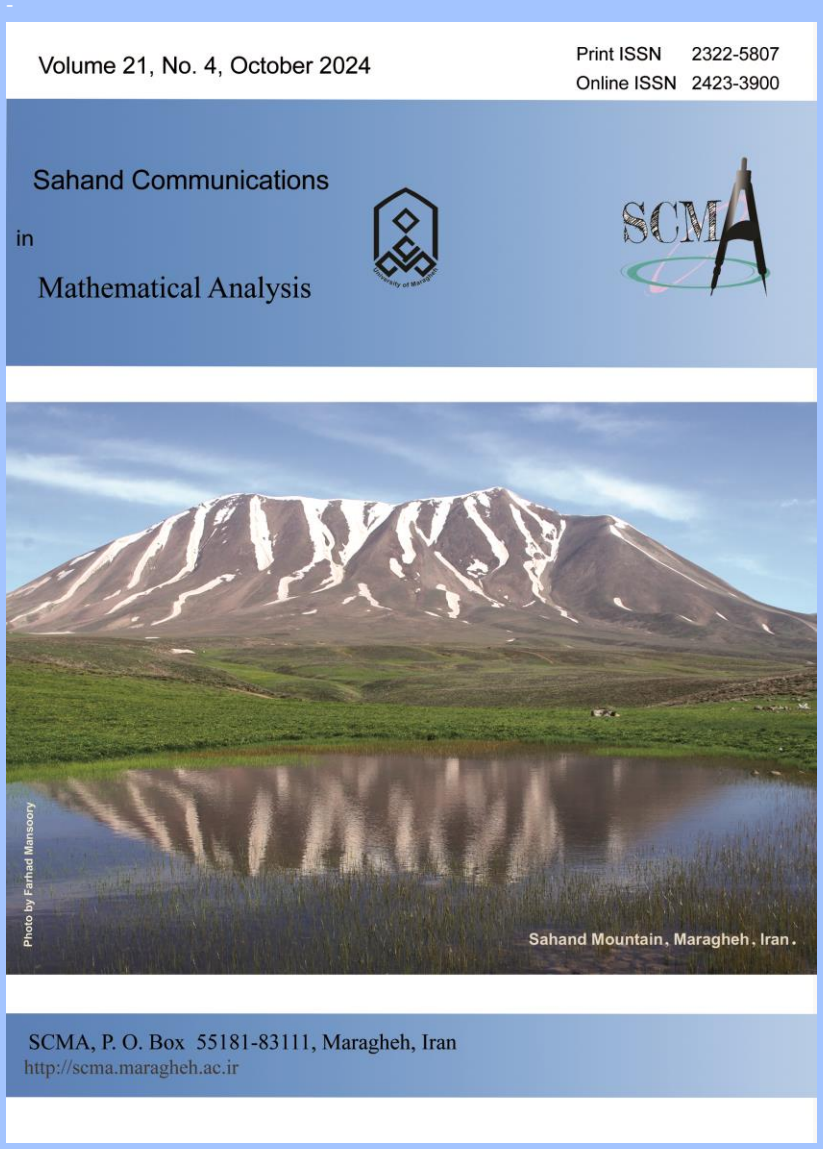
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K-b-Frames for Hilbert Spaces and the b-Adjoint Operator

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Abstract. This paper will generalize b-frames, a new concept of frames for Hilbert spaces, by K-b-frames. The idea is to take a sequence from a Banach space and see how it can be a frame for a Hilbert space. Instead of the scalar product, we will use a new product called the b-dual product, which is constructed via a bilinear mapping. We will introduce new results about this product, about b-frames and K-b-frames and we will also give some examples of both b-frames and K-b-frames that have never been given before. We will express the reconstruction formula of the elements of the Hilbert space. We will also study the stability and preservation of both b-frames and K-b-frames and to do so, we will give the equivalent of the adjoint operator according to the b-dual product.

1. Introduction

The history of Frames Theory began in 1952, when Duffin and Schaeffer introduced it to solve some nonharmonic Fourier series problems [7]. Frames were a key that solved several problems in many fields, such as physics, natural sciences and engineering. It is applied in signal and image processing and treatment, coding, system modeling and many other interesting fields (see [1, 4, 5])

It's worth nothing that D.Gabor introduced "Les Gaborettes" in 1946 which are a particular case of frames, but they were introduced for a physics purpose.

In 1986, I. Daubechies, A. Grossmann and Y. Meyer, revolutionized the Theory of Frames by giving redundancy properties. The interest

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was to find a reconstruction formula using frames to feed the needs of many problems in numerous fields.

In 2012, L. Gavruta introduced a more general case of frames, which is $K-$Frames and proved its existence in [8] and was able to extend many existing results to it and gave some new interesting results in [12]. For more results about K-frames see ([10, 11]).

In order to generalize frames for Hilbert spaces, M.Ismailov, F.Guliyeva and Y.Nasibov came up with the concept of b-frames. It is a concept where we can generate Hilbert frames by a bilinear mapping, $b : \mathcal{H}_1 \times \mathcal{B} \to \mathcal{H}_2$, such that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces and $\mathcal{B}$ is a Banach space. Via this bilinear mapping we construct a new product that we will call the $b-$dual product to define a new frame for a Hilbert space, but this time not from the Hilbert space itself; it is from the Banach space $\mathcal{B}$.

Moreover, M.Ismailov, F.Guliyeva and Y.Nasibov extended some well-known and important results of frame theory existing in the classical case to this more general one (see [9]). Except the definition needed to be completed and no example was given. Also, stability and preservation should have been studied.

In this paper, we will give a detailed definition of b-frames and generalize this definition by defining K-b-frames and giving some examples of both of them. In addition, we will introduce some new results about both b-frames and K-b-frames; we will explain in detail the $b-$dual product used to define this new concept of frames. We will give the reconstruction formula for both b-frames and K-b-frames. We will introduce the b-adjoint operator, manipulate the b-dual product to study the stability and preservation of both b-frames and K-b-frames.

2. Notations and Preliminaries

(i) By $\mathcal{C}(I)$, we denote the set of continuous functions mapping from an interval $I \subset \mathbb{C}$ to $\mathbb{C}$.
(ii) By $\mathcal{C}_n(I)$, we denote the set of piecewise-continuous functions from $I \subset \mathbb{C}$ to $\mathbb{C}$.
(iii) $\mathbb{1}_A$ denotes the characteristic function of the set A.
(iv) If $\mathcal{H}$ and $\mathcal{Z}$ are two normed spaces, then $\mathcal{L}(\mathcal{H}, \mathcal{Z})$ denotes the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{Z}$.
(v) If $T \in \mathcal{L}(\mathcal{H}, \mathcal{Z})$, then $T^* \in \mathcal{L}(\mathcal{Z}, \mathcal{H})$ is the adjoint operator of T and it verifies $\langle Tx, y\rangle_{\mathcal{Z}} = \langle x, T^*y\rangle_{\mathcal{H}}$, for all $x \in \mathcal{H}$ and $y \in \mathcal{Z}$.
(vi) $Id_{\mathcal{Z}}$ denotes the identity operator of $\mathcal{Z}$.
(vii) $\mathrm{ran}(T)$ denotes the range of T.
(viii) $\ker(T)$ denotes the kernel of T.

(ix) Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$, such that $\mathrm{ran}(T)$ is closed. $T^{\dagger}$ will denote the pseudo-inverse (or the moore-penrose inverse) of T, verifying $TT^{\dagger}h = h,\ \forall h \in \mathrm{ran}(T)$.

Theorem 2.1 ([6]). *Let $\mathcal{H}, \mathcal{H}_1$, and $\mathcal{H}_2$ be Hilbert spaces and let $S \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$. The following statements are equivalent:*

i) $\mathrm{ran}(S) \subset \mathrm{ran}(T)$;
ii) $SS^* \leq \lambda^2 TT^*$, *for some* $\lambda > 0$;
iii) *there exists* $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ *such that* $S = TU$.

Theorem 2.2 ([9]). *Let $T \in \mathcal{L}(\mathcal{H}, \mathcal{Z})$, where $\mathcal{H}$ and $\mathcal{Z}$ are two Hilbert spaces. Then,*

i) $T^* \in \mathcal{L}(\mathcal{Z}, \mathcal{H})$ *and* $\|T^*\| = \|T\|$;
ii) $\mathrm{ran}(T)$ *is closed if and only if* $\mathrm{ran}(T^*)$ *is closed;*
iii) T *is surjective if and only if there exists a constant* $c > 0$, *such that* $\|T^*z\|_{\mathcal{Z}} \geq c\|z\|_{\mathcal{Z}},\ \forall z \in \mathcal{Z}$.

Construction. Let $\mathcal{H}, \mathcal{Z}$ be two Hilbert spaces and let $\langle ., . \rangle_{\mathcal{H}}$, $\langle ., . \rangle_{\mathcal{Z}}$ be their corresponding scalar products respectively. We denote by $\|.\|_{\mathcal{H}}$ (resp. $\|.\|_{\mathcal{Z}}$) the norm of $\mathcal{H}$ (resp. $\mathcal{Z}$). Let $\mathcal{B}$ be a Banach space with norm $\|.\|$.

Now consider the bilinear mapping $\flat : \mathcal{H} \times \mathcal{B} \longrightarrow \mathcal{Z}$, satisfying the condition

$$\exists M > 0 : \|\flat(h, x)\|_{\mathcal{Z}} \leq M\|h\|_{\mathcal{H}}\|x\|,\ \forall h \in \mathcal{H},\ x \in \mathcal{B}. \tag{2.1}$$

Note that (2.1) means that the mapping $\flat$ is bounded and continuous.

Fix $z \in \mathcal{Z}$ and $x \in \mathcal{B}$ and consider the linear functional $\phi_{z,x} : \mathcal{H} \longrightarrow \mathbb{C}$, $\phi_{z,x}(h) = \langle \flat(h, x), z \rangle_{\mathcal{Z}}$. It is clear that $\phi_{z,x}$ is linear, also by (2.1) we have, for all $h \in \mathcal{H}$

$$\begin{aligned} |\phi_{z,x}(h)| &= |\langle \flat(h, x), z \rangle_{\mathcal{Z}}| \\ &\leq \|\flat(h, x)\|_{\mathcal{Z}}\|z\|_{\mathcal{Z}} \\ &\leq M\|h\|_{\mathcal{H}}\|x\|\|z\|_{\mathcal{Z}}. \end{aligned}$$

Hence, $\|\phi_{z,x}\| \leq M\|x\|\|z\|_{\mathcal{Z}}$. By the Riesz representation theorem, it follows that there exists a unique element $v \in \mathcal{H}$ such that $\phi_{z,x}(h) = \langle h, v \rangle_{\mathcal{H}}$ and this element will be called $\flat$-dual product of z and x and will be denoted by $\langle z/x \rangle$. In what follows we will study the properties of this product.
Set

$$\begin{aligned} \xi : \mathcal{Z} \times \mathcal{B} &\to \mathcal{H} \\ (z, x) &\mapsto \langle z/x \rangle \end{aligned}$$

Proposition 2.3. *The mapping ξ has the following properties:*

1) ξ *is sesquilinear.*
2) ξ *is bounded and continuous.*
3) *Suppose that* $\overline{\text{ran}}(\flat) = \mathcal{Z}$. *If for every* $x \in \mathcal{B}$, $\langle z/x\rangle = 0$ *then* $z = 0$. *We say that* ξ *is non-degenerated with respect to* z.
4) *Suppose that there exists* $m > 0$ *such that*

$$m\|h\|_{\mathcal{H}}\|x\| \leq \|\flat(h,x)\|_{\mathcal{Z}}\|z\|_{\mathcal{Z}}, \quad \forall x \in \mathcal{B}, \forall h \in \mathcal{H}.$$

If for every $z \in \mathcal{Z}$, $\langle z/x\rangle = 0$, *then* $x = 0$. *We say that* ξ *is non-degenerated with respect to* x.

