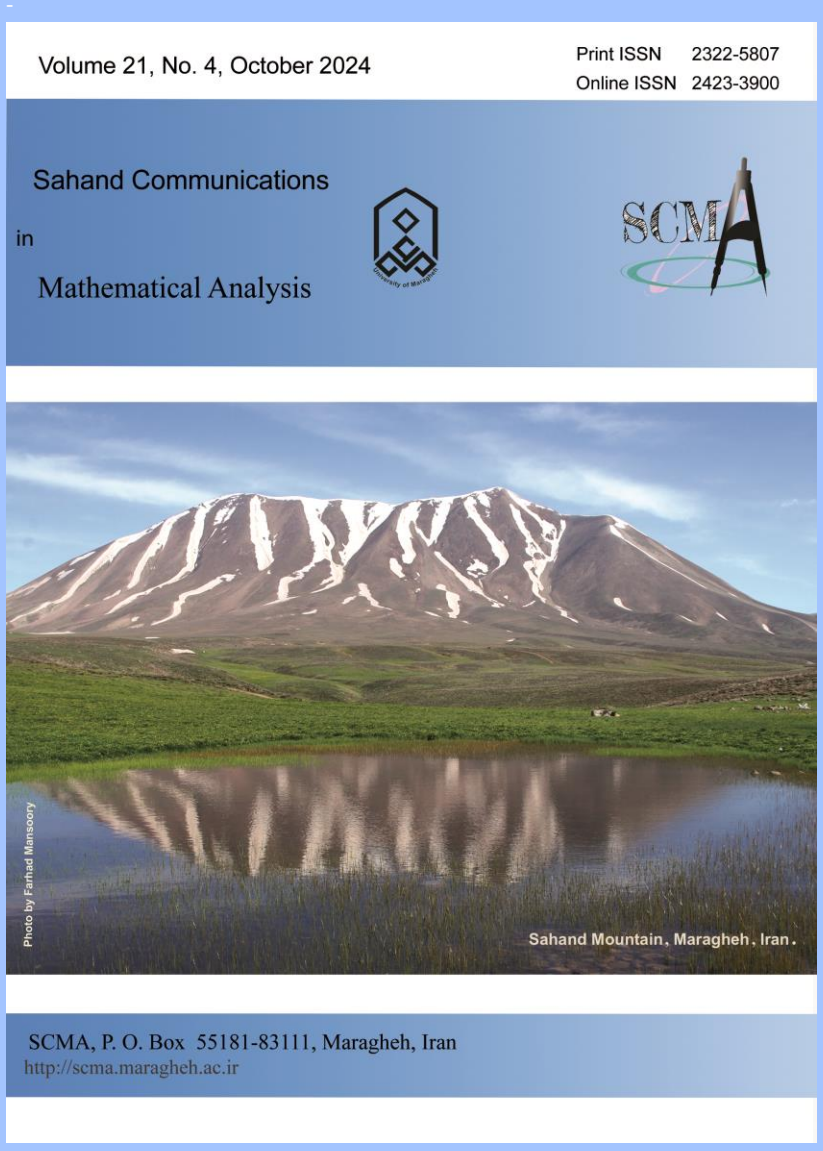
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A Version of the Hahn-Banach Theorem for R-Vector Spaces

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Abstract. Recently, R-metric spaces have been introduced to generalized metric spaces. This extension is based on the construction of a new universe with interesting properties. In this paper, some topological properties of R-metric spaces are studied and compared to the classical metric spaces via several examples. Also, some properties of a metric space with different relations are considered. Then, the elementary tools needed for the study of two important theorems of functional analysis are presented. For example, R-sequentially bounded sets, R-bounded sets, R-sequentially bounded functions and R-bounded functions are introduced in R-metric spaces. Moreover, a condition is given under which an R-continuous function is R-sequentially bounded. Finally, variants of the Heine–Borel theorem and the Hahn–Banach theorem are proved for R-metric spaces and R-vector spaces, respectively.

1. Introduction

Metric spaces, which constitute a class of topological spaces, play an important role in various branches of applied mathematics. In [1], Fréchet introduced metric spaces for the first time. These spaces have been considered and have been extended differently [2, 5] and also used by many researchers in various fields of knowledge.

Recently, Eshaghi et al. [2] introduced a new class of metric spaces, namely, R-metric spaces. Such spaces have been investigated in many studies, for instance [2–4, 6, 7]. In this paper, we study some topological properties of R-metric spaces and functions which are defined on them. Two important theorems of functional analysis, the Heine–Borel theorem

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and the Hahn–Banach theorem, are considered for R-metric spaces and R-vector spaces, respectively.

The paper is arranged as follows. In Section 2, we explain the basic terms and notions of R-metric spaces. Also, to clarify the differences of these properties in R-metric spaces and metric spaces, we present some examples. Section 3 aims to prove a variant of the Heine–Borel theorem for R-metric spaces. In this section, we introduce R-bounded and R-sequentially bounded sets and investigate the way these are related to each other. An important part of our investigation is devoted to a study of functions defined on R-metric spaces. Section 4 defines some properties of such functions, including the properties of being R-continuous, R-preserving, R-bounded and R-sequentially bounded. Also, we find some conditions for relations and functions under which the properties of R-metric spaces presented in the previous sections are preserved. Finally, in Section 5, we prove a variant of the Hahn–Banach theorem.

2. Basic Topological Properties of R-Metric Spaces

In this section, we review the preliminaries of R-metric spaces presented in [2]. Also, we provide some examples to illustrate some differences between R-metric spaces and metric spaces.

Let (X, d) be a metric space and R be a binary relation on X. The triple (X, d, R), or briefly X, is called an R-metric space. For example, if $(\mathbb{R}, d)$ is a metric space and R is any binary relation on $\mathbb{R}$, then $(\mathbb{R}, d, R)$ is an R-metric space.

Throughout this paper, we use X and M to denote non-empty sets and by a relation on each of the sets we mean a binary relation.

