}{\Phi(\overline{u})}$;
- $(b_2)$ $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(\overline{u})}{\Phi(\overline{u})}$.

*Assume also that, for each*

$$\lambda \in \Lambda = \left]\frac{3}{2}\frac{\Phi(\overline{u})}{\Psi(\overline{u})}, \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}, \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}\right\}\right[,$$

*the functional $\Phi - \lambda\Psi$ satisfies the $(PS)^{[r_2]}$-condition and*

- $(b_3)$ $\inf_{\zeta\in[0,1]} \Psi(\zeta u_1 + (1-\zeta)u_2) \geq 0$,

*for each $u_1, u_2 \in X$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$.*

*Then, for each $\lambda \in \Lambda$, the functional $\Phi - \lambda\Psi$ admits at least three critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.*

In order to introduce the results of this section, put

$$\Theta = \frac{\min\limits_{t\in[a,b]} e^{-G(t)}}{\max\limits_{t\in[a,b]} e^{-G(t)}},$$

and

$$F^\beta = \frac{\displaystyle\int_a^b \max_{|\xi|\leq\beta} F(t,\xi)dt}{\beta^2}, \quad \forall \beta > 0.$$

Now, we express the main theorem of this section.

**Theorem 4.4.** *Assume that there exist three positive constants $\eta$, $\theta_1$ and $\theta_2$ with $\theta_1 < \eta < \frac{m}{2\,M}\sqrt{\frac{k_1}{k_2}}\,\theta_2$ such that*

- $(i)$ $F^{\theta_1} < \dfrac{k_1\, m^2\, \Theta \displaystyle\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,\eta)dt}{3\, k_2\, M^2\eta^2}$,
- $(ii)$ $F^{\theta_2} < \dfrac{k_1\, m^2\, \Theta \displaystyle\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,\eta)dt}{6\, k_2\, M^2\eta^2}$.

*Then, for each*

$$\lambda \in \Lambda' = \left] \frac{6\,k_2\,M^2\eta^2}{(b-a)\min\limits_{t\in[a,b]} e^{-G(t)} \displaystyle\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,\eta)dt} \right.$$

$$\left. , \frac{k_1\,m^2}{(b-a)\max\limits_{t\in[a,b]} e^{-G(t)}} \min\left\{ \frac{2}{F^{\theta_1}}, \frac{1}{F^{\theta_2}} \right\} \right[ ,$$

*the problem* (1.1) *admits at least three distinct non-negative generalized solutions* $u_i \in X$ *such that* $\max\limits_{t\in[a,b]} |u_i(t)| < \theta_2$ , $i = 1, 2, 3$.

*Proof.* Fix $\lambda \in \Lambda'$ and take $X$ and $\Phi, \Psi : X \to \mathbb{R}$ as in the Section 2.

Our aim is to apply Theorem 4.3 to problem (1.1). We observe that the functionals $\Phi$ and $\Psi$ satisfy the assumptions of regularity requested in Theorem 4.3 and also according to proposition 4.2, the functional $I_\lambda$ satisfies the $(PS)^{[r]}$-condition for all $r > 0$. Put

$$r_1 := \frac{2\,k_1\,m^2}{b-a}\,{\theta_1}^2, \qquad r_2 := \frac{2\,k_1\,m^2}{b-a}\,{\theta_2}^2,$$

and

$$w(t) := \begin{cases} 4\,\eta\,\dfrac{t-a}{b-a}, & \text{if } t \in \left[a, a+\frac{b-a}{4}\right[, \\ \eta, & \text{if } t \in \left[a+\frac{b-a}{4}, b-\frac{b-a}{4}\right], \\ 4\,\eta\,\dfrac{b-t}{b-a}, & \text{if } t \in \left]b-\frac{b-a}{4}, b\right]. \end{cases} \tag{4.4}$$

We observe that $w \in X$ and in particular,

$$\| w \|^2 = \int_a^b |w'(t)|^2 dt = \frac{8\,\eta^2}{b-a},$$

and taking into account (2.6) one has

$$\frac{4\,k_1\,m^2\eta^2}{b-a} \le \Phi(w) \le \frac{4\,k_2\,M^2\eta^2}{b-a}.$$

Using the condition $\theta_1 < \eta < \frac{m}{2\,M}\sqrt{\frac{k_1}{k_2}}\,\theta_2$, we have $2r_1 < \Phi(w) < \frac{r_2}{2}$.