Now let $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ be a vector sequence, we will first give some necessary definitions :

Definition 2.4 ([9])**.** A sequence $\{x_i\}_{i\in\mathbb{N}}$ is said to be $\flat$-orthonormal in $\mathcal{Z}$ if

$$\langle \flat(h,x_i)/x_j\rangle = \delta_i^j h, \quad \forall h \in \mathcal{H},\ i,j \in \mathbb{N}.$$

Definition 2.5 ([9])**.** A sequence $\{x_i\}_{i\in\mathbb{N}}$ is said to be a $\flat$-basis for $\mathcal{Z}$ if for every $z \in \mathcal{Z}$, there exists a unique sequence $\{h_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$ such that:

$$z = \sum_{i=1}^{\infty} \flat\left(h_i, x_i\right).$$

if in addition $\{x_i\}_{i\in\mathbb{N}}$ is $\flat$-orthonormal, then $\{x_i\}_{i\in\mathbb{N}}$ is called $\flat$-orthonormal basis in $\mathcal{Z}$.

Definition 2.6 ([9])**.** Let $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$. $\{x_i\}_{i\in\mathbb{N}}$ is said to be $\flat$-complete in $\mathcal{Z}$ if and only if

$$\langle z/x_i\rangle = 0, \quad \forall i \in \mathbb{N} \quad \Rightarrow \quad z = 0. \tag{2.2}$$

Remark 2.7. If in addition to (2.1) the bilinear mapping $\flat$ verifies

$$\exists m > 0;\ m\|h\|_{\mathcal{H}}\|x\| \leq \|\flat(h,x)\|_{\mathcal{Z}}, \quad \forall h \in \mathcal{H}, \forall x \in \mathcal{B}. \tag{2.3}$$

and $\overline{Span}^{\mathcal{B}}\{x_i/i \in \mathbb{N}\} = \mathcal{B}$. Then $\{x_i\}_{i\in\mathbb{N}}$ is $\flat$-complete in $\mathcal{Z}$ if and only if $\overline{\text{ran}(\flat)}^{\mathcal{Z}} = \mathcal{Z}$.

Proposition 2.8 ([9])**.** *Let* $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$, *we have the following equivalence:*

$[\{x_i\}_{i\in\mathbb{N}}$ *is a* $\flat$*-orthonormal basis in* $\mathcal{Z}] \Leftrightarrow [\{x_i\}_{i\in\mathbb{N}}$ *is* $\flat$*-complete in* $\mathcal{Z}]$.

Example 2.9 (In finite dimension)**.** Let $\mathcal{H} = \mathbb{R}^4$ with basis $\{e_i\}_{1\leq i\leq 4}$, $\mathcal{Z} = \mathbb{R}^3$ with basis $\{v_1, v_2, v_3\}$ and $\mathcal{B} = \mathbb{R}^2$ with basis $\{f_1, f_2\}$. Let $b : \mathcal{H} \times \mathcal{B} \to \mathcal{Z}$ be as follows:

$$\begin{array}{llll} b(e_1,f_1) = u_1, & b(e_2,f_1) = u_3, & b(e_3,f_1) = 0, & (e_4,f_1) = 0, \\ b(e_1,f_2) = u_2, & b(e_2,f_2) = 0, & b(e_3,f_2) = 0, & b(e_4,f_2) = 0. \end{array}$$

for each $h = \sum_{i=1}^{4} h_i e_i \in \mathcal{H}$ and $x = x_1 f_1 + x_2 f_2 \in \mathcal{B}$ we have, $b(h,x) = h_1 x_1 u_1 + h_1 x_2 u_2 + h_2 x_1 u_3$, and

$$\begin{aligned} \|b(h,x)\|_{\mathcal{Z}}^2 &= \|h_1 x_1 u_1 + h_1 x_2 u_2 + h_2 x_1 u_3\|_{\mathcal{Z}}^2 \\ &= h_1^2 x_1^2 + h_1^2 x_2^2 + h_2^2 x_1^2 \\ &= \left(h_1^2 + h_2^2\right)\left(x_1^2 + x_2^2\right) - (h_2 x_2)^2 \end{aligned}$$

So

$$\|b(h,x)\|_{\mathcal{Z}} \leq \beta \|h\|_{\mathcal{H}} \|x\|.$$

where $\beta = 1$. By a simple calculus we obtain the b-dual product for every $z \in \mathcal{Z}$ and every $x \in \mathcal{B}$, $\langle z/x \rangle = (x_1 z_1 + x_2 z_2) e_1 + x_1 z_3 e_2$.

Chose $\{X_1, X_2\} \subset \mathcal{B}$ such that $X_1 = f_1 - f_2$ and $X_2 = f_1 + f_2$, we have

$$\begin{aligned} \begin{cases} \langle z/X_1 \rangle = 0 \\ \langle z/X_2 \rangle = 0 \end{cases} \quad &\Rightarrow \quad \begin{cases} z_1 - z_2 = 0 \\ z_1 + z_2 = 0 \\ z_3 = 0 \end{cases} \\ &\Rightarrow \quad z = 0 \end{aligned}$$

Then $\{X_1, X_2\}$ is a $\flat$-orthonormal basis in $\mathcal{Z}$.

2.1. **$\flat$-Frames and Some Examples.** Now, we will give a more detailed definition of b-frames, which were introduced in [9] and we will give some examples that have never been given in the literature to see how this concept works.

Definition 2.10. The sequence $\{x_i\}_{i \in \mathbb{N}} \subset \mathcal{B}$ is called a $\flat$-frame for $\mathcal{Z}$ if there exist constants $0 < A \leq B < \infty$, such that

$$A\|z\|_{\mathcal{Z}}^2 \leq \sum_{i=1}^{\infty} \| \langle z/x_i \rangle \|_{\mathcal{H}}^2 \leq B\|z\|_{\mathcal{Z}}^2, \quad \forall z \in \mathcal{Z}. \tag{2.4}$$

Constants A and B are called the bounds of $\flat$-frame.

If $A = B$, then $\{x_i\}_{i \in \mathbb{N}}$ is a tight $\flat$-frame for $\mathcal{Z}$.

If $A = B = 1$, then $\{x_i\}_{i \in \mathbb{N}}$ is Parseval $\flat$-frame for $\mathcal{Z}$.

If only the right side of (2.4) is verified, then we say that the sequence $\{x_i\}_{i \in \mathbb{N}}$ is $\flat$-Besselian in $\mathcal{Z}$ with bound B.

Example 2.11. Choose $\mathcal{H} = \mathbb{R}^3$, $\mathcal{B} = \mathbb{R}^2$ and $\mathcal{Z} = \mathbb{R}^4$ such that $(e_1, e_2, e_3), (f_1, f_2)$ and (u_1, u_2, u_3, u_4) are their canonical bases respectively. Define the bilinear mapping $b : \mathcal{H} \times \mathcal{B} \to \mathcal{Z}$ as follows

$$\begin{aligned} &b(e_1, f_1) = u_1, \quad b(e_2, f_1) = u_3, \quad b(e_3, f_1) = u_1, \\ &b(e_1, f_2) = u_2, \quad b(e_2, f_2) = u_4, \quad b(e_3, f_2) = u_2. \end{aligned}$$

for each $h = \sum_{i=1}^{3} h_i e_i$, and $x = \sum_{i=1}^{2} x_i f_i$, we obtain

$$b(h,x) = (h_1+h_3)x_1u_1 + (h_1+h_3)x_2u_2 + h_2x_1u_3 + h_2x_2u_4,$$

hence

$$\begin{aligned}\|b(h,x)\|_{\mathcal{Z}}^2 &= \left(h_1^2+h_2^2+h_3^2\right)\left(x_1^2+x_2^2\right) + 2h_1h_3\left(x_1^2+x_2^2\right)\\ &\leq 2\|h\|^2\|x\|^2.\end{aligned}$$

for $z \in \mathcal{Z}$ we have

$$\begin{aligned}\langle b(h,x), z\rangle &= \langle h, \langle z/x\rangle\rangle\\ &= h_1(x_1z_1+x_2z_2) + h_2(x_1z_3+x_2z_4)\\ &\quad + h_3(x_1z_1+x_2z_2).\end{aligned}$$

then

$$\langle z/x\rangle = (x_1z_1+x_2z_2)e_1 + (x_1z_3+x_2z_4)e_2 + (x_1z_1+x_2z_2)e_3.$$

Choose $y_1 = f_1+f_2,\ y_2 = f_1-f_2$ in $\mathcal{B}$. We obtain $\forall z \in \mathcal{Z}$:

$$\begin{aligned}\|z\|_{\mathcal{Z}}^2 &\leq \|\langle z/y_1\rangle\|_{\mathcal{H}}^2 + \|\langle z/y_2\rangle\|_{\mathcal{H}}^2\\ &= 4z_1^2+4z_2^2+2z_3^2+2z_4^2\\ &\leq 4\|z\|_{\mathcal{Z}}^2.\end{aligned}$$

Then $\{y_1, y_2\}$ is a b-frame for $\mathcal{Z}$ with bounds $A = 1$ and $B = 4$.

Example 2.12. Consider the Hilbert spaces, $\mathcal{H} = \mathcal{Z} = L^2(\mathbb{C})$ and the Banach space $\mathcal{B} = L^\infty(\mathbb{C})$ and consider the bilinear mapping

$$\begin{aligned}b : \mathcal{H}\times\mathcal{B} &\to \mathcal{Z}\\ (h,g) &\mapsto hg\end{aligned}$$

b is well defined and verifies (2.1), in fact

$$\begin{aligned}\|b(h,g)\|_{\mathcal{Z}} &= \|hg\|_2\\ &= \sqrt{\int_{\mathbb{C}} |h(t)g(t)|^2 dt}\\ &\leq \|g\|_\infty \sqrt{\int_{\mathbb{C}} |h(t)|^2 dt}\\ &\leq \|h\|_2\ \|g\|_\infty.\end{aligned}$$

One can verify that: $\langle f/g\rangle = \bar{g}f,\ \forall f \in \mathcal{Z}, \forall g \in \mathcal{B}$.

Let $\{g_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$, such that $g_i(t) = \frac{1}{i+1}$, $\forall i \in \mathbb{N}, \forall t \in \mathbb{C}$.

By a simple calculus we obtain,

$$\sum_{i\in\mathbb{N}} \|\langle f/g_i\rangle\|_{\mathcal{H}}^2 = \frac{\pi^2}{6}\|f\|_2^2.$$

Hence $\{g_i\}_{i\in\mathbb{N}}$ is a tight b-frame for $\mathcal{Z}$ with bound $A = B = \frac{\pi^2}{6}$.

3. Main Results

3.1. **K-$\flat$-Frames and Some Examples.** In this section, we generalize the concept of b-frames by the same idea that Gavruta generalized frames for Hilbert spaces by K-frames for Hilbert spaces in [8]. Recall first the definition given by Gavruta.

Definition 3.1 ([8])**.** Let $K \in \mathcal{L}(\mathcal{Z})$ and let K^* be its adjoint. A sequence $\{z_i\}_{i\in\mathbb{N}} \subset \mathcal{Z}$ is a K-frame for $\mathcal{Z}$ if and only if there exist constants $0 < A < B < \infty$, such that

$$A\,\|K^*z\|_{\mathcal{Z}}^2 \leq \sum_{i=1}^{\infty} |\langle z, z_i\rangle_{\mathcal{Z}}|^2 \leq B\|z\|_{\mathcal{Z}}^2, \quad \forall z \in \mathcal{Z}. \tag{3.1}$$

constants A and B are called lower and upper frame bound, respectively.