Definition 2.1 ([2])**.** Let (X, d, R) be an R-metric space.

(i) A sequence $\{x_n\}$ in X is called an R-sequence if $x_n R x_{n+k}$ for all $n, k \in \mathbb{N}$.
(ii) We say that an R-sequence $\{x_n\}$ in X converges to $x \in X$ and we write $x_n \xrightarrow{R} x$, if for every $\epsilon > 0$ there exists an integer N such that $d(x_n, x) < \epsilon$ for all $n \geq N$.
(iii) A point $x \in X$ is said to be an R-limit point for $E \subseteq X$ if there exists an R-sequence $\{x_n\}$ in E such that $x_n \neq x$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{R} x$. The set of all R-limit points of E is denoted by E'^R and the R-closure of E is the set $\overline{E}^R = E \cup E'^R$.
(iv) A set $E \subseteq X$ is called R-closed if $E'^R \subseteq E$. Also, $E \subseteq X$ is called R-open if E^c is R-closed.
(v) A set $K \subseteq X$ is called R-compact if every R-sequence in K has a convergent subsequence in K.

The following examples reveal some differences between above concepts and similar concepts in the classical case.

Example 2.2. Consider the R-metric space $((0,1],|.|,\leq)$. Let $\{x_n\}$ be an R-sequence in $(0,1]$. Clearly, $\{x_n\}$ is an increasing sequence in $(0,1]$ and hence, $\{x_n\}$ is convergent and $x_n \nrightarrow 0$. This shows that $0 \notin E'^R$ and then, $(0,1]$ is R-closed. But, this is not a closed set.

Example 2.3. Suppose that $X = \mathbb{R}$ is equipped with the Euclidean metric and the relation $R := \geq$ and $K = [0,1)$. It is well-known that K is not compact. But K is R-compact. To see this, let $\{x_n\}$ be an R-sequence in K. It is clear that $\{x_n\}$ is decreasing and bounded and accordingly convergent. Therefore, K is R-compact.

The following proposition presents some elementary and useful results for R-open and R-closed sets in R-metric spaces.

Proposition 2.4. *If X is an R-metric space, then the following statements are true.*

(i) *The sets X and $\emptyset$ are both R-closed and R-open.*
(ii) *Every closed subset of X is R-closed.*
(iii) *Every open subset of X is R-open.*
(v) *The intersection of a finite collection of R-open sets is an R-open set.*
(vi) *The union of an arbitrary collection of R-open sets is an R-open set.*

Proof. The proofs of (i), (ii) and (iii) are straightforward. To prove (v), let $\{U_k\}_{k=1}^m$ be a finite collection of R-open subsets of X. Suppose that x is an R-limit point of $W^c = (\bigcap_{k=1}^m U_k)^c$. Then, there exists an R-sequence $\{x_n\}$ in $W^c \setminus \{x\}$ such that $x_n \xrightarrow{R} x$. It can be easily shown that there exist $1 \leq j \leq m$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ in $U_j^c \setminus \{x\}$ such that $x_{n_k} \xrightarrow{R} x$. Then, x is an R-limit point for U_j^c and $x \in U_j^c$, because U_j^c is R-closed, hence $x \in W^c$. Thus, we conclude that W is R-open.

To prove (vi), let $\{U_j\}_{j\in J}$ be an arbitrary family of R-open sets. Let $U = \bigcup_j U_j$ and x be an R-limit point of $U^c = \bigcap_j U_j^c$. Then, there exists an R-sequence $\{x_n\}$ of $U^c \setminus \{x\} \subseteq U_j^c$ for every j such that $x_n \xrightarrow{R} x$. Therefore, for each $j \in J$, x is an R-limit point of U_j^c. Since every U_j^c is R-closed, $x \in U_j^c$ and so, $x \in U^c$. Thus, U is R-open. □

The following examples show that the conditions given in parts (v) and (vi) of Proposition 2.4 are necessary.

Example 2.5. Consider $(\mathbb{R}, |.|)$ and for $n \in \mathbb{N}$, $U_n = \left[0, 1 - \frac{1}{n}\right]$. By Proposition 2.4, each U_n is R-closed for every relation R on $\mathbb{R}$. Now, set $R := \leq$. Then, $[0, 1) = \cup_{n\in\mathbb{N}} U_n$ is not an R-closed set.

The last example shows that the union of an arbitrary collection of R-closed sets is not necessarily R-closed. Moreover, the intersection of an arbitrary collection of R-open sets is not necessarily R-open.

Example 2.6. Consider the set of open sets $A_n = \left(0, 1 + \frac{1}{n}\right)$ in the metric space $(\mathbb{R}, |.|)$. By Proposition 2.4, each A_n is R-open for every relation R on $\mathbb{R}$. But, $\cap_n A_n = (0, 1]$ is not R-open when $R := \geq$, because the complement of $(0, 1]$, $(0, 1]^c = (-\infty, 0] \cup (1, +\infty)$, is not R-closed.

The following proposition presents a useful property of R-open sets that will be needed in the sequel.

Proposition 2.7. *Let X be an R-metric space, $\{x_n\} \subseteq X$ be an R-sequence and $x \in X$. If for every R-open set W containing x there exists an integer N such that for every $n \geq N$, $x_n \in W$, then $\{x_n\}$ converges to x.*

Proof. Assume that $\{x_n\}$ does not converge to x. Then, there exists an open set W such that for any $m \in \mathbb{N}$, there exists $n_m \geq n_{m-1}$ with $x_{n_m} \notin W$. Since W is an open set, it is an R-open set. But, this contradicts the assumption. □

Definition 2.8. Let X be a metric space and, R_1 and R_2 be relations on X. We write $R_1 \subseteq R_2$ if xR_2y for every $x, y \in X$ such that xR_1y.

In the following proposition, we consider the concepts of being closed and compact in a metric space concerning relations R_1 and R_2 satisfying $R_1 \subseteq R_2$.