Now, for $i = 1, 2$ we see that

$$\sup_{u\in\Phi^{-1}(]-\infty,r_i[)} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r_i[)} \int_a^b e^{-G(t)} F(t,u(t))dt$$

$$\le \max_{t\in[a,b]} e^{-G(t)} \int_a^b \max_{|\xi|\le\theta_i} F(t,\xi)dt.$$

Hence, we have

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} \leq \frac{\max\limits_{t\in[a,b]} e^{-G(t)}\displaystyle\int_a^b \max_{|\xi|\leq\theta_1} F(t,\xi)dt}{\dfrac{2\,k_1\,m^2}{b-a}\,{\theta_1}^2} = \frac{(b-a)\max\limits_{t\in[a,b]} e^{-G(t)}}{2\,k_1\,m^2}\,F^{\theta_1} < \frac{1}{\lambda}. \tag{4.5}$$

Analogously, one has

$$\frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} \leq \frac{(b-a)\max\limits_{t\in[a,b]} e^{-G(t)}}{k_1\,m^2}\,F^{\theta_2} < \frac{1}{\lambda}. \tag{4.6}$$

On the other hand, we have

$$\frac{2}{3}\frac{\Psi(w)}{\Phi(w)} \geq \frac{2}{3}\frac{\displaystyle\int_a^b e^{-G(t)}F(t,w(t))dt}{\dfrac{4\,k_2\,M^2\eta^2}{b-a}} \geq \frac{1}{6}\frac{(b-a)\min\limits_{t\in[a,b]} e^{-G(t)}}{k_2\,M^2\eta^2}\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,\eta)dt > \frac{1}{\lambda}. \tag{4.7}$$

Therefore from (4.5), (4.6) and (4.7) we have,

$$\lambda\in\Lambda'\subseteq\left]\frac{2}{3}\frac{\Phi(w)}{\Psi(w)},\min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)},\frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[.$$

We still need to prove that condition $(b_3)$ is satisfied. For this purpose, suppose that $u_1,u_2\in X$ be two local minima of the functional $I_\lambda=\Phi-\lambda\Psi$, with $\Psi(u_1)\geq 0$ and $\Psi(u_2)\geq 0$. From Lemma 2.7, $u_1$ and $u_2$ are non-negative and so $\zeta u_1+(1-\zeta)u_2\geq 0$ for any $\zeta\in[0,1]$ and therefore due to the Remark 2.4 we have

$$\inf_{\zeta\in[0,1]}\Psi\left(\zeta u_1+(1-\zeta)u_2\right) = \inf_{\zeta\in[0,1]}\int_a^b e^{-G(t)}F\left(t,\zeta u_1(t)+(1-\zeta)u_2(t)\right)dt$$

$$\geq \min_{t\in[a,b]} e^{-G(t)} \left( \inf_{\zeta\in[0,1]} \int_a^b F(t, \zeta u_1(t) + (1-\zeta)u_2(t))dt \right)$$
$$\geq 0.$$

Finally, since $\lambda \in \Lambda' \subseteq \Lambda$, Theorem 4.3 (with $\overline{u} = w$) guarantees the conclusion. □

**Remark 4.5.** Assume that $f : \mathbb{R} \to \mathbb{R}$, be a continuous and non-negative function. Put $F(x) = \int_0^x f(\xi)d\xi$ for all $x \in \mathbb{R}$. Since $F'(x) = f(x) \geq 0$ for all $x \in \mathbb{R}$ then $F$ is non-decreasing and so

$$F^\beta = \frac{\int_a^b \max_{|\xi|\leq\beta} F(\xi)dt}{\beta^2}$$
$$= \frac{(b-a)F(\beta)}{\beta^2},$$

for all $\beta > 0$. Hence when $f$ does not depend on $t$, hypotheses $(i)$ and $(ii)$ in Theorem 4.4 become the following forms:

- $(I)$ $\dfrac{F(\theta_1)}{{\theta_1}^2} < \dfrac{k_1\, m^2\, \Theta\, F(\eta)}{6\, k_2\, M^2\, \eta^2}$ ,
- $(II)$ $\dfrac{F(\theta_2)}{{\theta_2}^2} < \dfrac{k_1\, m^2\, \Theta\, F(\eta)}{12\, k_2\, M^2\, \eta^2}$,

and the interval becomes

$$\left] \frac{12\, k_2\, M^2\, \eta^2}{(b-a)^2 \min\limits_{t\in[a,b]} e^{-G(t)}\, F(\eta)}, \frac{k_1\, m^2}{(b-a)^2 \max\limits_{t\in[a,b]} e^{-G(t)}} \min\left\{ \frac{2\,{\theta_1}^2}{F(\theta_1)}, \frac{{\theta_2}^2}{F(\theta_2)} \right\} \right[.$$

As a straightforward result of Theorem 4.4 and Remark 4.5, we have the following theorem.

