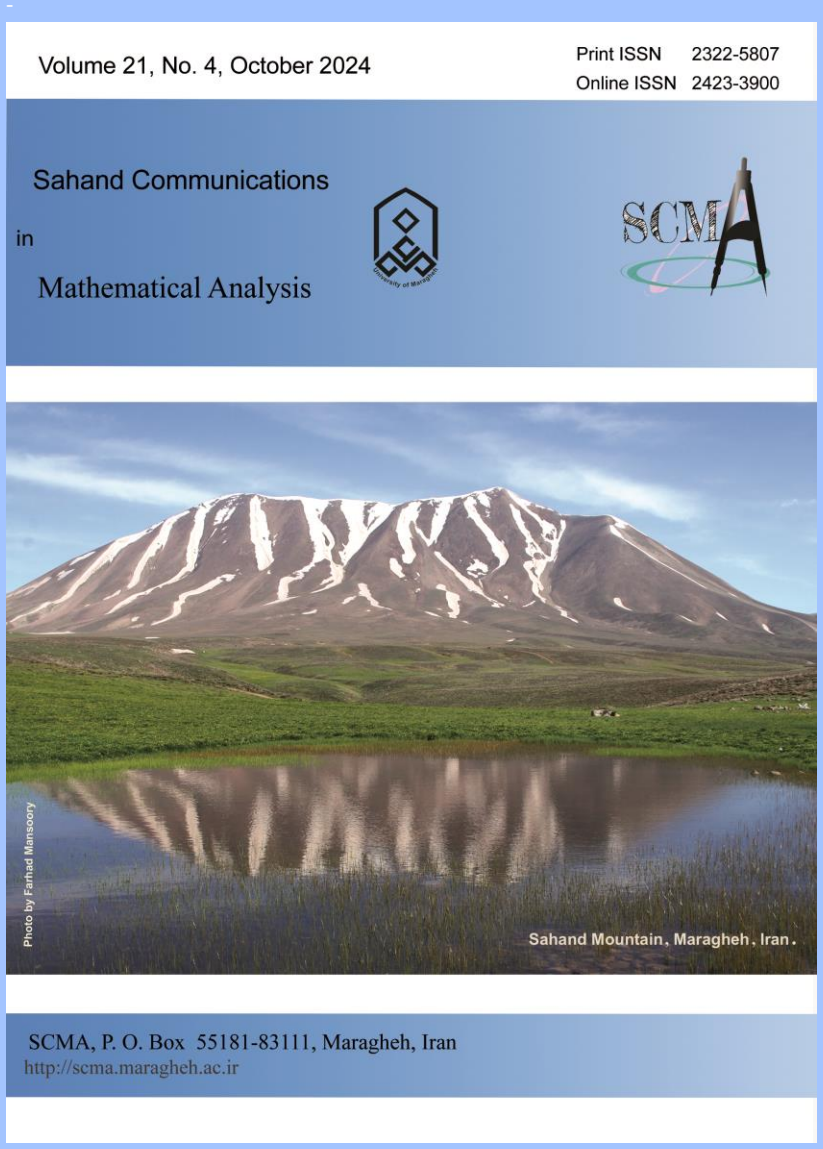
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Some Representations of the Modified Degenerate Gamma Matrix Function

İnci Ege[1*] and Kwara Nantomah[2]

Abstract. The main goal of this article is to extend the concept of special matrix functions through the study of modified degenerate Gamma matrix function. For this, the modified degenerate Gamma matrix function has been introduced and some of its properties have been established. The article also presents limit representations, asymptotic equality, and infinite product representations. The new function of matrix argument is an analogue of the Gamma matrix function, and the properties are analogues of those satisfied by the Gamma matrix function. Upon taking the limit as $\lambda \to 0^+$, the established results reduce to the results of the Gamma matrix function.

1. Introduction

Matrix functions have a major role in science and engineering. Computing a function f on a matrix of order n is an occurring problem in many application areas, [9]. Special matrix functions have been studied in the literature related to theoretical physics, spherical functions on certain symmetric spaces, and multivariate analysis in statistics, number theory, and group representation theory, [4, 5, 23, 24].

Recently, much research has been done on the matrix generalization of special functions, [1, 2, 8, 16, 21]. Since the Gamma function Γ is one of the most important special functions and has large applications in many fields such as statistics, analytic number theory, and physics, one of the fundamental matrix functions studied on it is the Gamma matrix function. On the other hand, in recent years, some extensions of

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the classical Gamma function have been discussed by many researchers, [3, 17–20].