If $K \equiv Id_{\mathcal{Z}}$, then $\{f_i\}_{i\in\mathbb{N}}$ is a frame for $\mathcal{Z}$. Now, let $K \in \mathcal{L}(\mathcal{Z})$ and let K^* be its adjoint, we define a K-$\flat$-frame as follows:

Definition 3.2. The sequence $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ is called a K-$\flat$-frame for $\mathcal{Z}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\,\|K^*z\|_{\mathcal{Z}}^2 \leq \sum_{i=1}^{\infty} \|\,\langle z/x_i\rangle\,\|_{\mathcal{H}}^2 \leq B\|z\|_{\mathcal{Z}}^2, \quad \forall z \in \mathcal{Z}. \tag{3.2}$$

The constant A (resp, B) is called left (resp, right) bound of the K-$\flat$-frame. When only the right inequation of (3.2) is fulfilled, then the sequence $\{x_i\}_{i\in N}$ is called K-$\flat$-Besselian in $\mathcal{Z}$ with bound B.

If $A = B$, then $\{x_i\}_{i\in\mathbb{N}}$ is a tight K-$\flat$-frame for $\mathcal{Z}$.

If $A = B = 1$, then $\{x_i\}_{i\in\mathbb{N}}$ is Parseval K-$\flat$-frame for $\mathcal{Z}$.

Remark 3.3. Note that a $\flat$-frame is a K-$\flat$-frame with $K \equiv Id_{\mathcal{Z}}$.

Example 3.4. Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{Z}_1$, and $\mathcal{Z}_2$ be Hilbert spaces. Let $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and let $\mathcal{B} = \mathcal{L}(\mathcal{H}_1, \mathcal{Z}_1) \oplus \mathcal{L}(\mathcal{H}_2, \mathcal{Z}_2)$ be a Banach space.

Consider the bilinear mapping $\flat : \mathcal{H} \times \mathcal{B} \longrightarrow \mathcal{Z}$ such that $\flat((h_1 \oplus h_2), (t_1 \oplus t_2)) = t_1(h_1) \oplus t_2(h_2)$. Let $z_1 \oplus z_2 \in \mathcal{Z}$, we have

$$\begin{aligned}\langle \flat((h_1 \oplus h_2), (t_1 \oplus t_2)), z_1 \oplus z_2\rangle_{\mathcal{Z}} &= \langle t_1(h_1) \oplus t_2(h_2), z_1 \oplus z_2\rangle_{\mathcal{Z}} \\ &= \langle h_1 \oplus h_2, t_1^*(z_1) \oplus t_2^*(z_2)\rangle_{\mathcal{H}}.\end{aligned}$$

We claim that

$$\begin{aligned}\langle z_1 \oplus z_2/t_1 \oplus t_2\rangle &= t_1^*(z_1) \oplus t_2^*(z_2)\\ &= \langle z_1/t_1\rangle \oplus \langle z_2/t_2\rangle.\end{aligned}$$

Now, suppose that $(t_1^i)_{i\in\mathbb{N}}$ is a $\flat$-frame for $\mathcal{Z}_1$, i.e, there exist constants $0 < A_1 \leq B_1 < \infty$, such that

$$A_1\|z_1\|_{\mathcal{Z}_1}^2 \leq \sum_{i\in\mathbb{N}} \|t_1^{i*}(z_1)\|_{\mathcal{H}_1}^2 \leq B_1\|z_1\|_{\mathcal{Z}_1}^2, \quad \forall z_1 \in \mathcal{Z}_1. \tag{3.3}$$

If we suppose that $(t_2^i)_{i\in\mathbb{N}} \cong \ominus$ (i.e. all the operators t_2^i are zero), then $(t_1^i \oplus t_2^i)_{i\in\mathbb{N}}$ isn't a $\flat$-frame for $\mathcal{Z}$, but if we set $K : \mathcal{Z} \longrightarrow \mathcal{Z}$ such that, for every $z = (z_1 \oplus z_2) \in \mathcal{Z}$, $K(z_1 \oplus z_2) = z_1$. Then we have $\|K^*(z)\|_{\mathcal{Z}}^2 = \|z_1\|_{\mathcal{Z}_1}^2$, because K is self adjoint.
So, for each $z_1 \in \mathcal{Z}_1, z_2 \in \mathcal{Z}_2$ we have,

$$\begin{aligned}A_1\|K^*(z_1 \oplus z_2)\|_{\mathcal{Z}}^2 &\leq \sum_{i\in\mathbb{N}} \|t_1^{i*}(z_1)\|_{\mathcal{Z}}^2\\ &= \sum_{i\in\mathbb{N}} \|t_1^{i*} \oplus t_2^{i*}(z_1 \oplus z_2)\|_{\mathcal{Z}}^2\\ &= \sum_{i\in\mathbb{N}} \|\langle z_1 \oplus z_2/t_1 \oplus t_2\rangle\|_{\mathcal{H}}^2\\ &\leq B_1\|z_1\|_{\mathcal{Z}_1}^2 \leq B_1\|z_1 \oplus z_2\|_{\mathcal{Z}}^2.\end{aligned}$$

Which shows that $\{t_1^i \oplus t_2^i\}_{i\in\mathbb{N}}$ is a K-$\flat$-frame for $\mathcal{Z}$ with bounds A_1 and B_1.

Example 3.5. Choose $\mathcal{H} = \mathbb{R}^3$, $\mathcal{B} = \mathbb{R}^2$ and $\mathcal{Z} = \mathbb{R}^4$ such that $(e_1, e_2, e_3), (f_1, f_2)$ and (u_1, u_2, u_3, u_4) are their canonical bases respectively. Define the bilinear form $b : \mathcal{H} \times \mathcal{B} \to \mathcal{Z}$ as follows

$$\begin{array}{lll} b(e_1, f_1) = u_1 - u_2, & b(e_2, f_1) = u_1 + u_2, & b(e_3, f_1) = u_3,\\ b(e_1, f_2) = u_1 + u_2, & b(e_2, f_2) = -u_1 + u_2, & b(e_3, f_2) = u_4 \end{array}$$

for $h = \sum_{i=1}^{3} h_i e_i$ and $x = x_1 f_1 + x_2 f_2$, we have

$$\begin{aligned}b(h, x) = &[h_1(x_1 + x_2) + h_2(x_1 - x_2)]u_1\\ &+ [h_1(-x_1 + x_2) + h_2(x_1 + x_2)]u_2 + h_3x_1u_3 + h_3x_2u_4.\end{aligned}$$

and,

$$\|b(h, x)\|_{\mathcal{Z}}^2 = 2\left(h_1^2 + h_2^2\right)\left(x_1^2 + x_2^2\right) + h_3^2\left(x_1^2 + x_2^2\right).$$

Hence

$$\|h\|_{\mathcal{H}}\|x\| \leq \|b(h, x)\|_{\mathcal{Z}} \leq \sqrt{2}\|h\|_{\mathcal{H}}\|x\|.$$

In the other hand $\mathrm{ran}(b) = \mathbb{R}^4 = \mathcal{Z}$.

Let $z = \sum_{i=1}^{4} z_i u_i \in \mathcal{Z}$, then

$$\langle b(h,x), z\rangle_{\mathcal{Z}} = \langle h, \langle z/x\rangle\rangle_{\mathcal{H}}.$$

and also

$$\begin{aligned}\langle b(h,x), z\rangle_{\mathcal{Z}} &= h_1[(x_1+x_2)z_1 + (-x_1+x_2)z_2] \\ &\quad + h_2[(x_1-x_2)z_1 + (x_1+x_2)z_2] + h_3[x_1z_3 + x_2z_4].\end{aligned}$$

It follows that,

$$\begin{aligned}\langle z/x\rangle &= [(x_1+x_2)z_1 + (-x_1+x_2)z_2]e_1 \\ &\quad + [(x_1-x_2)z_1 + (x_1+x_2)z_2]e_2 + [x_1z_3 + x_2z_4]e_3.\end{aligned}$$

Choose $x = f_1 + f_2$, we obtain

$$\langle z/x\rangle = 2z_1e_1 + 2z_2e_2 + (z_3+z_4)e_3.$$

Hence

$$\|\langle z/x\rangle\|_{\mathcal{H}}^2 = 4z_1^2 + 4z_2^2 + z_3^2 + z_4^2 + |z_3z_4| \leq 4\|z\|_{\mathcal{Z}}^2, \quad \forall z \in \mathcal{Z}.$$

Then, we can say that $\{x = f_1 + f_2\}$ is a b-Bessel sequence for $\mathcal{Z} = \mathbb{C}^4$. Now, consider the linear orthogonal projection on $\mathbb{C}u_1 \oplus \mathbb{C}u_2$:

$$\begin{aligned}K : \mathcal{Z} &\to \mathcal{Z} \\ (z_1, z_2, z_3, z_4) &\mapsto (z_1, z_2)\end{aligned}$$

Then, we easily get

$$\|K^*z\|_{\mathcal{Z}}^2 \leq \|\langle z/x\rangle\|_{\mathcal{H}}^2 \leq 4\|z\|_{\mathcal{Z}}^2.$$

And, $\{x = f_1 + f_2\}$ becomes a K-b-frame for $\mathcal{Z}$ with bounds $A = 1$ and $B = 4$.

Lemma 3.6. *If $\{x_i\}_{i\in\mathbb{N}}$ is a $\flat$-orthonormal basis for $\mathcal{Z}$ and $z \in \mathcal{Z}$, such that $z = \sum_{i\in\mathbb{N}} \flat(h_i, x_i)$, $h_i \in \mathcal{H}, \forall i \in \mathbb{N}$, then*

$$\sum_{i\in\mathbb{N}} \|h_i\|_{\mathcal{H}}^2 = \|z\|_{\mathcal{Z}}^2.$$

Proof. Assume that, $\{x_i\}_{i\in\mathbb{N}}$ is $\flat$-orthonormal, then for every i, j in $\mathbb{N}$, we have

$$\langle \flat(h_i, x_i)/x_i\rangle = \delta_i^j h_i.$$

Hence

$$\left\langle \sum_{i\in\mathbb{N}} \flat(h_i, x_i)/x_j \right\rangle = \sum_{i\in\mathbb{N}} \delta_i^j h_i.$$

And, as $\{x_i\}_{i\in\mathbb{N}}$ is a $\flat$-basis for $\mathcal{Z}$, then every $z \in \mathcal{Z}$ is uniquely represented as $z = \sum_{i\in\mathbb{N}} \flat\,(h_i, x_i)$. This implies that $\langle z/x\rangle = \sum_{i\in\mathbb{N}} \delta_i^j h_i$.
For $i = j$, we have $\langle z/x_j\rangle = h_j$. Hence

$$\langle h_j, \langle z/x_j\rangle\rangle_{\mathcal{H}} = \|h_j\|^2_{\mathcal{H}},$$

So,

$$\langle \flat(h_j, x_j), z\rangle_{\mathcal{Z}} = \|h_j\|^2_{\mathcal{H}},$$

And then,

$$\begin{aligned}\left\langle \sum_{j\in\mathbb{N}} \flat(h_j, x_j), z \right\rangle_{\mathcal{Z}} &= \sum_{j\in\mathbb{N}} \|h_j\|^2_{\mathcal{H}} \\ &= \|z\|^2_{\mathcal{Z}}.\end{aligned}$$

□

Theorem 3.7. *A sequence $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ is K-$\flat$-Besselian in $\mathcal{Z}$ with bound B if and only if the operator*

$$\begin{aligned} T : l_2(\mathcal{H}) &\to \mathcal{Z} \\ \{h_i\}_{i\in\mathbb{N}} &\mapsto \sum_{i\in\mathbb{N}} \flat\,(h_i, x_i)\end{aligned}$$

is well defined, bounded, has closed range and $\|T\| \leq \sqrt{B}$. Moreover , its adjoint operator $T^* : \mathcal{Z} \longrightarrow l_2(\mathcal{H})$ is determined by $T^*(z) = \{\langle z/x_i\rangle\}_{i\in\mathbb{N}}$.

