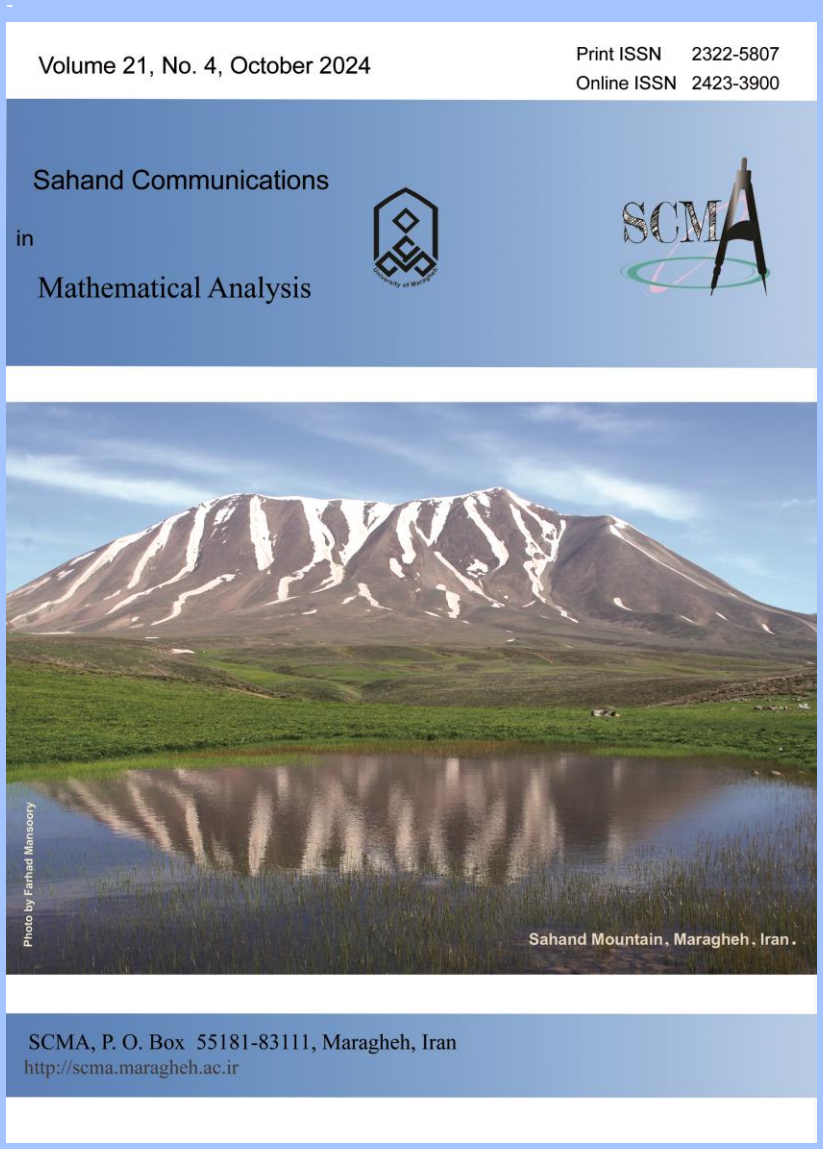
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Extremal Functions and Calderón's Reproducing Formula for the Laguerre-Bessel Two-Wavelet Transform

Ahmed Chana[1*] and Abdellatif Akhlidj[2]

Abstract. The main purpose of this paper is to study the wavelet transform associated with the Laguerre-Bessel transform and to provide an integral representation of the extremal function related to this transform. We then introduce the notion of a Laguerre-Bessel two-wavelet and present generalized versions of Parseval's, inversion and Calderón's type reproducing formulas related to this transform.

1. Introduction

The wavelet transform has a long story wich stared in 1984 with Mortel, a French petroleum engineer in connection with his study of seismic traces, the mathematical foundations were given by Grossman and Mortel in [3, 4]. Mortel defined a wavelet as a collection of functions constructed by using translation and dilatation of a single function ψ called the mother wavelet by :

$$\psi_{b,a}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$

where $a > 0$ is called the scaling parameter, which measure the degree of compression and $b \in \mathbb{R}$ is a translation parameter which determines the time location of the wavelet.