The classical Gamma function has several representations, [22]. It's integral form is one of the most used ones. Also, it can be expressed by an infinite product due to Weierstrass. Such a representation plays an important role in dealing with other special functions in Number Theory, [6, 14]. For example, the asymptotic behavior of Laguerre matrix polynomials has been proved by infinite product expression. Also, the degenerate Gamma function is used as a mathematical tool in engineering to solve ODEs and related initial value problems. These facts motivate us to define the modified degenerate Gamma matrix function and its infinite product representations.

The study is orginized as follows. In section 2, we give the basic definitions and results that are needed in this paper. In section 3, we introduce the modified degenerate Gamma function of matrix argument. We investigate certain properties and limit representations of the modified degenerate Gamma matrix function. We present a matrix extension of the classical Gamma function due to Weierstrass through an infinite product. Also in Section 3, we give an asymptotic equality and a duplication formula for the mentioned function.

2. Preliminaries

In this section, we summarize basic concepts, notations, and definitions which we need in this work. We recall some properties of the Riesz-Dunford functional calculus that may be found in [7, 12, 13]. Throughout this paper, we shall denote $\mathbb{Z}^+$ and $\mathbb{C}$ are the sets of positive integers and complex numbers respectively. $\mathbb{C}^{n\times n}$ is a vector space of all $n \times n$ square matrices with complex number entries z, where the real and imaginary parts of z are $Re(z)$ and $Im(z)$, respectively.
In [17], Kim et al. introduced a degenerate Gamma function, namely the modified degenerate Gamma function, denoted Γ_λ^*, defined as

$$\Gamma_\lambda^*(z) = \int_0^\infty t^{z-1}(1+\lambda)^{-\frac{t}{\lambda}} dt, \quad \lambda \in (0,1), Re(z) > 0. \tag{2.1}$$

Note that $\lim_{\lambda\to 0^+} \Gamma_\lambda^*(z) = \Gamma(z)$, where Γ is the Euler's classical Gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\, dt, \quad Re(z) > 0.$$

The modified degenerate Gamma function satisfies the following properties, [17]:

$$\Gamma_\lambda^*(z+1) = \frac{\lambda}{\ln(1+\lambda)} z\Gamma_\lambda^*(z),$$
$$\Gamma_\lambda^*(k+1) = \frac{\lambda^{k+1}\, k!}{\ln^{k+1}(1+\lambda)}, \quad k \in \mathbb{N}.$$

For a matrix $A \in \mathbb{C}^{n\times n}$ its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A for which we denote the real numbers $\alpha(A)$ and $\beta(A)$ as

$$\alpha(A) = \max\{Re(z) : z \in \sigma(A)\}, \qquad \beta(A) = \min\{Re(z) : z \in \sigma(A)\}$$

where $\alpha(A)$ refers to the spectral abscissa of A. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be positive stable if $\beta(A) > 0$. The two-norm of A, denoted by $\|A\|$, is defined by

$$\begin{aligned}\|A\| &= \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \\ &= \max\left\{\sqrt{\lambda} : \lambda \in \sigma(A^*A)\right\},\end{aligned}$$

where for any $x \in \mathbb{C}^n$, $\|x\| = (x^*x)^{1/2}$ is the Euclidean norm of x and A^* denotes the transposed conjugate of A.

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, defined in an open set ω of the complex plane, and $A \in \mathbb{C}^{n\times n}$ such that $\sigma(A) \subset \omega$, then from the properties of the matrix functional calculus [7], it follows that

$$f(A)g(A) = g(A)f(A).$$

Furthermore, if $B \in \mathbb{C}^{n\times n}$ is a matrix for which $\sigma(B) \in \omega$, and if $AB = BA$, then

$$f(A)g(B) = g(B)f(A).$$

Let $A \in \mathbb{C}^{n\times n}$ is a positive stable matrix. Then, the Gamma matrix function $\Gamma(A)$ is defined as follows:

$$\Gamma(A) = \int_0^\infty t^{A-I}e^{-t}dt, \quad t^{A-I} = \exp((A-I)\ln t) \tag{2.2}$$

where I is the identitiy matrix of order n, [12–15]. The general theory of matrix functions, as seen in [10], ensures that the Gamma matrix function $\Gamma(A)$ is well defined for a given positive stable matrix A in $\mathbb{C}^{n\times n}$.