Proof. Let $\{x_i\}_{i\in\mathbb{N}}$ be K-$\flat$-Besselian in $\mathcal{Z}$ with bound B. The serie $\sum_{i\in\mathbb{N}} \flat\,(h_i, x_i)$ converges for every $\{h_i\}_{i\in\mathbb{N}} \in l_2(\mathcal{H})$. Let $n, m \in \mathbb{N}$ and set $z_n = \sum_{i=1}^{n} \flat\,(h_i, x_i)$, we have

$$\begin{aligned}\|z_m - z_n\|_{\mathcal{Z}} &= \sup_{\|z\|=1} \left| \left\langle \sum_{i=n+1}^{m} \flat\,(h_i, x_i)\,, z \right\rangle_{\mathcal{Z}} \right| \\ &= \sup_{\|z\|=1} \left| \sum_{i=n+1}^{m} \langle h_i, \langle z/x_i\rangle\rangle_{\mathcal{H}} \right| \\ &\leq \sup_{\|z\|=1} \sum_{i=n+1}^{m} \|h_i\|_{\mathcal{H}} \|\langle z/x_i\rangle\|_{\mathcal{H}} \\ &\leq \sup_{\|z\|=1} \left(\sum_{i=n+1}^{m} \|\langle z/x_i\rangle\|^2_{\mathcal{H}} \right)^{\frac{1}{2}} \left(\sum_{i=n+1}^{m} \|h_i\|^2_{\mathcal{H}} \right)^{\frac{1}{2}} \\ &\leq \sqrt{B} \left(\sum_{i=n+1}^{m} \|h_i\|^2_{\mathcal{H}} \right)^{\frac{1}{2}}.\end{aligned}$$

Note that $\left(\sum_{i=n+1}^{m} \|h_i\|_{\mathcal{H}}^2\right)^{\frac{1}{2}}$ tends to 0 when n tends to ∞, hence $\{z_n\}_{n\in\mathbb{N}}$ verifies Cauchy criterion, then it converges. We claim that the operator T is well defined and $\|T\| \leq \sqrt{B}$.
Conversely, suppose that T is well defined and that $\|T\| \leq \sqrt{B}$. Let $\{h_i\}_{i\in\mathbb{N}} \subset l_2(\mathcal{H})$ and $z \in \mathcal{Z}$, we have

$$\begin{aligned}\langle T(\{h_i\}_{i\in\mathbb{N}}), z\rangle_{\mathcal{Z}} &= \sum_{i\in\mathbb{N}} \langle \flat\left(h_i, x_i\right), z\rangle_{\mathcal{Z}} \\ &= \sum_{i\in\mathbb{N}} \langle h_i, \langle z/x_i\rangle\rangle_{\mathcal{H}} \\ &= \langle \{h_i\}_{i\in\mathbb{N}}, \{\langle z/x_i\rangle\}_{i\in\mathbb{N}}\rangle_2 .\end{aligned}$$

which gives us the expression of the adjoint operator

$$T^*(z) = \{\langle z/x_i\rangle\}_{i\in\mathbb{N}}, \quad \forall z \in \mathcal{Z}.$$

We have

$$\begin{aligned}\sum_{i\in\mathbb{N}} \|\langle z/x_i\rangle\|_{\mathcal{H}}^2 &= \|T^*(z)\|_2^2 \\ &\leq \|T^*\|^2 \|z\|_{\mathcal{Z}}^2 \\ &= \|T\|^2 \|z\|_{\mathcal{Z}}^2 \\ &\leq B\|z\|_{\mathcal{Z}}^2.\end{aligned}$$

hence, $\{x_i\}_{i\in\mathbb{N}}$ is K-$\flat$-Besselian. □

T is called K-$\flat$-synthesis operator and T^* is called K-$\flat$-analysis operator.

Definition 3.8. Suppose that $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ is a K-$\flat$-frame for $\mathcal{Z}$ with bounds A and B, then the operator $S : \mathcal{Z} \longrightarrow \mathcal{Z}$ defined by:

$$S(z) = \sum_{i\in\mathbb{N}} \flat\left(\langle z/x_i\rangle, x_i\right), \quad \forall z \in \mathcal{Z}. \tag{3.4}$$

is called K-$\flat$-frame operator for $\{x_i\}_{i\in\mathbb{N}}$ and $S = TT^*$.

Theorem 3.9. *Let $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ be a K-$\flat$-frame for $\mathcal{Z}$ with bounds A and B and with K-$\flat$-frame operator S. Then S is bounded, positive and self-adjoint in $\mathcal{Z}$.*

Proof. for all $z \in \mathcal{Z}$,

$$\langle Sz, z\rangle_{\mathcal{Z}} = \left\langle \sum_{i\in\mathbb{N}} \flat\left(\langle z/x_i\rangle, x_i\right), z \right\rangle_{\mathcal{Z}}$$

$$= \sum_{i\in\mathbb{N}} \langle \flat\left(\langle z/x_i\rangle, x_i\right), z\rangle_{\mathcal{Z}}$$
$$= \sum_{i\in\mathbb{N}} \|\langle z/x_i\rangle\|^2_{\mathcal{H}};$$

which means that S is positive and as $S = TT^*$, $S^* = (TT^*)^* = TT^*$, so S is self-adjoint.

Finally, if we gather (3.2) and (**??**) we obtain that, $\exists 0 < A \leq B < \infty$, such that

$$\begin{aligned} A\langle K^*z, K^*z\rangle_{\mathcal{Z}} &= A\langle KK^*z, z\rangle_{\mathcal{Z}} \\ &\leq \langle Sz, z\rangle_{\mathcal{Z}} \\ &\leq B\langle z, z\rangle_{\mathcal{Z}}, \quad \forall z \in \operatorname{ran}(K), \end{aligned}$$

hence

$$AKK^* \leq S \leq BI_{\mathcal{Z}}. \qquad \square$$

Proposition 3.10. *Let $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ be a K-$\flat$-frame for $\mathcal{Z}$ with bounds A and B. If* $\operatorname{ran}(K)$ *is closed, then S is invertible on* $\operatorname{ran}(K)$ *and moreover*

$$B^{-1} \leq \left\|S^{-1}\right\| \leq A^{-1}\left\|K^{\dagger}\right\|^2.$$

Proof. Suppose that $\operatorname{ran}(K)$ is closed, then there exists $K^{\dagger} \in \mathcal{L}(\mathcal{Z})$ such that

$$KK^{\dagger}z = z, \quad \forall z \in \operatorname{ran}(K),$$

and by (**??**) we have

$$\begin{aligned} \langle Sz, z\rangle_{\mathcal{Z}} &= \sum_{i\in\mathbb{N}} \|\langle z/x_i\rangle\|^2_{\mathcal{H}} \\ &\geq A\left\|K^*z\right\|^2_{\mathcal{Z}}. \end{aligned}$$

And also $\langle Sz, z\rangle_{\mathcal{Z}} \leq \|Sz\|_{\mathcal{Z}}\|z\|_{\mathcal{Z}}$, thus $A\left\|K^*z\right\|^2_{\mathcal{Z}} \leq \|Sz\|_{\mathcal{Z}}\|z\|_{\mathcal{Z}}$.

In the other hand one has

$$\begin{aligned} \|z\|_{\mathcal{Z}} &= \left\|KK^{\dagger}z\right\|_{\mathcal{Z}} \\ &= \left\|\left(K^{\dagger}\right)^* K^*z\right\|_{\mathcal{Z}}, \quad \left(\text{since } \left(KK^{\dagger}\right)^* = Id_{\operatorname{ran}(K)}\right). \end{aligned}$$

Then we obtain for all $z \in \operatorname{ran}(K)$,

$$\|z\|^2_{\mathcal{Z}} \leq \|\left(K^{\dagger}\right)^*\|^2 \left\|K^*z\right\|^2_{\mathcal{Z}} = \left\|K^{\dagger}\right\|^2 \left\|K^*z\right\|^2_{\mathcal{Z}},$$

then

$$A\left\|K^{\dagger}\right\|^{-2} \|z\|^2_{\mathcal{Z}} \leq A\left\|K^*z\right\|^2_{\mathcal{Z}} \leq \|Sz\|_{\mathcal{Z}}\|z\|_{\mathcal{Z}},$$

then

$$A\left\|K^{\dagger}\right\|^{-2} \|z\|_{\mathcal{Z}} \leq \|Sz\|_{\mathcal{Z}}.$$

hence by iii), of Theorem 2.2, S^* is surjective and S is injective, but as S is self-adjoint, then S is invertible on $\mathrm{ran}(K)$. We also have

$$\begin{aligned}\left\|SS^{-1}z\right\|_{\mathcal{Z}} &= \|z\|_{\mathcal{Z}} \\ &\leq B\|S^{-1}z\|_{\mathcal{Z}} \\ &\leq B\left\|S^{-1}\right\|\|z\|_{\mathcal{Z}}\end{aligned}$$

then

$$\left\|S^{-1}\right\| \geq B^{-1}.$$

Finally, we have shown that

$$A\left\|K^{\dagger}\right\|_{\mathcal{L}(\mathcal{Z})}^{-2}\|z\|_{\mathcal{Z}} \leq \|Sz\|_{\mathcal{Z}},$$

hence

$$A\left\|K^{\dagger}\right\|^{-2}\|S^{-1}z\|_{\mathcal{Z}} \leq \left\|SS^{-1}z\right\|_{\mathcal{Z}}.$$

We conclude that

$$\left\|S^{-1}\right\| \leq A^{-1}\left\|K^{\dagger}\right\|^{2}, \quad \forall z \in \mathrm{ran}(K). \qquad \square$$

Remark 3.11. When S is invertible on $\mathrm{ran}(K)$, we have

$$SS^{-1}(z) = S^{-1}S(z) = z, \quad (\forall z \in \mathrm{ran}(K)).$$

If and only if

$$z = \sum_{i\in\mathbb{N}} \flat\left(\left\langle S^{-1}(z)/x_i\right\rangle, x_i\right), \quad (\forall z \in \mathrm{ran}(K)).$$

If and only if

$$z = \sum_{i\in\mathbb{N}} S^{-1}\left(\flat(\langle z/x_i\rangle, x_i\right), \quad (\forall z \in \mathrm{ran}(K)).$$

In case $\{x_i\}_{i\in\mathbb{N}}$ is a $\flat$-frame, then

$$\begin{aligned}z &= \sum_{i\in\mathbb{N}} \flat\left(\left\langle S^{-1}(z)/x_i\right\rangle, x_i\right) \\ &= \sum_{i\in\mathbb{N}} S^{-1}\left(\flat(\langle z/x_i\rangle, x_i\right), \quad (\forall z \in \mathcal{Z}).\end{aligned}$$

Theorem 3.12. *Let $K \in \mathcal{L}(\mathcal{Z})$ has closed range. The sequence $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ forms a K-$\flat$-frame for* $\mathrm{ran}(K)$ *if and only if the operator,*

$$\begin{aligned}T : l_2(\mathcal{H}) &\to \mathcal{Z} \\ \{h_i\}_{i\in\mathbb{N}} &\mapsto \sum_{i\in\mathbb{N}} \flat\left(h_i, x_i\right)\end{aligned}$$

is defined, bounded and surjective on $\mathrm{ran}(K)$.