The theory of wavelet has wide applications, ranging from signal analysis in geophysics and acoustics to quantum theory and pure mathematics (see [2, 3, 11]). Recently, the theory of wavelets and continuous

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wavelets transforms has been extended and generalized in the context of differential-differences operators [1, 6, 8].

The classical Fourier transform in $\mathbb{R}^d$ can be defined in many ways; its most basic formulation is given by the integral transform

$$\mathcal{F}(f)(\lambda) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\langle \lambda, x\rangle} f(x) \mathrm{d}x.$$

A lot of attention has been given to various generalization of the classical Fourier transform. This paper focuses on the generalized Fourier transform associated with the Laguerre-Bessel operator called the Laguerre-Bessel transform. More precisely, we consider a system of partial differential operator Δ_1 and Δ_2 defined on $\mathbb{K} := [0, +\infty) \times [0, +\infty)$, by

$$\begin{cases} \Delta_1 := \frac{\partial^2}{\partial t^2} + \frac{2\alpha}{t}\frac{\partial}{\partial t}, & \alpha \geq 0,\ t > 0, \\ \\ \Delta_2 := \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\Delta_1, & x > 0. \end{cases}$$

The eigenfunctions of this system are related to the Bessel and Laguerre functions and they satisfy a product formula which permits the developements of a new harmonic analysis associated with these operators.

Wavelet analysis has attracted attention for its ability to analyze rapidly changing transient signals. Any application using the Fourier like transform can be formulated using wavelets to provide more accurately localized temporal and frequency information. The reason for the extension from one wavelet to two wavelets comes from the extra degree of flexibility in signal analysis and imaging when the localization operators are used as time-varying filters. This paper is an attempt to fill this gap by extending one wavelet to two wavelets in the Laguerre-Bessel setting. The remainder of this paper is arranged as follows, in section 2 we review the main results concerning the harmonic analysis associated with the Laguerre-Bessel wavelet transform. In section 3 we give an integral representation of extremal function related to the Laguerre-Bessel wavelet transform. In section 4 we introduce the notion of a Laguerre-Bessel two-wavelet and we prove a generalized Parseval's, inversion and Calderón's reproducing formulas related to this transform.

2. Preliminaries

In this section we set some notations and we recall some results in harmonic analysis related to the Laguerre-Bessel transform, for more details we refer the reader to [5]. In the following we denote by

- $\mathbb{K} := [0, +\infty) \times [0, +\infty)$ equipped with the weighted Lebesgue measure μ_α given by

$$d\mu_\alpha(x,t) := \frac{x^{2\alpha+1}t^{2\alpha}dxdt}{\pi\Gamma\left(\alpha+1\right)\Gamma(\alpha+\frac{1}{2})}, \quad \alpha \geq 0,$$

where Γ is the Gamma function.

- $L^p_\alpha(\mathbb{K}), 1 \leq p \leq \infty$, the space of measurable functions on $\mathbb{K}$, satisfying

$$\|f\|_{p,\mu_\alpha} := \left(\int_{\mathbb{K}} |f(x,t)|^p d\mu_\alpha(x,t)\right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_{\infty,\mu_\alpha} := \operatorname{ess}\sup_{(x,t)\in\mathbb{K}} |f(x,t)| < \infty, \quad p = \infty.$$

- $\mathcal{S}_*(\mathbb{K})$ the space of function $f : \mathbb{R}^2 \to \mathbb{C}$, even with respect to the first variable, C^∞ on $\mathbb{R}^2$ and rapidly decreasing together with their derivations.
- $\mathcal{L}^{(\alpha)}_m(x)$ is the Laguerre function defined on $[0,+\infty[$ by

$$\mathcal{L}^{(\alpha)}_m(x) := \frac{e^{-\frac{x}{2}}L^{(\alpha)}_m(x)}{L^{(\alpha)}_m(0)},$$

where $L^{(\alpha)}_m$ is the Laguerre polynomial of degree m and order α given by

$$L^{(\alpha)}_m(x) = \sum_{k=0}^{m} \frac{(-1)^k\Gamma(m+\alpha+1)}{\Gamma(k+\alpha+1)}\frac{x^k}{k!(m-k)!}.$$