The reciprocal Gamma function denoted by $\Gamma^{-1}(z) = 1/\Gamma(z)$ is an entire function of the complex variable z. Thus the Riesz-Dunford functional calculus, [7], reveals that the image of $\Gamma^{-1}(z)$ acting on A denoted by $\Gamma^{-1}(A)$ is well-defined, and is the inverse of $\Gamma(A)$, that is

$\Gamma^{-1}(A) = (\Gamma(A))^{-1} = \dfrac{1}{\Gamma(A)}$. Moreover, if $A + nI$ is invertible for all integers $n \geq 0$, then $\Gamma(A)$ is invertible, its inverse concides with $\Gamma^{-1}(A)$, and

$$A(A+I)\cdots(A+(n-1)I)\Gamma^{-1}(A+nI) = \Gamma^{-1}(A), \quad n \geq 1,$$

[11]. Let $(z)_n$ be the Pochhammer symbol or scalar factorial function denoted by

$$(z)_n = z(z+1)\cdots(z+(n-1)), \quad n \geq 1, (z)_0 = 1. \tag{2.3}$$

Then by application of the matrix calculus to $(z)_n$, for any matrix $A \in \mathbb{C}^{n\times n}$ one gets matrix factorial function as

$$(A)_n = A(A+I)\cdots(A+(n-1)I), \quad n \geq 1, (A)_0 = 1. \tag{2.4}$$

If $A + nI$ is inverible for all integer $n \geq 0$, then

$$(A)_n = \Gamma(A+nI)\Gamma^{-1}(A).$$

The Gamma matrix function satisfies the limit expression

$$\Gamma(A) = \lim_{n\to\infty}(n-1)!(A)_n^{-1}n^A,$$

[12]. Also, an asymptotic behaviour of the Gamma matrix function is given in [6] as

$$\Gamma^{-1}(A) = Ae^{\gamma A}\prod_{n=1}^{\infty}\left[\left(I+\frac{1}{n}A\right)e^{-(1/n)A}\right],$$

where γ is the Euler-Mascheroni constant given by

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) \\ &\approx 0.5772.\end{aligned} \tag{2.5}$$

An alternative representation of Gamma matrix function in terms of infinite matrix product is given in [6] as

$$\Gamma(A) = A^{-1}\prod_{n=1}^{\infty}\left[\left(I+\frac{1}{n}A\right)^{-1}\left(1+\frac{1}{n}\right)^{A}\right].$$

If $f(A)$ is well defined and $S \in \mathbb{C}^{n\times n}$ is invertible, then, [9],

$$f(SAS^{-1}) = Sf(A)S^{-1}.$$

The exponential of a matrix $A \in \mathbb{C}^{n\times n}$ is defined in [9] as

$$e^A = \sum_{n=0}^{\infty}\frac{A^n}{n!}.$$

Using the Shur decomposition of $A \in \mathbb{C}^{n\times n}$, it follows that

$$\left\|e^{zA}\right\| = e^{z\alpha(A)} \left\{ \sum_{i=0}^{l-1} \frac{\left(\|A\| \sqrt{l} z\right)^i}{i!} \right\}, \quad (z \geq 0) \tag{2.6}$$

where $\alpha(A)$ is the spectral abscissa of A, [9].

Let A_n and B_n be sequences of matrix. Then, A_n is asymptotically equal to B_n (denoted $A_n \approx B_n$) if $\lim_{n\to\infty} A_n B_n^{-1} = I$, [14].

3. Main Results

In this section, firstly, we define and investigate certain properties of the modified degenerate Gamma matrix function.

Definition 3.1. Let A be a positive stable matrix in $\mathbb{C}^{n\times n}$ and $\lambda \in (0, 1)$. Then the modified degenerate Gamma matrix function can be defined as follows:

$$\Gamma_\lambda^*(A) = \int_0^\infty t^{A-I}(1+\lambda)^{-\frac{t}{\lambda}}\, dt, \tag{3.1}$$

where $t^{A-I} = \exp((A-I)\ln t)$ and $I \in \mathbb{C}^{n\times n}$ is the identity matrix.

Note that, the equation (3.1) is a matrix argument of the modified degenerate Gamma function given in the equation (2.1), and $\lim_{\lambda\to 0^+} \Gamma_\lambda^*(A) = \Gamma(A)$. Also, since

$$\Gamma_\lambda^*(I) = \int_0^\infty t^{O}(1+\lambda)^{-\frac{t}{\lambda}} dt,$$

where $O \in \mathbb{C}^{n\times n}$ is the zero matrix function and $t^O = exp(O \ln t) = I$, we have

$$\Gamma_\lambda^*(I) = \frac{\lambda}{\ln(1+\lambda)} I.$$

Let $\mathcal{M}$ be the Mellin transform operator of $f(t)$ defined by

$$\mathcal{M}\{f(t); A\} = \int_0^\infty f(t) t^{A-I} dt.$$

Then the modified degenerate Gamma matrix function (MDGMF) is the Mellin transform of $f(t) = (1+\lambda)^{-\frac{t}{\lambda}}$ in A, that is $\Gamma_\lambda^*(A) = \mathcal{M}\left\{ (1+\lambda)^{-\frac{t}{\lambda}}; A \right\}$.

Now we use this fact to give the following recurrence relation.