Proof. Let $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ be a K-♭-frame for $\mathcal{Z}$, then it is K-♭-Besselian in $\mathcal{Z}$. Hence, by Theorem 3.7 T is well defined and let $S = TT^*$ be the corresponding K-♭- frame operator, we proved that S is surjective on $\mathrm{ran}(K)$, then T is also surjective on $\mathrm{ran}(K)$.

Conversely, suppose that T is defined and surjective, then by theorem 3.7 $\{x_i\}_{i\in\mathbb{N}}$ is K-♭-Besselian and also T is surjective, then for each $z \in \mathrm{ran}(K)$, there exists a sequence $\{y_i\}_{i\in\mathbb{N}} \subset \ell^2(\mathcal{H})$ such that $T(\{y_i\}) = z$ and let $T^\dagger z = \{y_i\}$. $\forall z \in \mathrm{ran}(K)$, we have

$$\begin{aligned}
\|K^*z\|_{\mathcal{Z}}^4 &= |\langle K^*z, K^*z\rangle_{\mathcal{Z}}|^2 \\
&= |\langle z, KK^*z\rangle_{\mathcal{Z}}|^2 \\
&= |\langle T(\{y_i\}), KK^*z\rangle_{\mathcal{Z}}|^2 \\
&= \left|\left\langle \sum_{i=1}^{\infty} b(y_i, x_i), KK^*z \right\rangle_{\mathcal{Z}}\right|^2 \\
&\leq \|K\|^4 \left|\sum_{i=1}^{\infty} \langle y_i, \langle z/x_i\rangle\rangle_{\mathcal{H}}\right|_{\mathcal{H}}^2 \\
&\leq \|K\|^4 \left|\sum_{i=1}^{\infty} \|y_i\|_{\mathcal{H}} \|\langle z/x_i\rangle\|_{\mathcal{H}}\right|^2 \\
&\leq \|K\|^4 \left\|T^\dagger z\right\|_{\ell^2(\mathcal{H})}^2 \sum_{i=1}^{\infty} \|\langle z/x_i\rangle\|_{\mathcal{H}}^2 \\
&\leq \|K\|^4 \left\|T^\dagger\right\|^2 \|z\|_{\mathcal{Z}}^2 \sum_{i=1}^{\infty} \|\langle z/x_i\rangle\|_{\mathcal{H}}^2 \\
&\leq \|K\|^4 \left\|T^\dagger\right\|^2 \left\|K^\dagger\right\|^2 \|K^*z\|_{\mathcal{Z}}^2 \sum_{i=1}^{\infty} \|\langle z/x_i\rangle\|_{\mathcal{H}}^2.
\end{aligned}$$

Hence

(3.5)

$$\|K\|^{-4} \left\|T^\dagger\right\|^{-2} \left\|K^\dagger\right\|^{-2} \|K^*z\|_{\mathcal{Z}}^2 \leq \sum_{i=1}^{\infty} \|\langle z/x_i\rangle\|_{\mathcal{H}}^2, \quad \forall z \in \mathrm{ran}(K).$$

□

Theorem 3.13. *Let $K \in \mathcal{L}(\mathcal{Z})$ such that* $\mathrm{ran}(K)$ *is closed. A sequence $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ is a K-♭-frame for* $\mathrm{ran}(K)$ *with bounds A and B, if and only if the two following conditions are satisfied.*

1) $\{x_i\}_{i\in\mathbb{N}}$ *is ♭-complete in* $\mathrm{ran}(K)$.

2) *The K-$\flat$-synthesis operator is well defined and*

$$\frac{A}{\|K^\dagger\|^2}\sum_{i\in\mathbb{N}}^{\infty}\|h_i\|_{\mathcal{H}}^2 \leq \|T(\{h_i\}_{i\in\mathbb{N}})\|_{\mathcal{Z}}^2 \leq B\sum_{i\in\mathbb{N}}^{\infty}\|h_i\|_{\mathcal{H}}^2, \quad \forall\{h_i\}_{i\in\mathbb{N}}\in\mathcal{H}.$$

Proof. ($\Rightarrow$) Suppose that $\{x_i\}_{i\in\mathbb{N}}$ is a K-$\flat$-frame for $\mathrm{ran}(K)$ with bounds A and B. And, suppose that $\langle z/x_i\rangle = 0$, for all $i\in\mathbb{N}$, we have

$$A\,\|K^*z\|_{\mathcal{Z}}^2 \leq \sum_{i=1}^{\infty}\|\,\langle z/x_i\rangle\,\|_{\mathcal{H}}^2 \leq B\|z\|_{\mathcal{Z}}^2, \quad \forall z\in\mathrm{ran}(K).$$

which means that $K^*z = 0$. $\mathrm{ran}(K)$ is closed, then there exists $K^\dagger\in\mathcal{L}(\mathcal{Z})$ such that $KK^\dagger z = z,\ \forall z\in\mathrm{ran}(K)$. We have

$$\begin{aligned}\|z\|_{\mathcal{Z}}^2 &= |\langle z,z\rangle_{\mathcal{Z}}|\\ &= |\langle KK^\dagger z,z\rangle_{\mathcal{Z}}|\\ &= |\langle K^\dagger z,K^*z\rangle_{\mathcal{Z}}|\\ &= |\langle K^\dagger z,0\rangle_{\mathcal{Z}}|\\ &= 0.\end{aligned}$$

Hence $z=0$, which implies that $\{x_i\}_{i\in\mathbb{N}}$ is $\flat$-complete on $\mathrm{ran}(K)$. In addition, we claim that for every $z\in\mathrm{ran}(K)$, there exists a unique $\{h_i\}_{i\in\mathbb{N}}\subset\mathcal{H}$, such that $z=\sum\limits_{i\in\mathbb{N}}^{\infty}b\,(h_i,x_i) = T(\{h_i\}_{i\in\mathbb{N}})$, as long as T is surjective and that by Lemma 3.6, we have for $\{h_i\}_{i\in\mathbb{N}}\subset\mathcal{H}$,

$$\begin{aligned}\frac{A}{\|K^\dagger\|^2}\sum_{i\in\mathbb{N}}^{\infty}\|h_i\|_{\mathcal{H}}^2 7 &\leq A\,\|K^*z\|_{\mathcal{Z}}^2\\ &\leq \sum_{j\in\mathbb{N}}^{\infty}\|\langle\sum_{i\in\mathbb{N}}^{\infty}b\,(h_i,x_i)\,/x_j\rangle\|_{\mathcal{H}}^2\\ &\leq B\sum_{i\in\mathbb{N}}^{\infty}\|h_i\|_{\mathcal{H}}^2,\end{aligned}$$

hence

$$\frac{A}{\|K^\dagger\|^2}\sum_{i\in\mathbb{N}}^{\infty}\|h_i\|_{\mathcal{H}}^2 \leq \sum_{j\in\mathbb{N}}^{\infty}\|h_j\|_{\mathcal{H}}^2$$

$$\leq B\sum_{i\in\mathbb{N}}^{\infty}\|h_i\|_{\mathcal{H}}^2$$

and so

$$\frac{A}{\|K^\dagger\|^2}\sum_{i\in\mathbb{N}}^{\infty}\|h_i\|_{\mathcal{H}}^2 \leq \|z\|_{\mathcal{H}}^2 \leq B\sum_{i\in\mathbb{N}}^{\infty}\|h_i\|_{\mathcal{Z}}^2$$

then

$$\frac{A}{\|K^\dagger\|^2}\sum_{i\in\mathbb{N}}^{\infty}\|h_i\|_{\mathcal{H}}^2 \leq \sum_{j\in\mathbb{N}}^{\infty}\|T(\{h_i\}_{i\in\mathbb{N}})\|_{\mathcal{H}}^2 \leq B\sum_{i\in\mathbb{N}}^{\infty}\|h_i\|_{\mathcal{H}}^2.$$

($\Leftarrow$) Conversely, suppose that 1 and 2 hold, by Theorem 3.12, we only need to show that T is surjective on $\mathrm{ran}(K)$. Suppose that $T^*(z)=0$, then $\|T^*(z)\|_{\mathcal{H}} = \sum\limits_{i\in\mathbb{N}}\|\langle z/x_i\rangle\|_{\mathcal{H}}^2 = 0,\ \forall i\in\mathbb{N}$. This implies that $\|z\|_{\mathrm{ran}(K)}=0 \Rightarrow z=0$, we conclude that T^* is injective, which means that T is surjective on $\mathrm{ran}(K)$. □

3.2. **The $\flat$-Adjoint Operator.** In the study of Hilbert spaces, the operator theory took an important place, where bounded operators played an important role. More specifically, the adjoint operator is also called the Hermitian conjugate. Now, we want to generalize this notion according to the $\flat$-dual product defined in proposition 2.3 to give some results about b-frames preserving. In other words, we want to figure out under which conditions we have for a bounded linear operator U on a Banach space $\mathcal{B}$, the existence of a unique bounded operator V on a Hilbert space $\mathcal{Z}$ verifying

$$\langle z/Ux\rangle = \langle Vz/x\rangle, \quad \forall x\in\mathcal{B},\ \forall z\in\mathcal{Z}.$$

From now on, we will suppose that $\mathcal{H}$ is separable and $\{e_i\}_{i\in\mathbb{N}}$ is its orthonormal basis and that the bilinear mapping $\flat$ has a dense range and that $\{x_i\}_{i\in\mathbb{N}}\subset\mathcal{B}$ is a $\flat$-orthonormal basis in $\mathcal{Z}$ such that

$$\sum_{i\in\mathbb{N}}\|x_i\|<\infty.$$

Denote by $\mathcal{X}$ the subspace of $\mathcal{B}$ such that

$$\mathcal{X}=\overline{\mathrm{span}\{x_i/i\in\mathbb{N}\}}.$$

Proposition 3.14. *Let $U : \mathcal{X} \to \mathcal{X}$ be a bounded linear operator, let $\{h_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ and let $V : \mathcal{Z} \to \mathcal{Z}$ be an operator such that for each $z = \sum_{i\in\mathbb{N}} \flat\left(h_i, x_i\right) \in \mathcal{Z}$,*

$$\begin{aligned} V^*(z) &= \sum_{i\in\mathbb{N}} \\ &= \flat(h_i, Ux_i), \end{aligned}$$

Then V^* is well defined, bounded and linear.