Proposition 2.9. *Let X be a metric space and, R_1 and R_2 be relations on X such that $R_1 \subseteq R_2$. Then, the following statements are true.*

(i) *For $A \subset X$, the inclusion $\overline{A}^{R_1} \subseteq \overline{A}^{R_2}$ holds.*
(ii) *Every R_2-closed, R_2-open and R_2-compact sets are R_1-closed, R_1-open and R_1-compact, respectively.*

Proof. The proof is straightforward by the fact that every R_1-sequence is an R_2-sequence. □

The converse of the above proposition is not true. This is the content of our next example.

Example 2.10. Suppose that $X = \mathbb{R}$ and $A = (-1, 1]$. For $x, y \in \mathbb{R}$, write xR_1y if and only if $x \geq 0$ and $y \geq 0$ and also xR_2y if and only if $x \leq 0$ and $y \leq 0$, or $x \geq 0$ and $y \geq 0$. Clearly, $R_1 \subseteq R_2$. It is easy to show that A is R_1-closed and R_1-compact, but it is not R_2-compact and R_2-closed.

Next, we define the concept of R-connectedness and discuss its related notions.

Definition 2.11. Let C and D be subsets of an R-metric space X. We say that C and D are R-separated if $C \cap \overline{D}^R = D \cap \overline{C}^R = \emptyset$. Also, the pair (C, D) is said to be a separated pair. If the set $E \subseteq X$ is not an union of two non-empty separated sets (or the union of the elements of a separated pair), then E is called an R-connected set; otherwise, E is R-disconnected.

Remark 2.12. In an R-metric space, every R-connected set is a connected set. In fact, every pair of R-separated sets is a pair of separated sets and so, every disconnected set is an R-disconnected set.

Example 2.13. Consider $E = (1, 3) = (1, 2] \cup (2, 3)$ as a subset of $\mathbb{R}$ and let $R :=\leq$. Since $\overline{(2, 3)}^R = (2, 3]$, the set E is connected, but it is not R-connected.

Theorem 2.14. *Let R_1 and R_2 be relations on a metric space X with $R_1 \subseteq R_2$. Then, any R_1-connected set is also R_2-connected.*

Proof. Let E be an R_1-connected set and assume that E is not R_2-connected. Suppose that $E = A \cup B$, where $\overline{A}^{R_2} \cap B = A \cap \overline{B}^{R_2} = \emptyset$. By Remark 2.9, $\overline{A}^{R_1} \cap B \subseteq \overline{A}^{R_2} \cap B = \emptyset$, $A \cap \overline{B}^{R_1} \subseteq A \cap \overline{B}^{R_2} = \emptyset$, which is a contradiction. Thus, E is an R_2-connected set. □

Example 2.15. Let $(\mathbb{R}, |\ |)$ be given with relations R_1, R_2 defined by $xR_1y \Leftrightarrow x > 0, y > 0$ and $x \leq y$, $xR_2y \Leftrightarrow x > 0, y > 0$. It is clear that $R_1 \subseteq R_2$. Set $E = [1, 3]$. We see that $[1, 3]$ is R_2-connected, but E is not an R_1-connected set because $[1, 3] = [1, 2] \cup (2, 3]$.

3. A Version of Heine–Borel Theorem For R-Metric Spaces

The Heine–Borel theorem is one of the important theorems of functional analysis. The aim of this section is to prove a variant of the Heine–Borel theorem for R-metric spaces. To do so, we first introduce R-bounded sets and R-sequentially bounded sets and then study such sets in R-matric spaces.

Definition 3.1. Let X be an R-metric space and $A \subseteq X$.

(i) We say that A is an R-bounded set if there exists $M > 0$ such that $d(x, y) \leq M$ for every $x, y \in A$ with xRy. We call M an R-bound of A.
(ii) We call that A is an R-sequentially bounded set if every R-sequence in A is bounded. Here, by the boundary we mean the classical concept of boundary in metric spaces.

In what follows, we discuss the way that the concepts of boundedness, R-boundedness, R-sequential boundedness and R-compactness are related to each other.

(i) Clearly, every R-bounded set is R-sequentially bounded.
(ii) Every R-bounded set is not necessarily a bounded set. See Example 3.2.
(iii) Every R-compact, R-sequentially bounded set is not necessarily an R-bounded set. See Example 3.3.
(v) Every bounded or R-sequentially bounded set is not necessarily an R-bounded set. See Example 3.4.

Example 3.2. Suppose that $X = \mathbb{R}$ is equipped with the Euclidean metric and $A = [0, +\infty)$. Let $xRy \Leftrightarrow d(x, y) < 1$. Then, A is R-bounded but is not bounded.

Example 3.3. Suppose that $(\mathbb{R}\backslash\{0\}, |.|, R)$ is an R-metric space such that $xRy \Leftrightarrow y = \frac{1}{x}$. Then, X is not R-bounded because no integer M exists such that $d(x, y) = \left|x - \frac{1}{x}\right| = \left|\frac{x^2-1}{x}\right| \leq M$ for every xRy. Now, suppose that $\{x_n\}$ is an $R-$sequence. Then, $x_n = 1$ or $x_n = -1$ for any $n \in \mathbb{N}$. This shows that every R-sequence in this space is the constant sequence $\{1\}$ or $\{-1\}$. Thus, every R-sequence is bounded and convergent and so, $\mathbb{R}\backslash\{0\}$ is R-compact and R-sequentially bounded, but it is not R-bounded.

Example 3.4. Suppose that $([0, \infty), |.|, R)$ is an R-metric space such that $xRy \Leftrightarrow x > y$. The set $[0, \infty)$ is not R-bounded, because no integer M exists such that $|x - y| \leq M$ for any $x > y$. Assume that $\{x_n\}$ is an R-sequence. Then, $\{x_n\}$ is bounded because $|x_m - x_n| < 2x_1$ and hence, $[0, \infty)$ is R-sequentially bounded.

Now, we are ready to present some relations that allow us to conclude the boundedness of an R-bounded set.