Proposition 3.2. *Let A be a positive stable matrix in $\mathbb{C}^{n\times n}$. Then*

$$\Gamma_\lambda^*(A+I) = \frac{\lambda}{\ln(1+\lambda)} A\Gamma_\lambda^*(A) \tag{3.2}$$

for $\lambda \in (0,1)$.

Proof. Since

$$\mathcal{M}\{f^{'}(t); A+I\} = -A\mathcal{M}\{f(t); A\},$$

we get for $f(t) = (1+\lambda)^{-\frac{t}{\lambda}}$ that

$$-\frac{1}{\lambda}\int_0^\infty t^A \ln(1+\lambda)(1+\lambda)^{-\frac{t}{\lambda}}dt = -A\int_0^\infty t^{A-I}(1+\lambda)^{-\frac{t}{\lambda}}dt.$$

Hence by using the definition of MDGMF we get

$$-\frac{1}{\lambda}\ln(1+\lambda)\Gamma_\lambda^*(A+I) = -A\Gamma_\lambda^*(A), \tag{3.3}$$

and the result follows. □

As an application of the Proposition 3.2, we get the following corollary.

Corollary 3.3. *Let A be a positive stable matrix in $\mathbb{C}^{n\times n}$ and $A+nI$ is invertible for all integer $n \geq 0$. Then,*

$$\Gamma_\lambda^*(A+nI) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^n (A)_n \Gamma_\lambda^*(A) \tag{3.4}$$

for $\lambda \in (0,1)$.

In particular,

$$\Gamma_\lambda^*(nI+I) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{n+1} n!\, I. \tag{3.5}$$

Proof. The proof follows by induction on n. □

Corollary 3.4. *Let A be a positive stable matrix in $\mathbb{C}^{n\times n}$ and $A+nI$ is invertible for all integer $n \geq 0$. Then,*

$$\Gamma_\lambda^*(nI+I) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{n+1} \lim_{t\to\infty}\left\{\left(\frac{t}{t+n}\right)^n \left(\frac{t+n}{t}\right)_n\right\} I.$$

for $\lambda \in (0,1)$.

Proof. Writing $(t+n)/n$ in the equation (2.3) instead of z, we get

$$\begin{aligned}\left(\frac{t}{t+n}\right)^n \left(\frac{t+n}{t}\right)_n &= \left(\frac{t}{t+n}\right)^n \left(1+\frac{n}{t}\right)\left(2+\frac{n}{t}\right)\cdots\left(n+\frac{n}{t}\right)\\ &= \frac{t}{t+n}\,\frac{t+n}{t}\,\frac{t}{t+n}\,\frac{2t+n}{t}\cdots\frac{t}{t+n}\,\frac{nt+n}{t}\end{aligned}$$

$$= \frac{t+n}{t+n}\,\frac{2t+n}{t+n}\cdots\frac{nt+n}{t+n}.$$

Now, taking the limit of both sides of the last equation as $t \to \infty$, we have

$$\lim_{t\to\infty}\left\{\left(\frac{t}{t+n}\right)^n\left(\frac{t+n}{t}\right)_n\right\} = 1.2.\ldots.n = n!. \tag{3.6}$$

Then, by using the equations (3.5) and (3.6), we get the result. □

Lemma 3.5. *The inequalities*
i.

$$\left(1-\left(\frac{\ln(1+\lambda)t}{\lambda n}\right)^2\right)^k \geq 1-k\left(\frac{\ln(1+\lambda)t}{\lambda n}\right)^2, \tag{3.7}$$

ii.

$$\begin{aligned} 0 \leq (1+\lambda)^{-\frac{t}{\lambda}} - \left(1-\frac{\ln(1+\lambda)t}{\lambda n}\right)^n \\ \leq \frac{1}{n}(1+\lambda)^{-\frac{t}{\lambda}}\left(\frac{\ln(1+\lambda)t}{\lambda}\right)^2 \end{aligned} \tag{3.8}$$

are valid for $0 \leq t < \dfrac{\lambda n}{\ln(1+\lambda)}$, $n,k \in \mathbb{Z}^+$ *and* $\lambda \in (0,1)$.

Proof. For proving the inequality (3.7), we use induction method on k. For $k=1$, the inequality (3.7) holds. Now, let the inequality (3.7) holds for some m. Then we have,

$$\left(1-\left(\frac{\ln(1+\lambda)t}{\lambda n}\right)^2\right)^m \geq 1-m\left(\frac{\ln(1+\lambda)t}{\lambda n}\right)^2.$$