**Theorem 4.6.** *Let $f : \mathbb{R} \to [0, +\infty)$ be a continuous and nonzero function. Put $F(x) := \int_0^x f(\xi)d\xi$ for each $x \in \mathbb{R}$. Assume that*

$$\lim_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = \lim_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0.$$

*Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ the problem*

$$\begin{cases} T(u) = \lambda\, f(u(t)), & t \in (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{4.8}$$

*admits at least three non-negative solutions.*

*Proof.* Put

$$\lambda^* = \inf \left\{ \frac{12\, k_2 M^2 \eta^2}{(b-a)^2 \min\limits_{t\in[a,b]} e^{-G(t)}\, F(\eta)} \ , \eta > 0 \ , F(\eta) > 0 \right\},$$

and fix $\lambda > \lambda^*$. It is clear that $\lambda^* < +\infty$. Let $\eta > 0$ such that $F(\eta) > 0$ and

$$\lambda > \frac{12\, k_2 M^2 \eta^2}{(b-a)^2 \min\limits_{t\in[a,b]} e^{-G(t)}\, F(\eta)}.$$

Since $\lim\limits_{\xi\to 0^+} \dfrac{F(\xi)}{\xi^2} = 0$, there is $\theta_1 > 0$ such that $\theta_1 < \eta$ and $\dfrac{F(\theta_1)}{{\theta_1}^2} < \dfrac{k_1\, m^2\, \Theta\, F(\eta)}{6\, k_2\, M^2\, \eta^2}$. Again from the condition $\lim\limits_{\xi\to+\infty} \dfrac{F(\xi)}{\xi^2} = 0$ there is $\theta_2 > 0$ such that $\eta < \frac{m}{2\,M}\sqrt{\frac{k_1}{k_2}}\ \theta_2$ and $\dfrac{F(\theta_2)}{{\theta_2}^2} < \dfrac{k_1\, m^2\, \Theta\, F(\eta)}{12\, k_2\, M^2\, \eta^2}$.

Therefore, by applying Theorem 4.4 and also Remark 4.5, for every $\lambda \in (\lambda^*, +\infty)$ the problem (4.8) admits at least three non-negative solutions. □

The following example is suitable for an intuitive explanation of Theorem 4.6.

**Example 4.7.** Put $f(x) = x^6\, e^{-\,x^7}$. Hence, we have

$$F(x) = \int_0^x f(\xi) d\xi = \frac{1}{7}\left(1 - e^{-\,x^7}\right),$$

and we observe that

$$\lim_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = \lim_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0.$$

Now, consider the problem

$$\begin{cases} J(u) = \lambda\, u^6\, e^{-\,u^7}, & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{4.9}$$

where

$$J(u) = \left(2 + \cos\left(\int_0^1 e^{-\arctan t}\left(|u'(t)|^2 + \sqrt{t^2+3}\,|u(t)|^2\right) dt\right)\right) \times \left(-u'' + \frac{1}{t^2+1}\, u' + \sqrt{t^2+3}\, u\right).$$

From the comparison of problem (4.8) with problem (4.9), we find that $a = 0$, $b = 1$, $K(t) = 2 + \cos t$, $\forall t \geq 0$, $g(t) = \frac{1}{t^2+1}$ and

$h(t) = \sqrt{t^2+3}$ , $\forall\, t \in [0,1]$. According to the function $K$, we can consider $k_2 = 3$ and also we have $\min\limits_{t\in[a,b]} e^{-G(t)} = \min\limits_{t\in[0,1]} e^{-\arctan t} = e^{-\frac{\pi}{4}}$. Since $\operatorname{ess\,sup}\limits_{t\in[0,1]} h(t) \geq 0$, from Proposition 2.2 one has

$$M = \left[\max_{t\in[a,b]} e^{-G(t)} \left(1 + \operatorname*{ess\,sup}_{t\in[a,b]} h(t)\left(\frac{b-a}{\pi}\right)^2\right)\right]^{\frac{1}{2}}$$
$$= \left(1 + \frac{2}{\pi^2}\right)^{\frac{1}{2}}.$$

So by applying Theorem 4.6, for every

$$\lambda > \frac{252\left(1+\frac{2}{\pi^2}\right)}{e^{-\frac{\pi}{4}}} \left(\inf_{\eta>0} \frac{\eta}{1-e^{-\eta^7}}\right),$$

the problem (4.9) has at least three non-negative solutions.

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