Theorem 3.5. *Suppose that X is an R-metric space and $A \subseteq X$ is an R-bounded set such that, there exists $a_0 \in A$ with a_0Ra or aRa_0 for every $a \in A$. Then, A is bounded.*

Proof. Let $M > 0$ be an R-bound for A. Then, for every $a, b \in A$, we obtain the following inequalities by the property of R:

$$d(a, b) \leq d(a, a_0) + d(a_0, b) \leq 2M.$$

This means that A is bounded. □

Next, we try to study the properties of R-compact sets and present equivalent formulations of R-compactness for some relations.

Theorem 3.6. *Every R-compact set is R-sequentially bounded and R-closed.*

Proof. Assume that A is an R-compact set that is not R-sequentially bounded. Then, A has an unbounded R-sequence $\{x_n\}$. Let $x_k \in \{x_n\}$ be an arbitrary element of this sequence. Then, for $m = 1$ there exists $x_{k_1} \in \{x_n\}$ such that $k_1 > k$ and $d(x_{k_1}, x_k) > 1$. Also, for $m = 2$, there exists $x_{k_2} \in \{x_n\}$ with $k_2 > k_1$ and $d(x_{k_2}, x_k) > 2$. By induction, for any $m \in \mathbb{N}$ there exists x_{k_m} such that $k_m > k_{m-1}$ and $d(x_{k_m}, x_k) > m$. On the other hand, since A is R-compact, the R-sequence $\{x_{k_m}\}$ has a convergent subsequence. For the sake of convenience, we denote the subsequence again by $\{x_{k_m}\}$. The sequence $\{x_{k_m}\}$ is bounded because it is convergent. This contradicts the structure of $\{x_{k_m}\}$. Therefore, $\{x_n\}$ is R-sequentially bounded. The proof of the second part can be found in [2]. □

Theorem 3.7. *Let X be an R-metric space such that R is a symmetric and transitive relation. Suppose that $A \subseteq X$ is an R-compact set which has an element $a_0 \in A$ with the property that $a_0 R a$, for every $a \in A$. Then, A is bounded and so, R-bounded.*

Proof. It is enough to prove the existence of a $M > 0$ such that $d(a_0, a) \leq M$ for every $a \in A$. Otherwise, a sequence $\{x_m\}$ can be found such that $d(x_m, a_0) > m$, for $m \in \mathbb{N}$. The sequence $\{x_m\}$ is an R-sequence because R is symmetric and transitive and $a_0 R x_m$ for every m. By using the R-compactness of A, a convergent R-subsequence $\{x_{m_k}\}$ of $\{x_m\}$ can be found such that $x_{m_k} \xrightarrow{R} b$, for some $b \in A$. Therefore,

$$\exists N \in \mathbb{N} \ ; \ \forall n \geq N, \quad d(x_{m_n}, b) < 1.$$

Now,

$$d(a_0, b) > d(a_0, x_{m_n}) - d(x_{m_n}, b) > m_n - 1, \quad \forall m_n.$$

This is a contradiction, which completes the proof. □

Remark 3.8. Suppose that R is an arbitrary relation on $\mathbb{R}^k$. If $A \subseteq \mathbb{R}^k$ is R-bounded and R-closed, then every R-sequence in A is bounded. So, by the Bolzano–Weierstrass theorem, every sequence in A has a convergent subsequence. Therefore, A is R-compact.

4. Some Properties of Functions Defined on R-Metric Spaces

This section investigates some properties of functions defined on R-metric spaces. For this study, some notations are needed that are presented in the sequel. We use $\mathcal{F}_{R,S}(X,Y)$ to denote the set of all functions from an R-metric space X to an S-metric space Y. Also, we write $\mathcal{P}_{R,S}(X,Y)$ for the set of all (R,S)-preserving functions from (X,R) to (Y,S). Moreover, $\mathcal{C}_R(X,Y)$ is the set of all R-continuous functions between (X,R) and Y and so, $\mathcal{C}_{R,S}(X,Y) = \mathcal{C}_R(X,Y) \cap \mathcal{P}_{R,S}(X,Y)$. If R and R' are relations on X, then we write $\mathcal{F}_{R,R'}(X) := \mathcal{F}_{R,R'}(X,X)$, $\mathcal{F}_R(X) := \mathcal{F}_{R,R}(X,X)$, $\mathcal{P}_{R,R'}(X) := \mathcal{P}_{R,R'}(X,X)$, $\mathcal{P}_R(X) := \mathcal{P}_{R,R}(X,X)$ and $\mathcal{C}_R(X) := \mathcal{C}_R(X,X)$.

Definition 4.1 ([2])**.** Let X be an R-metric space, Y be an S-metric space and $f : X \to Y$ be a map.

(i) The map f is called (R,S)-preserving if $xR\,y$ implies $f(x)S\,f(y)$ for all $x, y \in X$.

(ii) The map f is said to be R-continuous at $x \in X$ if for every R-sequence $\{x_n\}$ in X with $x_n \xrightarrow{R} x$, it follows that $f(x_n) \to f(x)$. Also, f is said to be an R-continuous map on X if f is R-continuous at each $x \in X$.

Remark 4.2. If R is an equality relation on X, then every R-sequence is a constant sequence. Also, every function $f : X \to X$ is R-continuous and R-preserving.

It is easy to see that every continuous function is R-continuous, but not conversely. The following example illustrates the latter fact.

Example 4.3. Consider the R-metric space $(\mathbb{R}, |.|, R)$, where xRy if and only if $x, y \in \left(n + \frac{2}{3}, n + \frac{4}{5}\right)$ for some $n \in \mathbb{Z}$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = [x]$. It is well-known that this function is not continuous. But, it is R-continuous. To see this, let $\{x_n\}$ be a convergent R-sequence in $\mathbb{R}$ such that $x_n \xrightarrow{R} x$. Since $\{x_n\}$ is an R-sequence, there exists $m \in \mathbb{Z}$ such that $x, x_n \in \left[m + \frac{2}{3}, m + \frac{4}{5}\right]$ for any $n \in \mathbb{N}$ and then, $f(x_n) = f(x) = m$ for every $n \in \mathbb{N}$. This shows that $f \in \mathcal{C}_R(\mathbb{R})$.