Now, multiplying with $1-\left(\dfrac{\ln(1+\lambda)t}{\lambda n}\right)^2$ of each sides of the last inequality, we get

$$\begin{aligned} \left(1-\left(\frac{\ln(1+\lambda)t}{\lambda n}\right)^2\right)^{m+1} &\geq \left(1-m\left(\frac{\ln(1+\lambda)t}{\lambda n}\right)^2\right)\left(1-\left(\frac{\ln(1+\lambda)t}{\lambda n}\right)^2\right) \\ &= 1-(m+1)\left(\frac{\ln(1+\lambda)t}{\lambda n}\right)^2 + m\left(\frac{\ln(1+\lambda)t}{\lambda n}\right)^4. \end{aligned}$$

Then,

$$\left(1-\left(\frac{\ln(1+\lambda)t}{\lambda n}\right)^2\right)^{m+1} \geq 1-(m+1)\left(\frac{\ln(1+\lambda)t}{\lambda n}\right)^2.$$

Hence, the inequality (3.7) follows for all $k \in \mathbb{Z}^+$.

Since

$$(1+\lambda)^{\frac{t}{\lambda n}} = 1 + \frac{\ln(1+\lambda)t}{\lambda n} + \frac{\ln^2(1+\lambda)t^2}{2\lambda^2 n^2} + O(t^3) \tag{3.9}$$

we have

$$1 + \frac{\ln(1+\lambda)t}{\lambda n} \leq (1+\lambda)^{\frac{t}{\lambda n}}. \tag{3.10}$$

On the other hand, since $0 \leq t < \dfrac{\lambda n}{\ln(1+\lambda)}$ we can write

$$\frac{1}{1 - \ln\dfrac{(1+\lambda)t}{\lambda n}} = 1 + \frac{\ln(1+\lambda)t}{\lambda n} + \frac{\ln^2(1+\lambda)t^2}{\lambda^2 n^2} + O(t^3). \tag{3.11}$$

Then, combining the equations (3.9) and (3.11) we get

$$(1+\lambda)^{\frac{t}{\lambda n}} \leq \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^{-1}.$$

It follows that

$$1 + \frac{\ln(1+\lambda)t}{\lambda n} \leq (1+\lambda)^{\frac{t}{\lambda n}} \leq \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^{-1}. \tag{3.12}$$

Then,

$$\left(1 + \frac{\ln(1+\lambda)t}{\lambda n}\right)^{-n} \geq (1+\lambda)^{-\frac{t}{\lambda}} \geq \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^{n}.$$

From the last inequality, we get

$$(1+\lambda)^{-\frac{t}{\lambda}} - \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^{n} \geq 0. \tag{3.13}$$

Also,

$$\begin{aligned}(1+\lambda)^{-\frac{t}{\lambda}} &- \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^{n} \\ &= (1+\lambda)^{-\frac{t}{\lambda}} \left[1 - (1+\lambda)^{\frac{t}{\lambda}} \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^{n}\right],\end{aligned}$$

and since

$$-(1+\lambda)^{\frac{t}{\lambda}} \leq -\left(1 + \frac{\ln(1+\lambda)t}{\lambda n}\right)^{n},$$

by using the equation (3.12) we have

$$\begin{aligned}(1+\lambda)^{-\frac{t}{\lambda}} &- \left(1-\frac{\ln(1+\lambda)t}{\lambda n}\right)^n \\ &\leq (1+\lambda)^{-\frac{t}{\lambda}}\left[1-\left(1+\frac{\ln(1+\lambda)t}{\lambda n}\right)^n\left(1-\frac{\ln(1+\lambda)t}{\lambda n}\right)^n\right] \\ &= (1+\lambda)^{-\frac{t}{\lambda}}\left[1-\left(1-\left(\frac{\ln(1+\lambda)t}{\lambda n}\right)^2\right)^n\right].\end{aligned}$$

Now, by using the equation (3.7) for $k=n$ we get

$$\begin{aligned}(1+\lambda)^{-\frac{t}{\lambda}} &- \left(1-\frac{\ln(1+\lambda)t}{\lambda n}\right)^n \\ &\leq (1+\lambda)^{-\frac{t}{\lambda}}\left[1-\left(1-\frac{1}{n}\left(\frac{\ln(1+\lambda)t}{\lambda}\right)^2\right)\right] \\ &= \frac{1}{n}(1+\lambda)^{-\frac{t}{\lambda}}\left(\frac{\ln(1+\lambda)t}{\lambda}\right)^2.\end{aligned} \tag{3.14}$$

Hence, combining the inequalities (3.13) and (3.14) the inequality (3.8) follows. □

Now, we give the following limit representation of the MDGMF.

Theorem 3.6. *Let A be a positive stable matrix in $\mathbb{C}^{n\times n}$ and $A+nI$ is invertible for all integer $n\geq 0$. Then,*

$$\Gamma_\lambda^*(A) = \lim_{n\to\infty} n!\left(\frac{\lambda n}{\ln(1+\lambda)}\right)^A (A)_{n+1}^{-1} \tag{3.15}$$

for $\lambda\in(0,1)$.