- $\hat{\mathbb{K}} := [0,+\infty)\times\mathbb{N}$ equipped with weighted Lebesgue measure γ_α given by

$$d\gamma_\alpha(\lambda,m) = \frac{\lambda^{3\alpha+1}}{2^{2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)}L^{(\alpha)}_m(0)d\lambda\otimes\delta_m,$$

where δ_m is the Dirac measure at m and $d\lambda$ is the classical Lebesgue measure in $[0,+\infty)$.

- $L^p_\alpha\left(\hat{\mathbb{K}}\right)$ with $p \in [1,+\infty]$ the space of measurable functions on $\hat{\mathbb{K}}$ satisfying

$$\|g\|_{p,\gamma_\alpha} := \left(\int_{\hat{\mathbb{K}}} |g(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,$$
$$\|g\|_{\infty,\gamma_\alpha} := \operatorname{ess}\sup_{(\lambda,m)\in\hat{\mathbb{K}}} |g(\lambda,m)| < \infty, \quad p = \infty.$$

2.1. **The Eigenfunctions of the Partial Differential Operators Δ_1 and Δ_2.** For $(\lambda, m) \in \hat{\mathbb{K}}$ we consider the following Cauchy problem

$$(S): \begin{cases} \Delta_1(u) = -\lambda^2 u, \\ \Delta_2(u) = -2\lambda(2m+\alpha+1)u \\ u(0,0) = 1; \quad \frac{\partial u}{\partial x}(0,0) = \frac{\partial u}{\partial t}(0,0) = 0. \end{cases}$$

From [5], the Cauchy problem (S) admits a unique solution given by

$$\varphi_{\lambda,m}(x,t) = j_{\alpha-\frac{1}{2}}(\lambda t)\mathcal{L}_m^{(\alpha)}\left(\lambda x^2\right) \text{ for } (x,t) \in \mathbb{K} \text{ and } (\lambda, m) \in \hat{\mathbb{K}}$$

where j_α is the spherical Bessel function see [7]. The function $\varphi_{\lambda,m}$ is infinitely differentiable on $\mathbb{R}^2$, even with respect to each variable and we have the following important result

$$\sup_{(x,t)\in\mathbb{K}} |\varphi_{\lambda,m}(x,t)| = 1. \tag{2.1}$$

Furthermore this function is multiplicative on $\mathbb{K}$ in the sense

$$\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,s) = \int_{\mathbb{K}} \varphi_{\lambda,m}(z,v) K_\alpha[(x,t),(y,s),(z,v)] d\mu_\alpha(z,v) \tag{2.2}$$

where K_α is a positive kernel given explicitly in [5].

The product formula (2.2) permits to define a translation operator, a convolution product and to develop a new harmonic analysis associated to the operators Δ_1 and Δ_2.

2.2. **The Laguerre-Bessel Transform.**

Definition 2.1. The Laguerre-Bessel transform $\mathcal{F}_\alpha$ is defined on $L^1_\alpha(\mathbb{K})$ for $(\lambda, m) \in \hat{\mathbb{K}}$ by

$$\mathcal{F}_\alpha(f)(\lambda, m) = \int_{\mathbb{K}} \varphi_{\lambda,m}(x,t) f(x,t) d\mu_\alpha(x,t).$$

Some basic properties of this transform are as follows, for the proofs, we refer the reader to [5].