Here, the aim is to examine the way that the continuity of a function defined on a metric space is related to different relations on the space.

Proposition 4.4. *Let X be a metric space and, R_1 and R_2 be relations on X such that $R_1 \subseteq R_2$. If $f : X \to X$ is an R_2-continuous function on X, then f is R_1-continuous.*

Proof. Assume that $\{x_n\}$ is an R_1-sequence in X with $x_n \xrightarrow{R_1} x$. Then $x_n \xrightarrow{R_2} x$, because $R_1 \subseteq R_2$. On the other hand, f is an R_2-continuous function. Thus, $f(x_n) \to f(x)$. Therefore, f is R_1-continuous on X. □

Theorem 4.5. *Let (X, R), (Y, S) and (Z, T) be metric spaces with their respective relations. If $f \in \mathcal{C}_{R,S}(X, Y)$ and $g \in \mathcal{C}_{S,T}(f(X), Z)$, then $g \circ f \in \mathcal{C}_{R,T}(X, Z)$.*

Proof. By the given hypotheses, $f : X \to Y$ and $g : f(X) \to Z$ are (R, S)-preserving and (S, T)-preserving functions, respectively. It is evident that $g \circ f : X \to Z$ is (R, T)-preserving. Taking an R-sequence $\{x_n\}$ in X such that $x_n \xrightarrow{R} x$ and given that f and g are (R, S)-preserving and (S, T)-preserving, respectively and also f and g are R-continuous and $S-$continuous functions, as per Definition 4.1, we have $f(x_n) \xrightarrow{R} f(x)$ and $g(f(x_n)) \xrightarrow{R} g(f(x))$. Therefore, $g \circ f$ is an R-continuous function and consequently, $g \circ f \in \mathcal{C}_{R,T}(X, Z)$. □

Similar to the classical case, we obtain equivalent formulations of (R, R')-continuity. The following theorem and corollary present such formulations.

Theorem 4.6. *Let X be an R-metric space and Y be an S-metric space. Also, assume that $f \in \mathcal{P}_{R,S}(X, Y)$. Then, $f \in \mathcal{C}_{R,S}(X, Y)$ if and only if $f^{-1}(V)$ is an R-open subset of X for every S-open subset V of Y.*

Proof. Suppose that V is an S-open subset of Y. Then, V^c is S-closed. We prove that the set $\big(f^{-1}(V)\big)^c = f^{-1}(V^c)$ is R-closed. Let x_0 be an R-limit point of $f^{-1}(V^c)$. Then, there exists an R-sequence $\{x_n\} \subseteq f^{-1}(V^c)$ such that $x_n \neq x_0$, for $n \in \mathbb{N}$ and $x_n \xrightarrow{R} x_0$. Since f is R-continuous, $f(x_n) \to f(x_0)$. But, $\{f(x_n)\} \subseteq V^c$ and $\{f(x_n)\}$ is an S-sequence, because $f \in \mathcal{P}_{R,S}(X, Y)$. Therefore, we obtain $f(x_0) \in V^c$ and hence, $x_0 \in f^{-1}(V^c)$. So, $f^{-1}(V^c)$ is R-closed.

Conversely, let $\{x_n\} \subseteq X$ be an R-sequence with $x_n \xrightarrow{R} x_0$ for some $x_0 \in X$, and let $V \subseteq Y$ be an S-open set including $f(x_0)$. Since $f^{-1}(V)$ is an R-open set and $x_0 \in f^{-1}(V)$, there exists an integer N such that for any $n > N$, $\ x_n \in f^{-1}(V)$, by [2, Theorem 2.4]. Hence for any $n > N$, $\ f(x_n) \in V$. Therefore, $f(x_n) \to f(x_0)$, by Proposition 2.7 and this completes the proof. □

Corollary 4.7. *Let X be an R-metric space and Y be an S-metric space. Also, assume that $f \in \mathcal{P}_{R,S}(X, Y)$. Then, $f \in \mathcal{C}_{R,S}(X, Y)$ if and only if $f^{-1}(W)$ is R-closed for every S-closed set W.*

An important task in the classical theory of metric spaces is to characterize those functions that preserve a certain topological property. The

next theorem presents a condition under which a function defined between R-metric spaces transforms an R-connected set to an R-connected set.

Theorem 4.8. *Suppose that $f \in \mathcal{P}_R(X) \cap C_R(X)$, and E is an R-connected set. Then $f(E)$ is also an R-connected set.*

Proof. Suppose that $f(E)$ is R-disconnected and (A, B) is a separated pair for $f(E)$. Now, set $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$. Hence, $E = G \cup H$ and $G, H \neq \emptyset$. Therefore, $G \subseteq f^{-1}\left(\overline{A}^R\right)$ and $H \subseteq f^{-1}(B)$. It is clear that $E = G \cup H$ and, G and H are disjoint. By Corollary 4.7, we obtain the inclusion $\overline{G}^R \subseteq f^{-1}\left(\overline{A}^R\right)$ and then, $f\left(\overline{G}^R\right) \cap f(H) \subseteq \overline{A}^R \cap B = \emptyset$. Now, $\overline{G}^R \cap H = \emptyset$ and therefore, (G, H) is a separated pair for E. But, this is a contradiction. Thus, $f(E)$ is R-connected. □

The following theorem considers those operators that preserve the property of R-compactness.