Proof. Let

$$f(z) = \int_0^{\lambda n/\ln(1+\lambda)}\left(1-\frac{\ln(1+\lambda)s}{\lambda n}\right)^n s^{z-1}ds.$$

Then making the substitution $\dfrac{\ln(1+\lambda)s}{\lambda n} = t$ we get

$$g(t) = \left(\frac{\lambda n}{\ln(1+\lambda)}\right)^z \int_0^1 (1-t)^n t^{z-1}\,dt = \left(\frac{\lambda n}{\ln(1+\lambda)}\right)^z n!\,(z)_{n+1}^{-1}.$$

It follows that

$$f(z) = \left(\frac{\lambda n}{\ln(1+\lambda)}\right)^z n!\,(z)_{n+1}^{-1}. \tag{3.16}$$

Now, by application of the matrix functional calculus in (3.16) we get

$$f(A) = \int_0^{\lambda n/\ln(1+\lambda)} \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^n t^{A-1}\, dt \tag{3.17}$$
$$= \left(\frac{\lambda n}{\ln(1+\lambda)}\right)^A n!\ (A)_{n+1}^{-1}.$$

Then, by the definition of MDGMF and the equation (3.17) we get

(3.18)

$$\begin{aligned}
&\Gamma_\lambda^*(A) - \left(\frac{\lambda n}{\ln(1+\lambda)}\right)^A n!\ (A)_{n+1}^{-1} \\
&\quad = \int_0^\infty t^{A-I}(1+\lambda)^{-\frac{t}{\lambda}}\, dt - \int_0^{\lambda n/\ln(1+\lambda)} \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^n t^{A-1}\, dt \\
&\quad = \int_0^{\lambda n/\ln(1+\lambda)} t^{A-I}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^n\right] dt \\
&\qquad + \int_{\lambda n/\ln(1+\lambda)}^\infty t^{A-I}(1+\lambda)^{-\frac{t}{\lambda}}\, dt.
\end{aligned}$$

We want to show that

$$\lim_{n\to\infty}\left\{\Gamma_\lambda^*(A) - \left(\frac{\lambda n}{\ln(1+\lambda)}\right)^A n!\ (A)_{n+1}^{-1}\right\} = 0$$

or equivalent to that

(3.19)

$$\begin{aligned}
&\lim_{n\to\infty}\left\{\int_0^{\lambda n/\ln(1+\lambda)} t^{A-I}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^n\right] dt\right\} \\
&\qquad + \lim_{n\to\infty}\left\{\int_{\lambda n/\ln(1+\lambda)}^\infty t^{A-I}(1+\lambda)^{-\frac{t}{\lambda}}\, dt\right\} = 0.
\end{aligned}$$

For this, firstly we prove that

(3.20)

$$\lim_{n\to\infty}\left\{\int_0^{\lambda n/\ln(1+\lambda)} t^{A-I}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^n\right] dt\right\} = 0.$$

By the inequality (3.8) we have,

$$\left\| \int_0^{\lambda n/\ln(1+\lambda)} t^{A-I} \left[(1+\lambda)^{-\frac{t}{\lambda}} - \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^n \right] dt \right\| \tag{3.21}$$
$$\leq \frac{1}{n} \frac{\ln^2(1+\lambda)}{\lambda^2} \int_0^{\lambda n/\ln(1+\lambda)} \left\| t^{A+I} \right\| (1+\lambda)^{-\frac{t}{\lambda}} dt.$$

Now, writing $t^{A-I} = e^{(A-I)\ln t}$, using the equation (2.6) and $\ln t \leq t$ for $t > 0$ we can write

$$\left\| t^{A+I} \right\| \leq t^{\alpha(A)+1} \left[\sum_{i=0}^{l-1} \frac{\left[(\|A\| + 1)\sqrt{l}\, t \right]^i}{i!} \right]. \tag{3.22}$$

Then by the inequalities (3.21) and (3.22) we have

$$\frac{1}{n} \frac{\ln^2(1+\lambda)}{\lambda^2} \int_0^{\lambda n/\ln(1+\lambda)} \left\| t^{A+I} \right\| (1+\lambda)^{-\frac{t}{\lambda}} dt$$
$$\leq \frac{1}{n} \frac{\ln^2(1+\lambda)}{\lambda^2} \left\{ \sum_{i=0}^{l-1} \frac{\left[(\|A\| + 1)\sqrt{l}\, t \right]^i}{i!} \int_0^{\lambda n/\ln(1+\lambda)} t^{\alpha(A)+1+i} (1+\lambda)^{-\frac{t}{\lambda}} dt \right\}.$$