Proof. V^* is well defined on $\mathcal{Z}$ since $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ is a $\flat$-orthonormal basis in $\mathcal{Z}$ and it is linear, indeed: for each $z = \sum_{i\in\mathbb{N}} \flat\left(h_i, x_i\right)$, and $z' = \sum_{i\in\mathbb{N}} \flat\left(h'_i, x_i\right) \in \mathcal{Z}$ and let $\gamma \in \mathbb{C}$, we have

$$\begin{aligned} V^*(\gamma z + z') &= V^*\left(\gamma \sum_{i\in\mathbb{N}} \flat\left(h_i, x_i\right) + \sum_{i\in\mathbb{N}} \flat\left(h'_i, x_i\right)\right) \\ &= V^*\left(\sum_{i\in\mathbb{N}} \flat\left(\gamma h_i + h'i, x_i\right)\right) \\ &= \sum_{i\in\mathbb{N}} \flat\left(\gamma h_i + h'i, Ux_i\right) \\ &= \gamma V^*(z) + V^*(z') \end{aligned}$$

V^* is also bounded, in fact we have for all $z = \sum_{i\in\mathbb{N}} \flat\left(h_i, x_i\right) \in \mathcal{Z}$,

$$\begin{aligned} \|V^*z\|_{\mathcal{Z}} &= \left\| V^*\left(\sum_{i\in\mathbb{N}} \flat\left(h_i, x_i\right)\right)\right\|_{\mathcal{Z}} \\ &= \left\|\sum_{i\in\mathbb{N}} V^*(\flat\left(h_i, x_i\right))\right\|_{\mathcal{Z}} \\ &= \left\|\sum_{i\in\mathbb{N}} \flat(h_i, Ux_i)\right\|_{\mathcal{Z}} \\ &\leq \sum_{i\in\mathbb{N}} \|\flat(h_i, Ux_i)\|_{\mathcal{Z}} \\ &\leq \beta\|U\| \sum_{i\in\mathbb{N}} \|h_i\|_{\mathcal{H}}\|x_i\| \\ &\leq \beta\|U\| \sup_{i\in\mathbb{N}} \|h_i\|_{\mathcal{H}} \sum_{i\in\mathbb{N}} \|x_i\| \end{aligned}$$

we have already shown that for all $z = \sum_{i\in\mathbb{N}} \flat\,(h_i, x_i) \in \mathcal{Z}$,

$$\|z\|_{\mathcal{Z}}^2 = \sum_{i\in\mathbb{N}} \|h_i\|_{\mathcal{H}}^2,$$

hence $\|z\|_{\mathcal{Z}}^2 \geq \sup_{i\in\mathbb{N}} \|h_i\|_{\mathcal{H}}^2 = (\sup_{i\in\mathbb{N}} \|h_i\|_{\mathcal{H}})^2$, which means that $\|z\|_{\mathcal{Z}} \geq \sup_{i\in\mathbb{N}} \|h_i\|_{\mathcal{H}}$. In consequence we have

$$\begin{aligned}\|V^*z\|_{\mathcal{Z}} &\leq \beta\|U\| \sum_{i\in\mathbb{N}} \|x_i\|\|z\|_{\mathcal{Z}} \\ &< \infty.\end{aligned}$$

□

Theorem 3.15. *Let $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ be a $\flat-$orthonormal basis in $\mathcal{Z}$. Let $U : \mathcal{X} \to \mathcal{X}$ be a bounded linear operator, there exists a unique operator $V \in \mathcal{L}(\mathcal{Z})$ such that, $\forall z \in \mathcal{Z}$,*

$$\langle Vz/x_i\rangle = \langle z/Ux_i\rangle, \quad \forall i \in \mathbb{N}. \tag{3.6}$$

Proof. Let's provide $\mathcal{H}$ with its orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ and let $h = \sum_{j\in\mathbb{N}} \alpha_j e_j \in \mathcal{H}$, note that if $z_k = \sum_{j\in\mathbb{N}} \flat(h_j, x_j)$ with $h_j = 0$ if $j \neq k$ and $h_k = h$, then z_k can be also written as $z_k = \flat(h, x_k) = \sum_{j\in\mathbb{N}} \alpha_j \flat(e_j, x_k)$
Let V the adjoint operator of V^* on $\mathcal{Z}$. Let $z \in \mathcal{Z}$, we have

$$\begin{aligned}\langle V^*(\flat(h, x_i)), z\rangle_{\mathcal{Z}} &= \langle \flat(h, Ux_i), z\rangle_{\mathcal{Z}} \\ &= \langle h, \langle z/Ux_i\rangle\rangle_{\mathcal{H}}.\end{aligned} \tag{3.7}$$

In the other hand

$$\begin{aligned}\langle V^*(\flat(h, x_i)), z\rangle_{\mathcal{Z}} &= \langle \flat(h, x_i), Vz\rangle_{\mathcal{Z}} \\ &= \langle h, \langle Vz/x_i\rangle\rangle_{\mathcal{H}}.\end{aligned} \tag{3.8}$$

from (3.7) and (3.8) we obtain for all $h \in \mathcal{H}$,

$$\langle h, \langle z/Ux_i\rangle\rangle_{\mathcal{H}} = \langle h, \langle Vz/x_i\rangle\rangle_{\mathcal{H}},$$

then

$$\forall i \in \mathbb{N}, \quad \langle z/Ux_i\rangle = \langle Vz/x_i\rangle.$$

Suppose now that there exists W also verifying (3.9), then we have

$$\langle z/Ux_i\rangle = \langle Vz/x_i\rangle = \langle Wz/x_i\rangle, \quad \forall i \in \mathbb{N},$$

so

$$\langle (V - W)z/x_i\rangle = 0, \quad \forall i \in \mathbb{N},$$

then

$$V = W, \quad \text{on } \mathcal{X}.$$

hence V is unique. □

Corollary 3.16. *Let $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ be a $\flat-$orthonormal basis in $\mathcal{Z}$. Let $U : \mathcal{X} \to \mathcal{X}$ be a bounded linear operator, there exists a unique operator $V \in \mathcal{L}(\mathcal{Z})$ such that,*

$$\langle Vz/x\rangle = \langle z/Ux\rangle, \quad \forall x \in \mathcal{X}, \forall z \in \mathcal{Z}. \tag{3.9}$$

We call the unique operator verifying (3.9) on $\mathcal{Z}$, the $\flat$-adjoint of U and we denote it by $U^{\flat}$.

Example 3.17. If we go back to the example (2.9) and we consider the bounded linear operator $U : \mathcal{B} \to \mathcal{B}$ such that $U(f_1) = f_1 + f_2$ and $U(f_2) = 0$. We have

$$\begin{aligned} U(x) &= x_1U(f_1) + x_2U(f_2) \\ &= x_1f_1 + x_1f_2. \end{aligned}$$

Recall that, $\langle z/x\rangle = (x_1z_1 + x_2z_2)e_1 + x_1z_3e_2$. So

$$\langle z/Ux\rangle = x_1(z_1 + z_2)e_1 + x_1z_3e_2.$$

Let $V : \mathcal{Z} \to \mathcal{Z}$ such that $\langle Vz/x\rangle = \langle z/Ux\rangle$, we can write V as

$$\begin{aligned} Vz &= \begin{pmatrix} \alpha_1^1 & \alpha_2^1 & \alpha_3^1 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \\ &= \sum_{i=1}^{3} \alpha_i^1 z_i u_1 + \sum_{i=1}^{3} \alpha_i^2 z_i u_2 + \sum_{i=1}^{3} \alpha_i^3 z_i u_3. \end{aligned}$$

So

$$\langle Vz/x\rangle = \left(x_1 \sum_{i=1}^{3} \alpha_i^1 z_i + x_2 \sum_{i=1}^{3} \alpha_i^2 z_i \right) e_1 + x_1 \sum_{i=1}^{3} \alpha_i^3 z_i e_2.$$

Then

$$\left(x_1 \sum_{i=1}^{3} \alpha_i^1 z_i + x_2 \sum_{i=1}^{3} \alpha_i^2 z_i \right) e_1 + x_1 \sum_{i=1}^{3} \alpha_i^3 z_i e_2 = x_1(z_1 + z_2)e_1 + x_1z_3e_2.$$

By a simple calculus we obtain the b-adjoint of U,

$$U^b = V = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, $U^bz = (z_1 + z_2)u_1 + z_3u_3$, $\forall z \in \mathcal{Z}$.

Example 3.18. Let $\mathcal{H} = \overline{\ell^2(\mathbb{C}) \cap \ell^1(\mathbb{C})}^{\ell^2(\mathbb{C})}$, $\mathcal{Z} = \ell^2(\mathbb{C})$ and $\mathcal{B} = \ell^\infty(\mathbb{C})$. And let

$$\begin{aligned} U : \mathcal{B} &\to \mathcal{B} \\ \{x_n\}_{n\in\mathbb{N}} &\mapsto \{x_{n+1}\}_{n\in\mathbb{N}} \end{aligned}$$

we have $\|U\| = 1$ and consider the bilinear mapping

$$\phi : \mathcal{H} \times \mathcal{B} \to \mathcal{Z}$$
$$(\{h_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}) \mapsto \{h_n b_n\}_{n\in\mathbb{N}}$$

It's clear that: $\|\phi\left(\{h_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}\right)\|_2 \leq \|\{h_n\}_{n\in\mathbb{N}}\|_{\mathcal{H}}\|\{b_n\}_{n\in\mathbb{N}}\|_\infty$. And we have for every $\{z_n\}_{n\in\mathbb{N}} \subset \mathcal{Z}$

$$\begin{aligned}\langle\phi\left(\{h_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}\right), \{z_n\}_{n\in\mathbb{N}}\rangle_2 &= \{h_n b_n \bar{z_n}\}_{n\in\mathbb{N}} \\ &= \left\{h_n \overline{\bar{b_n} z_n}\right\}_{n\in\mathbb{N}} \\ &= \left\langle \{h_n\}_{n\in\mathbb{N}}, \{\bar{b_n} z_n\}_{n\in\mathbb{N}}\right\rangle_{\mathcal{H}}.\end{aligned}$$

hence for every $\{z_n\}_{n\in\mathbb{N}} \subset \mathcal{Z}$ and every $\{b_n\}_{n\in\mathbb{N}} \subset \mathcal{B}$, we have, $\langle\{z_n\}_{n\in\mathbb{N}}, \{b_n\}\rangle_{\mathcal{H}} = \{\bar{b_n} z_n\}_{n\in\mathbb{N}}$. Whence

$$\begin{aligned}&\langle\phi\left(\{h_n\}_{n\in\mathbb{N}}, U(\{b_n\}_{n\in\mathbb{N}})\right), \{z_n\}_{n\in\mathbb{N}}\rangle_2 \\ &= \left\langle \{h_n\}_{n\in\mathbb{N}}, \left\{\overline{b_{n+1}} z_n\right\}_{n\in\mathbb{N}}\right\rangle_{\mathcal{H}} \\ &= \langle\{h_n\}_{n\in\mathbb{N}}, \{\sigma_n\}_{n\in\mathbb{N}^*}\rangle_{\mathcal{H}} \\ &= \langle\{h_n\}_{n\in\mathbb{N}}, \langle V(\{z_n\}_{n\in\mathbb{N}})/\{b_n\}_{n\in\mathbb{N}}\rangle\rangle_{\mathcal{H}}\end{aligned}$$

Where $\sigma_k = \bar{b}_k z_{k-1}$, for $k > 0$ and $\sigma_0 = 0$ and

$$V : \mathcal{Z} \to \mathcal{Z}$$
$$\{z_n\}_{n\in\mathbb{N}} \mapsto \{0, z_0, z_1, \ldots, z_n, \ldots\}$$

$V = U^\flat$ is the $\flat$-adjoint operator of U on $\ell^2(\mathbb{C})$. (Note that the b-adjoint of U in $\ell^2(\mathbb{C})$ coincides with the adjoint operator of U in $\ell^\infty(\mathbb{C})$).