Theorem 4.9. *Let X be an R-metric space and Y be an S-metric space. Also, suppose that $f \in \mathcal{C}_{R,S}(X, Y)$ is injective and $f^{-1} \in \mathcal{P}_{S,R}(f(X), X)$. If $E \subseteq X$ is R-compact, then $f(E)$ is S-compact.*

Proof. Consider an arbitrary S-sequence $\{y_n\}$ in $f(E)$. For $n \in \mathbb{N}$, there exists $x_n \in E$ such that $f(x_n) = y_n$. The sequence $\{x_n\} = \left\{f^{-1}(y_n)\right\}$ is an R-sequence in the R-compact set E, because $f^{-1} \in \mathcal{P}_{S,R}(f(X), X)$. Thus, there exists a convergent R-subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ such that $x_{n_j} \xrightarrow{R} x$ for some $x \in E$. By $f \in \mathcal{C}_{R,S}(X, Y)$, we obtain $f(x_{n_j}) \to f(x)$. It is clear that $\{y_{n_j}\} = \left\{f\left(x_{n_j}\right)\right\}$ is a convergent S-subsequence and so, $f(E)$ is S-compact. □

Definition 4.10. Let X be an R-metric space and Y be an S-metric space. Also, assume that $f \in \mathcal{F}_{R,S}(X, Y)$. If $f(A)$ is S-bounded for every R-bounded subset A of X, then f is said to be an (R, S)-bounded function. Also, f is R-sequentially bounded if $\{f(x_n)\}$ is bounded for any bounded R-sequence $\{x_n\}$ of X. If $Y = X$ and $S = R$, then we speak of an R-bounded function instead of an (R, S)-bounded function.

The following example shows that a function may be R-bounded and R-sequentially bounded, but not R-continuous.

Example 4.11. Suppose that $((-\infty, 2], |.|, \leq)$ and $(\mathbb{R}, |.|, \leq)$ are R-metric spaces and $f : (-\infty, 2] \to \mathbb{R}$ is defined by $f(x) = [x]$. It is clear that f is R-bounded and R-sequentially bounded. But, f is not R-continuous, because for the $R-$sequence $\left\{1 - \frac{1}{n}\right\}$ we can write $1 - \frac{1}{n} \xrightarrow{R} 1$, but $f\left(1 - \frac{1}{n}\right) = 0 \nrightarrow 1$.

An R-sequentially bounded, R-continuous function is not necessarily R-bounded. This is the content of the next example.

Example 4.12. Let $X = [0, +\infty)$ be equipped with the Euclidean metric and for any $x, y \in X$, xRy if and only if $y = 2x$. Suppose that $f : [0, +\infty] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0, +\infty) \\ 1 & \text{if } x = 0. \end{cases}$$

By the definition of R, every R-sequence in X is a constant sequence and accordingly, f is R-sequentially bounded and R-continuous. Now, suppose that f is R-bounded. Then, for $A = \left[0, \frac{1}{2}\right]$ there exists $M \geq 0$ such that $|f(a) - f(a\prime)| \leq M$ for any $f(a), f(a\prime) \in f(A)$. Moreover, $n \in \mathbb{N}$ can be found such that $\frac{1}{nM} \in A$ and $n > 1$. Set $a := \frac{1}{2nM}$ and $a\prime := \frac{1}{nM}$. We see that $f(a\prime)Rf(a)$, but $|f(a) - f(a\prime)| = nM \geq M$. This is a contradiction. Therefore, f is not R-bounded.

In the following example, we show that a linear function may be R-bounded but not R-sequentially bounded and R-continuous.

Example 4.13. Let $\mathbb{R}[0, 1]$ be the set of all polynomials P such that $P(0) = 0$, with the supremum norm over $[0, 1]$. Consider the relation R and the function Λ on $\mathbb{R}[0, 1]$ as follows:

$$fRg \text{ if and only if } f(x) \geq g(x) \quad \forall x \in [0, 1] \text{ and } \Lambda(f) = f'.$$

We claim that Λ is a linear R-bounded function, but Λ is not R-sequentially bounded and Λ is not R-continuous. Let $A \subseteq \mathbb{R}[0, 1]$ be an R-bounded set. Then, there exists $M > 0$ such that for every $f, g \in A$ with fRg, the inequality $\|f - g\|_\infty \leq M$ holds. Suppose that $f', g' \in \Lambda(A)$ *and* $f'(x) \geq g'(x)$ *for all* $x \in [0, 1]$. Then for any $t \in (0, 1)$,

$$\int_0^t f'(x)dx \geq \int_0^t g'(x)dx.$$

Hence, $f(t) \geq g(t)$ for all $t \in [0, 1]$. Therefore, $\|f - g\|_\infty \leq M$, by the fact that A is R-bounded. Now, we show that $\|f' - g'\|_\infty \leq 2M$. If $\|f' - g'\|_\infty > 2M$, then there exists $z \in [0, 1]$ such that $|(f' - g')(z)| > 2M$. So $(f - g)'(z) > 2M$, because $f' \geq g'$. Then for any $0 \leq z < x \leq 1$,

$$\int_0^x (f - g)'(t)dt > \int_0^x 2Mdt.$$

Hence,

$$(f - g)(x) > 2Mx, \quad (0 \leq z < x \leq 1).$$

Therefore,

$$\|f - g\|_\infty \geq 2M.$$

This contradicts $\|f-g\|_\infty \leq M$. Thus, $\|f'-g'\|_\infty \leq 2M$ and Λ is R-bounded. But Λ is not R-sequentially bounded. To see this, consider the R-sequence $\{f_n\}$ such that $f_n(x)=x^n$. For every integer n, $\| f_n \|_\infty = 1$ holds and so, $\{f_n\}$ is a bounded R-sequence. But $\{\Lambda(f_n)\} = \left\{nx^{n-1}\right\}$ is not a bounded sequence. Furthermore, Λ is not R-continuous. Indeed, if for $n \in \mathbb{N}$ we define $f_n(x) = \frac{x^n}{n}$, then the R-sequence $\{f_n\}$ converges to 0, but $\{\Lambda(f_n)\}$ does not converge to 0, because $\{\Lambda(f_n)\}$ converges to

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 0 \\ 1 & \text{if } x = 1. \end{cases}$$

An R-normed space $(X, \|.\|, R)$ is a normed space $(X, \|.\|)$ over a field F with a relation R on X. In the following theorem, we show that an R-continuous linear function between R-normed spaces with some relations is R-sequentially bounded.