Proposition 2.2. (i) *For every $f \in L^1_\alpha(\mathbb{K})$, we have*

$$\|\mathcal{F}_\alpha(f)\|_{\infty,\gamma_\alpha} \leq \|f\|_{1,\mu_\alpha}. \tag{2.3}$$

(ii) *For $f \in \left(L^1_\alpha \cap L^2_\alpha\right)(\mathbb{K})$ such that $\mathcal{F}_\alpha(f) \in L^1_\alpha\left(\hat{\mathbb{K}}\right)$ we have*

$$f(x,t) = \int_{\hat{\mathbb{K}}} \varphi_{\lambda,m}(x,t)\mathcal{F}_\alpha(f)(\lambda,m) d\gamma_\alpha(\lambda,m), \ \ a.e\ (x,t) \in \mathbb{K}. \tag{2.4}$$

(iii) *(Parseval formula) For all* $f, g \in L^2_\alpha(\mathbb{K})$ *we have*

$$\langle f, g\rangle_\alpha = \langle \mathcal{F}_\alpha(f), \mathcal{F}_\alpha(g)\rangle_\gamma , \tag{2.5}$$

where

$$\langle f, g\rangle_\alpha = \int_{\mathbb{K}} f(x,t)\overline{g(x,t)}d\mu_\alpha(x,t)$$

$$\langle \mathcal{F}_\alpha(f), \mathcal{F}_\alpha(g)\rangle_\gamma = \int_{\hat{\mathbb{K}}} \mathcal{F}_\alpha(f)(\lambda,m)\overline{\mathcal{F}_\alpha(g)(\lambda,m)}d\gamma_\alpha(\lambda,m).$$

In particular we have

$$\|f\|_{2,\mu_\alpha} = \|\mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha} . \tag{2.6}$$

(iv) *(Plancherel theorem) The Laguerre-Bessel transform* $\mathcal{F}_\alpha$ *can be extended to an isometric isomorphism from* $L^2_\alpha(\mathbb{K})$ *into* $L^2_\alpha\left(\hat{\mathbb{K}}\right)$.

2.3. **The Translation Operator Associated with the Laguerre-Bessel Transform.** The product formula (2.2) permits to define the translation operator as follows

Definition 2.3. Let $(x,t),(y,s) \in \mathbb{K}$ and $f \in \mathcal{S}_*(\mathbb{K})$ the translation operator is defined by

$$\tau_\alpha^{(x,t)}(f)(y,s) = \int_{\mathbb{K}} f(z,v)K_\alpha[(x,t),(y,s),(z,v)]d\mu_\alpha(z,v).$$

The following proposition summarizes some properties of the Laguerre-Bessel translation operator see [5].

Proposition 2.4. *For all* $(x,t),(y,s) \in \mathbb{K}$, $f \in C_*(\mathbb{K})$ *we have:*

(i)

$$\tau_\alpha^{(x,t)}(f)(y,s) = \tau_\alpha^{(y,s)}(f)(x,t). \tag{2.7}$$

(ii)

$$\int_{\mathbb{K}} \tau_\alpha^{(x,t)}(f)(y,s)d\mu_\alpha(y,s) = \int_{\mathbb{K}} f(y,s)d\mu_\alpha(y,s). \tag{2.8}$$

(iii) *For* $f \in L^p_\alpha(\mathbb{K})$ *with* $p \in [1;+\infty]$ $\tau_\alpha^{(x,t)}(f) \in L^p_\alpha(\mathbb{K})$ *and we have*

$$\left\|\tau_\alpha^{(x,t)}(f)\right\|_{p,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha}. \tag{2.9}$$

(iv) *For* $f \in L^1_\alpha(\mathbb{K})$, $\tau_\alpha^{(x,t)}(f) \in L^1_\alpha(\mathbb{K})$ *and we have*

$$\mathcal{F}_\alpha\left(\tau_\alpha^{(x,t)}(f)\right)(\lambda,m) = \varphi_{\lambda,m}(x,t)\mathcal{F}_\alpha(f)(\lambda,m), \quad \forall(\lambda,m) \in \hat{\mathbb{K}}. \tag{2.10}$$

By using the generalized translation, we define the generalized convolution product of $f, g \in \mathcal{S}_*(\mathbb{K})$ by

$$(f *_\alpha g)(x,t) = \int_{\mathbb{K}} \tau_\alpha^{(x,t)}(f)(y,s) g(y,s) d\mu_\alpha(y,s).$$