Then we get,

$$0 \leq \left\| \int_0^{\lambda n/\ln(1+\lambda)} t^{A-I} \left[(1+\lambda)^{-\frac{t}{\lambda}} - \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^n \right] dt \right\|$$
$$\leq \frac{1}{n} \frac{\ln^2(1+\lambda)}{\lambda^2} \left\{ \sum_{i=0}^{l-1} \frac{\left[(\|A\| + 1)\sqrt{l}\, t \right]^i}{i!} \int_0^{\lambda n/\ln(1+\lambda)} t^{\alpha(A)+1+i} (1+\lambda)^{-\frac{t}{\lambda}} dt \right\}.$$

Since, $A \in \mathbb{C}^{n\times n}$ is a positive stable matrix and $0 \leq i \leq l-1$, then $\alpha(A) + 2 + i > 0$. It follows by the definition of the function Γ_λ^* that

$$\Gamma_\lambda^*(\alpha(A) + 2 + i) = \int_0^\infty t^{\alpha(A)+1+i} (1+\lambda)^{-\frac{t}{\lambda}} dt$$
$$< \infty$$

Then, we have

$$\lim_{n\to\infty} \left\| \int_0^{\lambda n/\ln(1+\lambda)} t^{A-I} \left[(1+\lambda)^{-\frac{t}{\lambda}} - \left(1 - \frac{\ln(1+\lambda)t}{\lambda n}\right)^n \right] dt \right\| = 0. \tag{3.23}$$

Hence, we get the equality (3.20). Also, we have

$$\lim_{n\to\infty}\int_{\lambda n/\ln(1+\lambda)}^{\infty} t^{A-I}(1+\lambda)^{-\frac{t}{\lambda}}\,dt=0,$$

since

$$\int_{0}^{\infty} t^{A-I}(1+\lambda)^{-\frac{t}{\lambda}}\,dt<0.$$

Thus the equality (3.19) holds, that is

$$\lim_{n\to\infty}\left\{\Gamma_{\lambda}^{*}(A)-\left(\frac{\lambda n}{\ln(1+\lambda)}\right)^{A} n!\,(A)_{n+1}^{-1}\right\}=0,$$

and the result follows. □

Remark 3.7. Since the reciprocal of the modified degenerate Gamma function denoted by $(\Gamma_{\lambda}^{*})^{-1}(z)=1/\Gamma_{\lambda}^{*}(z)$ is an entire function of z, then $(\Gamma_{\lambda}^{*})^{-1}(A)$ is a well-defined matrix. Furthermore, the previous theorem shows that $(\Gamma_{\lambda}^{*})^{-1}(A)$ is defined for all matrices satisfy conditions of the Theorem 3.6.

Now, we give asymptotic equal sequences of matrix including MDGMF with the following corollary.

Corollary 3.8. *Let A be a positive stable matrix in $\mathbb{C}^{n\times n}$ and $A+nI$ is invertible for all integer $n\geq 0$. Then,*

$$\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{A+n+1}\frac{\Gamma_{\lambda}^{*}(A+(n+1)I)}{\Gamma(n+1)\,I}\approx n^{A} \tag{3.24}$$

for $\lambda\in(0,1)$.

Proof. By using the equations (3.15), (2.4) and (3.4) respectively, we have

$$\begin{aligned}\Gamma_{\lambda}^{*}(A)&=\lim_{n\to\infty} n!\left(\frac{\lambda n}{\ln(1+\lambda)}\right)^{A}(A)_{n+1}^{-1}\\&=\lim_{n\to\infty} n!\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{A} n^{A}\left[\Gamma_{\lambda}^{*}(A+(n+1)I)\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n+1}(\Gamma_{\lambda}^{*})^{-1}(A)\right]^{-1}\\&=\lim_{n\to\infty} n!\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{A} n^{A}\,\Gamma_{\lambda}^{*}(A)\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{n+1}(\Gamma_{\lambda}^{*})^{-1}(A+(n+1)I).\end{aligned}$$

Then, using the well-known equality $\Gamma(n+1)=n!$ we get

$$I=\lim_{n\to\infty}\Gamma(n+1)\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{A+n+1} n^{A}\,(\Gamma_{\lambda}^{*})^{-1}(A+(n+1)I)$$

$$= \lim_{n\to\infty} \Gamma(n+1)\, I \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{A+n+1} n^A \left(\Gamma_\lambda^*\right)^{-1} (A+(n+1)I)$$

or equivalently

$$I = \lim_{n\to\infty} \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{A+n+1} \frac{\Gamma_\lambda^*(A+(n+1)I)}{\Gamma(n+1)\, I}\, n^{-A},$$

and by the concept of asymptotic equality, the result follows. □

Now, we give Euler's infinite product representation for the MDGMF.