Theorem 4.14. *Let X be an R-normed space, Y be a normed space and $\Lambda : X \to Y$ be an R-continuous linear function. Assume that R is a relation on X such that $\left\{e\frac{x_n}{t_n}\right\}$ is R-sequence where $\{x_n\}$ is an R-sequence for any $\{t_n\} \subseteq \mathbb{R}^+$. Then Λ is R-sequentially bounded.*

Proof. Assume that Λ is not $R-$sequentially bounded. Then, there exists a bounded $R-$sequence $\{x_n\}$ such that $\{\Lambda(x_n)\}$ is not bounded. This shows that $\left\{\Lambda\left(\frac{x_n}{\|x_n\|}\right)\right\}$ is not bounded. So, for every $m \in \mathbb{N}$ there exists $x_{n_m} \in \{x_n\}$ with $\|(x_{n_m})\| \geq m \|x_{n_m}\|$. Since $\left\{\frac{x_{n_m}}{m\|x_{n_m}\|}\right\}$ is an R-sequence and

$$\frac{x_{n_m}}{m\|x_{n_m}\|} \to 0,$$

by the R-continuity of f we obtain

$$\Lambda\left(\frac{x_{n_m}}{m\|x_{n_m}\|}\right) \to 0.$$

This contradiction shows that Λ is R-sequentially bounded. □

In what follows, we present some results for R-continuous functions and R-sequentially bounded functions with different relations.

Remark 4.15. Let X be a metric space with relations R_1 and R_2 such that $R_1 \subseteq R_2$ and $f : X \to X$. If f is an R_2-sequentially bounded function or an R_2-continuous function, then f is also R_1-sequentially bounded or R_1-continuous, respectively. To see this, assume that $\{x_n\}$ is a bounded R_2-sequence. Then $\{x_n\}$ is a bounded R_1-sequence and so, $\{f(x_n)\}$ is bounded. Therefore, f is R_1-sequentially bounded. Also, if f is R_2-continuous, then f is R_1-continuous, too. In fact, if $\{x_n\}$ is an R_1-sequence with $x_n \xrightarrow{R_1} x$ for some $x \in X$, then $\{x_n\}$ is also an

R_2-sequence and hence, $\{f(x_n)\}$ converges to $f(x)$. Therefore, f is R_2-continuous.

Example 4.16. Consider $X = \mathbb{R}^2$ with relations R_1 and R_2 defined by $R_1 := \{(x, y) : x \geq 0, y \geq 0\}$ and $R_2 := \{(x, y) : y \geq 0\}$. It is clear that $R_1 \subseteq R_2$.

(1) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f(x,y) = \begin{cases} (x,y) & \text{if} \quad x \geq 0, y \geq 0 \\ \left(\frac{1}{\sqrt{x^2+y^2}}, -1\right) & O.W. \end{cases}$$

If $A \subseteq \mathbb{R}^2$ is R_1-bounded, then $f(A) = A$ is also R_1-bounded and so, f is R_1-bounded. Similarly, f is R_1-sequentially bounded. Set $B := \{(x, y) : x^2 + y^2 \leq 4\}$. Clearly, B is R_2-bounded, but $f(B)$ is not R_2-bounded and hence, f is not R_2-bounded. Also, f is not R_2-sequentially bounded because for the bounded R_2-sequence $\left\{\left(-\frac{1}{n}, \frac{1}{n}\right)\right\}$, the sequence $\left\{f\left(-\frac{1}{n}, \frac{1}{n}\right)\right\} = \left\{\left(-\frac{n}{\sqrt{2}}, -1\right)\right\}$ is not bounded.

Moreover, f is R_1-continuous. To see this, assume that $\{(x_n, y_n)\}$ is an R_1-sequence such that $(x_n, y_n) \xrightarrow{R_1} (a, b)$. Then the x_n's, y_n's, a and b are non-negative and we obtain the desired result as follows:

$$f((x_n, y_n)) = (x_n, y_n) \to (a, b) = f(a, b).$$

But, f is not R_2-continuous. In fact, for the R_2-sequence $\left\{\left(-\frac{1}{n}, \frac{1}{n}\right)\right\}$ we can write $\left(-\frac{1}{n}, \frac{1}{n}\right) \xrightarrow{R_2} (0, 0)$, but the sequence $\left\{f\left(-\frac{1}{n}, \frac{1}{n}\right)\right\} = \left\{\left(-\frac{n}{\sqrt{2}}, -1\right)\right\}$ does not converge to $f(0, 0) = (0, 0)$.

(2) Suppose that a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$g((x,y)) = \begin{cases} \left(\frac{1}{\sqrt{x^2+y^2}}, -1\right) & \text{if } x > 0, y > 0 \\ (x,y) & O.W. \end{cases}$$

By using methods similar to those used in part (1), we observe that the function g is R_2-bounded, but it is not R_1-bounded.

5. A Version of the Hahn–Banach Theorem for R-Vector Spaces

An $R-$vector space (X, R) is a vector space over a field F with a relation R on X. If R has the following properties, then we say that the actions of the vector space X are invariant under R.

(i) $(x_1+y_1)R\,(x_2+y_2)$, where x_1Rx_2, y_1Ry_2 and $x_i, y_i \in X$, $i = 1, 2$.

(ii) $(\alpha x)R\,(\beta y)$, where xRy for $x, y \in X$ and $\alpha, \beta \in F$.

Example 5.1. Let $\mathbb{R}[x]$ be the set of all polynomials over $\mathbb{R}$. Let fRg if and only if the degree of f equals to that of g or $f = 0$ or $g = 0$. It is clear that the actions of the vector space $\mathbb{R}[x]$ are invariant under R.

The Hahn-Banach theorem holds significant importance and applicability in mathematics. In this study, we aim to explore this theorem within the context of R-vector spaces, establishing specific conditions for the validity of a version of the Hahn-Banach theorem in these spaces.