This convolution is commutative, associative and it satisfies the following properties

Proposition 2.5. (i) *(Young's inequality) For all $p, q, r \in [1; +\infty]$ such that: $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and for all $f \in L^p_\alpha(\mathbb{K}), g \in L^q_\alpha(\mathbb{K})$ the function $f *_\alpha g$ belongs to the space $L^r_\alpha(\mathbb{K})$ and we have*

$$\|f *_\alpha g\|_{r,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha} \|g\|_{q,\mu_\alpha}. \tag{2.11}$$

(ii) *For $f, g \in L^2_\alpha(\mathbb{K})$ the function $f *_\alpha g$ belongs to $L^2_\alpha(\mathbb{K})$ if and only if the function $\mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g)$ belongs to $L^2_\alpha\left(\hat{\mathbb{K}}\right)$ and in this case we have*

$$\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g). \tag{2.12}$$

(iii) *For $f, g \in L^2_\alpha(\mathbb{K})$ then we have*

$$\int_{\mathbb{K}} |f *_\alpha g(x,t)|^2 d\mu_\alpha(x,t) = \int_{\hat{\mathbb{K}}} |\mathcal{F}_\alpha(f)(\lambda,m)|^2 |\mathcal{F}_\alpha(g)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m), \tag{2.13}$$

where both integrals are simultaneously finite or infinite.

2.4. **Wavelet Transform Associated with the Laguerre-Bessel Operator.** In this section we define and give some results for the wavelet transform associated with the Laguerre-Bessel transform.

Notation. we denote by

• $L^p_\alpha(\mathbb{R}_+ \times \mathbb{K})$, $1 \leq p \leq +\infty$ the space of measurable functions on $\mathbb{R}_+ \times \mathbb{K}$ satisfying

$$f_{p,\theta_\alpha} := \begin{cases} \left(\int_{\mathbb{R}_+ \times \mathbb{K}} |f(a,x,t)|^p \mathrm{d}\theta_\alpha(a,x,t)\right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty[, \\ \operatorname{ess\,sup}_{(a,x,t) \in \mathbb{R}_+ \times \mathbb{K}} |f(a,x,t)|, & \text{if } p = +\infty, \end{cases}$$

where θ_α is the measure defined on $\mathbb{R}_+ \times \mathbb{K}$ by

$$\mathrm{d}\theta_\alpha(a,x,t) := \frac{\mathrm{d}a \mathrm{d}\mu_\alpha(x,t)}{a},$$

for all $a > 0$, and $(x,t) \in \mathbb{K}$.

Definition 2.6. Let $g \in L^2(\mathbb{K})$, we say that g is a Laguerre-Bessel wavelet on $\mathbb{K}$ if for all $m \in \mathbb{N}$ we have

$$0 < C_g := \int_0^{+\infty} |\mathcal{F}_\alpha(g)(\lambda, m)|^2 \frac{\mathrm{d}\lambda}{\lambda} < \infty. \tag{2.14}$$

Example 2.7. The function $E_r, r > 0$, defined by

$$E_r(x,t) := \int_{\hat{\mathbb{K}}} \varphi_{\lambda,m}(x,t) \exp\left(-r\lambda^2 \left(m + \frac{\alpha+1}{2}\right)^2\right) d\gamma_\alpha(\lambda, m), \quad (x,t) \in \mathbb{K}$$

satisfies

$$\mathcal{F}_\alpha\left(E_r\right)(\lambda, m) = \exp\left(-r\lambda^2 \left(m + \frac{\alpha+1}{2}\right)^2\right).$$

The function

$$g(x,t) := -\frac{d}{dr} E_r(x,t), \quad (x,t) \in \mathbb{K},$$

is a Laguerre-Bessel wavelet on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$ and we have $C_g = \frac{1}{4r^2}$.