Remark 3.19. The opposite of Corollary 3.16 is not true; in fact, for a bounded linear operator $V : \mathcal{Z} \to \mathcal{Z}$ we cannot always find a bounded linear operator $U : \mathcal{B} \to \mathcal{B}$ such that

$$\langle Vz/x\rangle = \langle z/Ux\rangle, \quad \forall z \in \mathcal{Z}, \forall x \in \mathcal{B}.$$

We can see this by giving the simple following example:

Example 3.20. Let $\mathcal{H} = \mathbb{R}^3$, $\mathcal{B} = \mathbb{R}^2$ and $\mathcal{Z} = \mathbb{R}^4$ and let (e_1, e_2, e_3), (f_1, f_2) and (u_1, u_2, u_3, u_4) be their canonical bases respectively. Consider the bilinear mapping $b : \mathcal{H} \times \mathcal{B} \to \mathcal{Z}$ such that

$$\begin{array}{lll} b(e_1, f_1) = u_1, & b(e_2, f_1) = u_3, & b(e_3, f_1) = u_1 - u_2, \\ b(e_1, f_2) = u_2, & b(e_2, f_2) = u_4, & b(e_3, f_2) = u_1 + u_2. \end{array}$$

Let $h = \sum_{i=1}^{3} h_i e_i \in \mathcal{H}$ and $x = x_1 f_1 + x_2 f_2 \in \mathcal{B}$, we have

$$b(h, x) = [(h_1+h_3)x_1+h_3x_2]u_1+[(h_1+h_3)x_2-h_3x_1]u_2+h_2x_1u_3+h_2x_2u_4,$$

and for each $z = \sum_{i=1}^{4} z_i u_i \in \mathcal{Z}$, we have

$$\langle b(h,x), z\rangle = h_1(x_1z_1+x_2z_2)+h_2(x_1z_3+x_2z_4)+h_3[(x_1+x_2)z_1+(x_2-x_1)z_2],$$

So we claim that for all $z \in \mathcal{Z}$ and $x \in \mathcal{B}$,

$$\langle z/x\rangle = (x_1z_1+x_2z_2)e_1+(x_1z_3+x_2z_4)e_2+[(x_1+x_2)z_1+(x_2-x_1)z_2]e_3.$$

Now consider the bounded linear operator $V : \mathcal{Z} \to \mathcal{Z}$, defined as $V(u_1) = u_2$ and $V(u_2) = V(u_3) = V(u_4) = 0$. We have $\langle Vz/x\rangle = x_2z_1e_1 + (x_2 - x_1)z_1e_3$. Suppose that there exists a bounded linear operator $U : \mathcal{B} \to \mathcal{B}$ such that $\langle Vz/x\rangle = \langle z/Ux\rangle$. U will be such as $U(x_1) = \alpha x_1 + \beta x_2$ and $U(x_2) = \gamma x_1 + \delta x_2$. Then we have

$$\begin{aligned}\langle z/Ux\rangle = {} & [(\alpha x_1 + \beta x_2)z_1 + (\gamma x_1 + \delta x_2)z_2]e_1 \\ & + [(\alpha x_1 + \beta x_2)z_3 + (\gamma x_1 + \delta x_2)z_4]e_2 \\ & + ([(\alpha+\gamma)x_1 + (\beta+\delta)x_2]z_1 + [(\alpha-\gamma)x_1 + (\beta-\delta)x_2]z_2)\, e_3.\end{aligned}$$

We obtain $\alpha = \delta = \gamma = 0$ and $\beta = 1$, which is contradictory; because if it's not then we will have $\beta x_2 z_3 e_2 \neq 0, \forall x_2 \in \mathcal{B}, \forall z_3 \in \mathcal{Z}$, which is not true. Hence we can't find U.

3.3. **Stability and Preserving.** Let $\{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}}$ be two sequences in $\mathcal{B}$ and suppose that the bilinear mapping $\flat$ has a dense range. We have the following results:

Proposition 3.21 (Stability by summing)**.** *If $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are $\flat$-frames for $\mathcal{Z}$ and if $T_x(\{h_i\}_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} \flat\,(h_i, x_i)$, and $T_y(\{h_i\}_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} \flat(h_i, y_i)$, are respectively their corresponding $\flat$-synthesis operators for $\{h_i\}_{i\in\mathbb{N}} \subset \ell^2(\mathcal{H})$, then $\{x_i+y_i\}_{i\in\mathbb{N}}$ is a $\flat-$frame for $\mathcal{Z}$ and its $\flat$-synthesis operator is given by*

$$T_{x+y}(\{h_i\}_{i\in\mathbb{N}}) = T_x(\{h_i\}_{i\in\mathbb{N}}) + T_y(\{h_i\}_{i\in\mathbb{N}}), \quad \forall\{h_i\}_{i\in\mathbb{N}} \subset \ell^2(\mathcal{H}).$$

And moreover

$$\|T_{x+y}\| \leq \sqrt{B_x} + \sqrt{B_y}.$$

Proof. Suppose that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are $\flat-$frames for $\mathcal{Z}$ with upper bounds B_x and B_y, respectively. Let $\{h_i\}_{i\in\mathbb{N}} \subset \ell^2(\mathcal{H})$, we have

$$\begin{aligned}T_{x+y}(\{h_i\}_{i\in\mathbb{N}}) &= T_x(\{h_i\}_{i\in\mathbb{N}}) + T_y(\{h_i\}_{i\in\mathbb{N}}) \\ &= \sum_{i=1}^{\infty} \flat\,(h_i, x_i) + \sum_{i=1}^{\infty} \flat(h_i, y_i) \\ &= \sum_{i=1}^{\infty} \flat(h_i, x_i + y_i).\end{aligned}$$

T_{x+y} is well defined, because we have

$$\left\|\sum_{i=1}^{\infty}\flat(h_i,x_i+y_i)\right\| \leq \left\|\sum_{i=1}^{\infty}\flat\left(h_i,x_i\right)\right\| + \left\|\sum_{i=1}^{\infty}\flat(h_i,y_i)\right\|.$$

And, it is clear that $\|T_{x+y}\| \leq \|T_x\| + \|T_y\| \leq \sqrt{B_x} + \sqrt{B_y} < \infty$.
Now let $z \in \mathcal{Z}$,

$$\begin{aligned}\langle T_{x+y}(\{h_i\}_{i\in\mathbb{N}}), z\rangle_{\mathcal{Z}} &= \sum_{i=1}^{\infty}\langle\flat(h_i,x_i+y_i),z\rangle_{\mathcal{Z}}\\ &= \sum_{i=1}^{\infty}\langle h_i,\langle z/x_i+y_i\rangle\rangle_{\mathcal{H}}\\ &= \langle\{h_i\}_{i\in\mathbb{N}},\{\langle z/x_i+y_i\rangle\}_{i\in\mathbb{N}}\rangle_{\mathcal{H}}.\end{aligned}$$

hence the $\flat$-analysis operator T^* is given by

$$T^*(z) = \{\langle z/x_i+y_i\rangle\}_{i\in\mathbb{N}}, \quad \forall z \in \mathcal{Z}.$$

Now let $z, z' \in \mathcal{Z}$, such that $T^*(z) = T^*(z')$, it means that

$$\forall i \in \mathbb{N}, \quad \langle z/x_i+y_i\rangle = \langle z'/x_i+y_i\rangle,$$

and

$$\forall h \in \mathcal{H}, \forall i \in \mathbb{N}, \quad \langle b(h,x_i+y_i),z\rangle_{\mathcal{Z}} = \langle b(h,x_i+y_i),z'\rangle_{\mathcal{Z}},$$

then

$$\forall h \in \mathcal{H}, \forall i \in \mathbb{N}, \quad \left\langle b(h,x_i+y_i), z-z'\right\rangle_{\mathcal{Z}} = 0.$$

As $\mathrm{ran}(\flat)$ is dense, then $\mathcal{Z} = \overline{\mathrm{ran}}(\flat) \oplus \ker(\flat) \Rightarrow \ker(\flat) = \{0\}$, whence necessarily $z = z'$. Which implies that T is surjective, then by Theorem 12 of [9] $\{x_i+y_i\}_{i\in\mathbb{N}}$ is a $\flat-$frame for $\mathcal{Z}$. □

Corollary 3.22. *If $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are K-$\flat$-frames for $\mathcal{Z}$ and if $T_x(\{h_i\}_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}}\flat\left(h_i,x_i\right)$, and $T_y(\{h_i\}_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}}\flat(h_i,y_i)$, are respectively their corresponding K-$\flat$-synthesis operators for $\{h_i\}_{i\in\mathbb{N}} \subset \ell^2(\mathcal{H})$. Then $\{x_i+y_i\}_{i\in\mathbb{N}}$ is a K-$\flat$-frame for* $\mathrm{ran}(K)$ *and its K-$\flat$-synthesis operator is given by*

$$T_{x+y}(\{h_i\}_{i\in\mathbb{N}}) = T_x(\{h_i\}_{i\in\mathbb{N}}) + T_y(\{h_i\}_{i\in\mathbb{N}}), \quad \forall\{h_i\}_{i\in\mathbb{N}} \subset \ell^2(\mathcal{H})$$

And moreover $\|T_{x+y}\| \leq \sqrt{B_x} + \sqrt{B_y}$.

Proof. The proof is simillar to the one of proposition 3.21, except that in this case the K-$\flat$-synthesis operator cannot be surjective unless if K has a closed range and the result won't be valid outside $\mathrm{ran}(K)$. □

Proposition 3.23. *Let $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ be a K-$\flat$-frame for $\mathcal{Z}$ and let $Q \in \mathcal{L}(\mathcal{Z})$ such that* $\mathrm{ran}(Q) \subseteq \mathrm{ran}(K)$. *Then $\{x_i\}_{i\in\mathbb{N}}$ is also a Q-$\flat$-frame for $\mathcal{Z}$.*

Proof. Suppose that $\{x_i\}_{i\in\mathbb{N}}$ is a K-$\flat$-frame for $\mathcal{Z}$, then there exist $0 < A \leq B < \infty$, such that

$$A\,\|K^*z\|_{\mathcal{Z}}^2 \leq \sum_{i=1}^{\infty} \|\,\langle z/x_i\rangle\,\|_{\mathcal{H}}^2 \leq B\|z\|_{\mathcal{Z}}^2, \quad \forall z \in \mathrm{ran}(K).$$

let $Q \in \mathcal{L}(\mathcal{H})$, such that $\mathrm{ran}(Q) \subseteq \mathrm{ran}(K)$, then by Theorem 2.1, there exists $\lambda > 0$ such that $TT^* \leq \lambda^2 KK^*$, then $\forall z \in \mathrm{ran}(Q)$ we have for $\lambda > 0$, $\langle QQ^*z, z\rangle_{\mathcal{Z}} \leq \langle \lambda^2 KK^*z, z\rangle_{\mathcal{Z}}$, which implies that

$$\begin{aligned}
& \langle QQ^*z, z\rangle_{\mathcal{Z}} \leq \lambda^2 \langle KK^*z, z\rangle_{\mathcal{Z}} \\
\Rightarrow \quad & \langle Q^*z, Q^*z\rangle_{\mathcal{Z}} \leq \lambda^2 \langle K^*z, k^*z\rangle_{\mathcal{Z}} \\
\Rightarrow \quad & \|Q^*z\|^2 \leq \lambda^2\,\|K^*z\|^2 \\
\Rightarrow \quad & \frac{A}{\lambda^2}\|Q^*z\|^2 \leq \|K^*z\|^2 \leq \sum_{i=1}^{\infty} \|\,\langle z/x_i\rangle\,\|_{\mathcal{H}}^2.
\end{aligned}$$

We also have

$$\sum_{i=1}^{\infty} \|\,\langle z/x_i\rangle\,\|_{\mathcal{H}}^2 \leq B\|z\|_{\mathcal{Z}}^2.$$

Hence $\{x_i\}_{i\in\mathbb{N}}$ is a Q-$\flat$-frame with bounds $\frac{A}{\lambda^2}$ and B. □