Theorem 5.2. *Let X be a real R-vector space such that the actions of X are invariant under R, p be a seminorm on X and A be a subspace of X. If R is reflexive and $\Lambda : A \to \mathbb{R}$ is a linear functional such that $|\Lambda(x_n)| \leq p(x_n)$ for any R-sequence $\{x_n\} \subseteq A$, then there exists a linear functional $\overline{\Lambda} : X \to \mathbb{R}$ such that $|\overline{\Lambda}(x_n)| \leq p(x_n)$ for any R-sequence $\{x_n\} \subseteq X$ and $\overline{\Lambda}\big|_A = \Lambda$.*

Proof. Let $\mathcal{P}$ be the set of all linear functionals Θ on some vector subspace B such that $A \subseteq B$, $|\Theta(x_n)| \leq p(x_n)$ for any R-sequence $\{x_n\} \subset B$ and $\Theta\big|_A = f$. That $\Lambda \in \mathcal{P}$ implies $\mathcal{P} \neq \emptyset$. We introduce a relation R' on $\mathcal{P}$ as follows:

$$\Theta_1 R' \Theta_2 \text{ if and only if } D_{\Theta_1} \subseteq D_{\Theta_2} \text{ and } \Theta_2\big|_{D_{\Theta_1}} = \Theta_1.$$

The family $\mathcal{P}$ is partially ordered. We claim that every chain $\mathcal{Q} \subseteq \mathcal{P}$ has an upper bound. To see this, assume that $\mathcal{Q}$ is an arbitrary chain in $\mathcal{P}$. Define Γ on $\cup_{\Theta \in \mathcal{Q}} D_\Theta$ by $\Gamma(x) = \Theta(x)$ whenever $x \in D_\Theta$ for some $\Theta \in \mathcal{Q}$. Clearly, Γ is well-defined and $|\Gamma(x_n)| \leq p(x_n)$ for any R-sequence $\{x_n\} \subset \cup_{\Theta \in \mathcal{Q}} D_\Theta$. Hence, $\Gamma \in \mathcal{Q}$. Therefore, by Zorn's Lemma, $\mathcal{P}$ has a maximal element. Let $\overline{\Lambda} \in \mathcal{P}$ be the maximal element of $\mathcal{P}$. It is enough to show that $D_{\overline{\Lambda}} = X$. Suppose that $D_{\overline{\Lambda}} \neq X$ and $y \in X \backslash D_{\overline{\Lambda}}$. Set $X_1 = D_{\overline{\Lambda}} + \mathbb{R}y$. For arbitrary R-sequences $\{x_n\}$ and $\{y_n\}$ in $D_{\overline{\Lambda}}$, we can write

$$\begin{aligned} \overline{\Lambda}(x_n) - \overline{\Lambda}(y_n) &= \overline{\Lambda}(x_n - y_n) \\ &\leq p(x_n - y_n) \\ &\leq p(x_n - y) + p(y_n - y). \end{aligned}$$

Then, $\overline{\Lambda}(x_n) - p(x_n - y) \leq \overline{\Lambda}(y_n) + p(y_n - y)$ for arbitrary R-sequences $\{x_n\}$ and $\{y_n\}$ in $D_{\overline{\Lambda}}$. Therefore, there exists $a \in \mathbb{R}$ such that $a \in [m, M]$, where

$$m = \sup\left\{\overline{\Lambda}(x_n) - p(x_n - y) : R - \text{sequence} \{x_n\} \subset D_{\overline{\Lambda}}\right\},$$

and

$$M = \inf\left\{\overline{\Lambda}(y_n) - p(y_n - y) : R - \text{sequence} \{y_n\} \subset D_{\overline{\Lambda}}\right\}.$$

Now, define $\Gamma : X_1 \to \mathbb{R}$ by $\Gamma(d+ty) = \overline{\Lambda}(d) + ta$, for any $d \in D_{\overline{\Lambda}}$ and $t \in \mathbb{R}$. Assume that $\{x_n + t_n y\}$ is an arbitrary R-sequence in X_1, where $\{x_n\}$ is an R-sequence in $D_{\overline{\Lambda}}$ and $(t_n) \subseteq \mathbb{R}\backslash\{0\}$. Then,

$$\overline{\Lambda}\left(\frac{-x_n}{t_n}\right) - p\left(\frac{x_n}{t_n} + y\right) \leq a \leq \overline{\Lambda}\left(\frac{-x_n}{t_n}\right) + p\left(\frac{x_n}{t_n} + y\right).$$

If $t_n > 0$, then

$$\overline{\Lambda}(-x_n) - p(x_n + t_n y) \leq t_n a \leq \overline{\Lambda}(-x_n) + p(x_n + t_n y).$$

Hence,

$$-p(x_n + t_n y) \leq \overline{\Lambda}(x_n) + t_n a \leq p(x_n + t_n y).$$

Therefore,

$$|\Gamma(x_n + t_n y)| \leq p(x_n + t_n y). \tag{5.1}$$

If $t_n < 0$, the inequality (5.1) can be proved by a method similar to the one used in the case $t_n > 0$. Therefore, $\overline{\Lambda}$ is not a maximal element of $\mathcal{P}$, which is a contradiction. Hence, $D_{\overline{\Lambda}} = X$. □

The extension of the Hahn–Banach theorem to a complex R-vector space can be obtained similarly. Therefore, we omit the proof.

Theorem 5.3. *Let X be a complex R-vector space such that the actions of X are invariant under R, p be a seminorm on X and A be a subspace of X. If R is reflexive and $\Lambda : A \to \mathbb{C}$ is a linear functional such that $|\Lambda(x_n)| \leq p(x_n)$ for any R-sequence $\{x_n\} \subseteq A$, then there exists a linear functional $\overline{\Lambda} : X \to \mathbb{C}$ such that $|\overline{\Lambda}(x_n)| \leq p(x_n)$ for any R-sequence $\{x_n\} \subseteq X$ and $\overline{\Lambda}\big|_A = \Lambda$.*

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Notes on Contributor(s)

The authors approved the final manuscript. Also, the authors have contributed equally to this work and share the first authorship.

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