Let $a > 0$, we define the dilatation operator D_a of a measurable function g on $\mathbb{K}$ by

$$D_a(g)(x,t) := \frac{1}{a^{3\alpha+2}} g\left(\frac{x}{\sqrt{a}}, \frac{t}{a}\right), \quad \text{a.e on } \mathbb{K} \tag{2.15}$$

the dilatation operator D_a satisfies the following properties

- For all $g \in L^p_\alpha(\mathbb{K})$ we have $D_a(g) \in L^p_\alpha(\mathbb{K})$ and

$$\|D_a(g)\|_{p,\mu_\alpha} = a^{(\frac{1}{p}-1)(3\alpha+2)} \|g\|_{p,\mu_\alpha}. \tag{2.16}$$

- For all $g \in L^2(\mathbb{K})$ we have

$$\mathcal{F}_\alpha(D_a(g))(\lambda, m) = \mathcal{F}_\alpha(g)(a\lambda, m). \tag{2.17}$$

- Let g be a Laguerre-Bessel wavelet on $\mathbb{K}$ in $L^p(\mathbb{K})$ with $1 \leq p \leq \infty$, for all $a > 0$,$(x,t) \in \mathbb{K}$ we define the function

$$g_{a,x,t}(y,s) = \tau_\alpha^{(x,t)}\left(D_a(g)\right)(y,s). \tag{2.18}$$

- By the relation (2.9) and (2.16) we have $g_{a,x,t} \in L^p_\alpha(\mathbb{K})$ and

$$\|g_{a,x,t}\|_{p,\mu_\alpha} \leq a^{(\frac{1}{p}-1)(3\alpha+2)} \|g\|_{p,\mu_\alpha}. \tag{2.19}$$

Definition 2.8. Let g be a Laguerre-Bessel wavelet on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$ the continuous wavelet transform associated with the Laguerre-Bessel operator is defined for a function $f \in L^2_\alpha(\mathbb{K})$ and $(a,x,t) \in \mathbb{R}_+ \times \mathbb{K}$ by

$$\Phi^\alpha_g(f)(a,x,t) := \int_{\mathbb{K}} f(y,s)\overline{g_{a,x,t}(y,s)} d\mu_\alpha(y,s). \tag{2.20}$$

Remark 2.9. The Laguerre-Bessel wavelet transform can be written as

$$\Phi_g^\alpha(f)(a,x,t) = \left(\overline{D_a(g)} *_\alpha f\right)(x,t) = \langle f, g_{a,x,t}\rangle_\alpha. \tag{2.21}$$

Using the relation (2.11) we have the following result.

Proposition 2.10. *Let $p,q,r \in [1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$ and for $f \in L^p_\alpha(\mathbb{K})$,$g \in L^q_\alpha(\mathbb{K})$ and $a>0$ the function*

$$(x,t) \longmapsto \Phi_g^\alpha(f)(a,x,t),$$

belongs to $L^r(\mathbb{K})$ and we have

$$\left\|\Phi_g^\alpha(f)(a,.,.)\right\|_{r,\mu_\alpha} \leq a^{\left(\frac{1}{q}-1\right)(3\alpha+2)}\|f\|_{p,\mu_\alpha}\|g\|_{q,\mu_\alpha}.$$

Proof. By the relation (2.21) we have

$$\Phi_g^\alpha(f)(a,.,.) = \overline{D_a(g)} *_\alpha f,$$

using Young's inequality (2.11) we find that

$$\left\|\Phi_g^\alpha(f)(a,.,.)\right\|_{r,\mu_\alpha} = \left\|\overline{D_a(g)} *_\alpha f\right\|_{r,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha}\|D_a(g)\|_{q,\mu_\alpha},$$

the relation (2.21) gives the desired result.