Theorem 3.24. *Let $\{y_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ be a $\flat$-orthonormal basis in $\mathcal{Z}$ and let $K \in \mathcal{L}(\mathcal{Z})$ have a closed range, then $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ forms a K-$\flat$-frame for* $\mathrm{ran}(K)$ *if and only if there exists a bounded surjective operator $\Omega : \mathcal{Z} \to \mathcal{Z}$ such that:*

$$\Omega(\flat(h, y_i)) = \flat(h, x_i), \quad \forall h \in \mathcal{H},\ \forall i \in \mathbb{N}.$$

Proof. ($\Rightarrow$) Let $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ and let $\Omega : \mathcal{Z} \to \mathcal{Z}$ be a bounded surjective operator such that $\Omega(\flat(h, y_i)) = \flat(h, x_i),\ \forall h \in \mathcal{H},\ \forall i \in \mathbb{N}$. Let $z \in \mathcal{Z}$, it is clear that $\langle z/x_i\rangle = \langle \Omega^*z/y_i\rangle$. Which implies that

$$\begin{aligned}
\sum_{i=1}^{\infty} \|\langle z/x_i\rangle\|_{\mathcal{H}}^2 &= \sum_{i=1}^{\infty} \|\langle \Omega^*z/y_i\rangle\|_{\mathcal{H}}^2 \\
&= \sum_{i=1}^{\infty} \langle \langle \Omega^*z/y_i\rangle\,, \langle \Omega^*z/y_i\rangle \rangle_{\mathcal{H}} \\
&= \sum_{i=1}^{\infty} \langle \flat\left(\langle \Omega^*z/y_i\rangle\,, y_i\right), \Omega^*z\rangle_{\mathcal{Z}} \\
&= \|\Omega^*z\|_{\mathcal{Z}}^2.
\end{aligned}$$

First we have $\|\Omega^* z\|_{\mathcal{Z}} \leq \|\Omega\|\|z\|_{\mathcal{Z}}$. In the other hand, Ω is surjective, then by iii of Theorem 2.2, there exists $A > 0$ such that $\|\Omega^* z\| \geq A\|z\|_{\mathcal{Z}} \geq \dfrac{A}{\|K^*\|}\|K^* z\|$.

($\Leftarrow$) Conversely, suppose that $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ is a K-$\mathfrak{b}$-frame for $\mathrm{ran}(K)$ then by Theorem 3.12 the synthesis operator

$$T\left(\{h_i\}_{i\in\mathbb{N}}\right) = \sum_{i\in\mathbb{N}} \mathfrak{b}\left(h_i, x_i\right), \quad \forall h_i \in \mathcal{H}.$$

is well defined and surjective on $\mathrm{ran}(K)$. Now define $\tau : \mathrm{ran}(K) \to \ell^2(\mathcal{B})$ by $\tau(z) = \{\langle z/y_i\rangle\}_{i\in\mathbb{N}}$, $\forall z \in \mathrm{ran}(K)$. We have $\tau(\mathfrak{b}(h, y_j)) = \{\langle \mathfrak{b}(h, y_j)/y_i\rangle\}_{i\in\mathbb{N}} = \{\delta_{ij}h\}_{i\in\mathbb{N}}$. Now considere the operator $\Omega : \mathrm{ran}(K) \to \mathrm{ran}(K)$, such that $\Omega = T\tau$. It is easy to check that Ω is bounded and surjective and that

$$\begin{aligned}\Omega(\mathfrak{b}(h, y_i)) &= T(\{\delta_{ij}h\}_{i\in\mathbb{N}}) \\ &= \mathfrak{b}(h, x_i), \quad \forall h \in \mathcal{H}, \forall i \in \mathbb{N}.\end{aligned}$$

□

Theorem 3.25. *Suppose that $\{x_i\}_{i\in\mathbb{N}}$ is a $\mathfrak{b}$-orthonormal basis for $\mathcal{Z}$ and let $U : \mathcal{X} \to \mathcal{X}$ be a continuous bounded linear operator. If $\{x_i\}_{i\in\mathbb{N}}$ is a $\mathfrak{b}$-frame for $\mathcal{Z}$ with bounds A and B, then $\{Ux_i\}_{i\in\mathbb{N}}$ is a $(U^{\mathfrak{b}*})$-$\mathfrak{b}$-frame for $\mathcal{Z}$ with bounds $A' = A$ and $B' = B\|U^{\mathfrak{b}}\|^2$. Where $U^{\mathfrak{b}*}$ designs the adjoint operator of $U^{\mathfrak{b}}$ on $\mathcal{Z}$.*

Proof. $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ is a $\mathfrak{b}$-frame for $\mathcal{Z}$, then for all $z \in \mathcal{Z}$,

$$A\|z\|_{\mathcal{Z}}^2 \leq \sum_{i=1}^{\infty} \|\langle z/x_i\rangle\|_{\mathcal{H}}^2 \leq B\|z\|_{\mathcal{Z}}^2.$$

We have by Theorem 3.15

$$\sum_{i=1}^{\infty} \|\langle z/Ux_i\rangle\|_{\mathcal{H}}^2 = \sum_{i=1}^{\infty} \left\|\left\langle U^{\mathfrak{b}} z/x_i\right\rangle\right\|_{\mathcal{H}}^2$$

hence

$$A\left\|U^{\mathfrak{b}} z\right\|_{\mathcal{Z}}^2 \leq \sum_{i=1}^{\infty} \|\langle z/Ux_i\rangle\|_{\mathcal{H}}^2 \leq B\left\|U^{\mathfrak{b}} z\right\|_{\mathcal{Z}}^2. \tag{3.10}$$

which implies that

$$A\left\|\left(U^{\mathfrak{b}*}\right)^* z\right\|_{\mathcal{Z}}^2 \leq \sum_{i=1}^{\infty} \|\langle z/Ux_i\rangle\|_{\mathcal{H}}^2 \leq B\left\|U^{\mathfrak{b}}\right\| \|z\|_{\mathcal{Z}}^2, \quad \forall z \in \mathcal{Z}.$$

□

Corollary 3.26. *Suppose that $\{x_i\}_{i\in\mathbb{N}}$ is $\flat$-complete in $\mathcal{Z}$ and let $U : \mathcal{X} \to \mathcal{X}$ be a continuous bounded linear operator. Let $U^\flat$ be the $\flat$-adjoint operator of U in $\mathcal{Z}$ and suppose that it has a closed range. If $\{x_i\}_{i\in\mathbb{N}}$ is a $\flat$-frame for $\mathcal{Z}$, with bounds A and B, then $\{Ux_i\}_{i\in\mathbb{N}}$ is also a $\flat$-frame for $\mathcal{Z}$ with bounds $A' = \left\|U^{\flat\dagger}\right\|^{-2}$ and $B' = B\left\|U^\flat\right\|^2$. Where $U^{\flat\dagger}$ is the pseudo-inverse of $U^\flat$ on $\operatorname{ran}\left(U^\flat\right)$.*

Proof. Suppose that $\operatorname{ran}\left(U^\flat\right)$ is closed then

$$\exists U^{\flat\dagger} \in \mathcal{L}(\mathcal{Z}); U^\flat U^{\flat\dagger} z = z, \quad \forall z \in \operatorname{ran}\left(U^\flat\right),$$

so $\|z\|_{\mathcal{Z}} = \|U^\flat U^{\flat\dagger} z\|_{\mathcal{Z}}$, then we have for all $z \in \operatorname{ran}\left(U^\flat\right)$:

$$\|z\|_{\mathcal{Z}}^2 \leq \left\|U^\flat z\right\|_{\mathcal{Z}}^2 \left\|U^{\flat\dagger} z\right\|^2 \quad \Rightarrow \quad \left\|U^{\flat\dagger} z\right\|^{-2} \|z\|_{\mathcal{Z}}^2 \leq \left\|U^\flat z\right\|_{\mathcal{Z}}^2.$$

using (3.10) we prove the result. □

Theorem 3.27. *Let $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ be a $\flat$-orthonormal basis in $\mathcal{Z}$ and let $U : \mathcal{X} \to \mathcal{X}$ be a continuous bounded linear operator and let $K \in \mathcal{L}(\mathcal{Z})$. Suppose that $\{x_i\}_{i\in\mathbb{N}}$ is a K-$\flat$-frame for $\mathcal{Z}$, with bounds A and B, Then $\{Ux_i\}_{i\in\mathbb{N}}$ is a $(U^{\flat *}K)$-$\flat$-frame for $\mathcal{Z}$ with bounds $A' = A$ and $B' = B\|U^\flat\|^2$. where $U^{\flat *}$ designs the adjoint operator of $U^\flat$ in $\mathcal{Z}$.*

Proof. $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ is a K-$\flat$-frame for $\mathcal{Z}$ with bounds A and B, then

$$A\left\|K^* z\right\|_{\mathcal{Z}}^2 \leq \sum_{i=1}^{\infty} \|\langle z/x_i\rangle\|_{\mathcal{H}}^2 \leq B\|z\|_{\mathcal{Z}}^2.$$

We have by Theorem 3.15

$$\sum_{i=1}^{\infty} \|\langle z/Ux_i\rangle\|_{\mathcal{H}}^2 = \sum_{i=1}^{\infty} \left\|\left\langle U^\flat z/x_i\right\rangle\right\|_{\mathcal{H}}^2$$

hence

$$A\left\|K^*\left(U^\flat z\right)\right\|_{\mathcal{Z}}^2 \leq \sum_{i=1}^{\infty} \|\langle z/Ux_i\rangle\|_{\mathcal{H}}^2 \leq B\left\|U^\flat z\right\|_{\mathcal{Z}}^2.$$

which implies that

$$A\left\|\left(U^{\flat *}K\right)^* z\right\|_{\mathcal{Z}}^2 \leq \sum_{i=1}^{\infty} \|\langle z/Ux_i\rangle\|_{\mathcal{H}}^2 \leq B\left\|U^b\right\| \|z\|_{\mathcal{Z}}^2. \qquad \square$$

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References

1. R. Balan, *Stability theorems for Fourier frames and wavelet Riesz bases*, J. Fourier Anal. Appl., 3 (5) (1997), pp. 499–504.
2. P.G. Casazza and O. Christensen, *Approximation of the inverse frame operator and applications to Gabor frames*, J. Approx. Theory, 103 (2) (2000), pp. 338–356.
3. O. Christensen, *An introduction to frames and Riesz bases*, Birkhäuser, Switzerland, 2016.
4. I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., 27 (5) (1986), pp. 1271–1283.
5. I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, Pa, USA, 1992.
6. R.G. Douglas, *On majorization, factorization and range inclusion of operators on Hilbert space*, Proc. Am. Math. Soc., 17 (2) (1966), pp. 413–415.
7. R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Am. Math. Soc., 72 (1952), pp. 341–366.
8. L. Gavruta, *Frames for Operators*, Appl. Comput. Harmon. Anal., 32 (2012), pp. 139–144.
9. M. Ismailov, F. Guliyeva and Y. Nasibov, *On a Generalization of the Hilbert Frame Generated by the Bilinear Mapping*, J. Funct. Spaces, 2016 (2016).
10. S. Kabbaj and M. Rossafi, $*$-*K-g-Frames in Hilbert* $\mathcal{A}$-*Modules*, J. Linear Topol. Algebra, 7 (2018), pp. 63-71.
11. S. Kabbaj and M. Rossafi, $*$-*K-Operator Frame for* $End^*_{\mathcal{A}}(\mathcal{H})$, Asian-Eur. J. Math., 13 (3) (2020).
12. X. Xiao, Y. Zhu and L. Gavruta, *Some properties of K-frames in Hilbert spaces*, Result. Math., 63 (2013), pp. 1243-1255.

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