Theorem 3.9. *Let A be a positive stable matrix in $\mathbb{C}^{n\times n}$ and $A+nI$ is invertible for all integer $n \geq 0$. Then,*

$$\Gamma_\lambda^*(A) = A^{-1} \left(\frac{\lambda}{\ln(1+\lambda)}\right)^A \prod_{k=1}^{\infty} k\,(A+kI)^{-1} \left(1+\frac{1}{k}\right)^A \tag{3.25}$$

for $\lambda \in (0,1)$.

Proof. Let us define

$$C_n = \left(\frac{\lambda n}{\ln(1+\lambda)}\right)^A n!\, [A(A+I)\cdots(A+nI)A]^{-1}.$$

Then,

(3.26)

$$\begin{aligned}
C_n &= n! \left(\frac{\lambda}{\ln(1+\lambda)}\right)^A n^A \left[(A+nI)\ldots(A+I)A\right]^{-1} \\
&= n! \left(\frac{\lambda}{\ln(1+\lambda)}\right)^A n^A \frac{(n+1)^A}{(n+1)^A} \left[A^{-1}(A+I)^{-1}\ldots(A+nI)^{-1}\right] \\
&= A^{-1} \left(\frac{\lambda}{\ln(1+\lambda)}\right)^A \left(\frac{n}{n+1}\right)^A \times \\
&\quad \times \left\{ \left((A+I)^{-1}2^A\right) \left(2(A+2I)^{-1}\left(\frac{3}{2}\right)^A\right) \ldots \left(n(A+nI)^{-1}\left(\frac{n+1}{n}\right)^A\right)\right\} \\
&= A^{-1} \left(\frac{\lambda}{\ln(1+\lambda)}\right)^A \left(\frac{n}{n+1}\right)^A \prod_{k=1}^{n} k\,(A+kI)^{-1} \left(\frac{k+1}{k}\right)^A.
\end{aligned}$$

Now, observing

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^A &= e^{A\ln\left(\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\right)} \\
&= e^O \\
&= I,
\end{aligned}$$

and taking the limit as $n \to \infty$ of both sides of the equation (3.26) and using the Theorem 3.6, we get

$$\begin{aligned}\Gamma_\lambda^*(A) &= \lim_{n\to\infty} C_n \\ &= A^{-1}\left(\frac{\lambda}{\ln(1+\lambda)}\right)^A \prod_{k=1}^{\infty} k\,(A+kI)^{-1}\left(\frac{k+1}{k}\right)^A,\end{aligned}$$

and the result follows. □

In the following, we give an infinite product expansion for the MDGMF due to Weierstrass.

Theorem 3.10. *Let A be a positive stable matrix in $\mathbb{C}^{n\times n}$ and $A+nI$ is invertible for all integer $n \geq 0$. Then,*

$$(\Gamma_\lambda^*)^{-1}(A) = A\left(\frac{\ln(1+\lambda)}{\lambda}\right)^A e^{\gamma A} \prod_{k=1}^{\infty}\left(\frac{A}{k}+I\right)^{-1} e^{-A/k} \tag{3.27}$$

for $\lambda \in (0,1)$, where γ is the Euler-Mascheroni constant given by (2.5).

Proof. Since

$$0 < \left(\frac{A}{k}+I\right)^{-1} e^{-A/k} < 1,$$

the infinite product in the equation (3.27) is convergent. Also, using the definition of MDGMF as limit expression in the Theorem 3.6 we get,

$$\begin{aligned}(\Gamma_\lambda^*)^{-1}(A) &= \left(\frac{\ln(1+\lambda)}{\lambda}\right)^A \lim_{n\to\infty} \frac{A(A+I)\cdots(A+nI)}{n!\,n^A} \\ &= A\left(\frac{\ln(1+\lambda)}{\lambda}\right)^A \lim_{n\to\infty} (A+I)(\frac{A}{2}+I)\cdots(\frac{A}{n}+I)e^{-A\ln n}.\end{aligned}$$

Then, by using

$$e^{A\ln n} = e^{A\left(\ln n - \left(1+\frac{1}{2}+\cdots\frac{1}{n}\right)\right)} e^{A\left(1+\frac{1}{2}+\cdots\frac{1}{n}\right)},$$

we get the result. □

4. Conclusion

Since the description of the Gamma function, many researchers have worked on new extensions of the Gamma function. One of the useful extensions of the Gamma function is the degenerate Gamma function. It is used to solve ODEs and related initial value problems. Also, applications of special matrix functions have appeared in the literature such as the Gamma matrix function. Motivated by these works, we introduce the modified degenerate Gamma function of matrix argument.

We investigate certain properties and limit representations of the modified degenerate Gamma matrix function. We present a matrix extension of the classical Gamma function due to Weierstrass through an infinite product. Also, we give an asymptotic equality and a duplication formula for the mentioned function.

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