From [5] we have the following results □

Theorem 2.11. *Let g be a Laguerre-Bessel wavelet on $\mathbb{K}$ in $L^2(\mathbb{K})$ then we have*

(i) *(Plancherel formula) For all $f \in L^2(\mathbb{K})$ we have*

$$\int_{\mathbb{R}_+\times\mathbb{K}} |\Phi_g^\alpha(f)(a,x,t)|^2 \mathrm{d}\theta_\alpha(a,x,t) = C_g \int_{\mathbb{K}} |f(y,s)|^2 \, d\mu_\alpha(y,s). \tag{2.22}$$

(ii) *(Inversion formula) for $f \in L^1_\alpha(\mathbb{K})$ such that $\mathcal{F}_\alpha(f) \in L^1_\alpha\left(\hat{\mathbb{K}}\right)$ we have*

$$f(x,t) = \frac{1}{C_g}\int_0^\infty \left(\int_{\mathbb{K}} \Phi_g^\alpha(f)(a,y,s) g_{a,x,t}(y,s) d\mu_\alpha(y,s)\right)\frac{da}{a}. \tag{2.23}$$

Remark 2.12. For $f,g \in L^2_\alpha(\mathbb{K})$, Hölder's inequality and the relation (2.22) we find that

$$\|\Phi_g^\alpha(f)\|_{\infty,\theta_\alpha} \leq \|f\|_{2,\mu_\alpha}\|g\|_{2,\mu_\alpha}. \tag{2.24}$$

We have the following Lieb's type inequality

Theorem 2.13. *Let g be a Laguerre-Bessel wavelet on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$, for all $f \in L^2_\alpha(\mathbb{K})$ the function $\Phi_g(f)$ belongs to $L^p_\alpha(\mathbb{R}_+ \times \mathbb{K})$ for all $p \in [2; +\infty[$ and we have*

$$\left\|\Phi^\alpha_g(f)\right\|_{p,\theta_\alpha} \leq (C_g)^{\frac{1}{p}} \|g\|^{1-\frac{2}{p}}_{2,\mu_\alpha} \|f\|_{2,\mu_\alpha}. \tag{2.25}$$

Proof. By using Plancherel's formula (2.6) we have

$$\|\Phi^\alpha_g(f)\|_{2,\theta_\alpha} = \sqrt{C_g}\|f\|_{2,\mu_\alpha}, \tag{2.26}$$

by the relations (2.24),(2.26) and the Riesz-Thorin interpolation theorem [10], we find that Φ^α_g extends uniquely as a continuous map from $L^2_\alpha(\mathbb{K})$ into $L^p_\alpha(\mathbb{R}_+ \times \mathbb{K})$ for all $p \in [2, +\infty[$ and we have

$$\left\|\Phi^\alpha_g(f)\right\|_{p,\theta_\alpha} \leq (C_g)^{\frac{1}{p}} \|g\|^{1-\frac{2}{p}}_{2,\mu_\alpha} \|f\|_{2,\mu_\alpha}. \qquad \square$$

The next result shows that for $g \in L^2_\alpha(\mathbb{K})$ a Laguerre-Bessel wavelet, the image space $\Phi^\alpha_g\left(L^2_\alpha(\mathbb{K})\right)$ is a reproducing kernel Hilbert space.

Proposition 2.14. *Let g be a Laguerre-Bessel wavelet on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$ then $\Phi^\alpha_g\left(L^2_\alpha(\mathbb{K})\right)$ is a reproducing kernel Hilbert space in $L^2_\alpha(\mathbb{R}_+ \times \mathbb{K})$ with kernel function given by*

$$\mathrm{K}_g((b,y,s),(a,x,t)) = \frac{1}{C_g}\left(D_b(g) * \overline{g_{a,x,t}}\right)(y,s),$$

furthermore this kernel is bounded and we have

$$|\mathrm{K}_g\left((b,y,s),(a,x,t)\right)| \leq \frac{\|g\|^2_{2,\mu_\alpha}}{C_g},$$

for $(a,x,t),(b,y,s) \in \mathbb{R}_+ \times \mathbb{K}$.

Proof. Using the relations (2.9) and (2.22) we find that

$$\begin{aligned}\Phi^\alpha_g(f)(a,x,t) &= \langle f, g_{a,x,t}\rangle_\alpha \\ &= \left\langle \Phi^\alpha_g(f), \mathrm{K}_g(.,(a,x,t))\right\rangle_{\theta_\alpha},\end{aligned}$$

where

$$\mathrm{K}_g(.,(a,x,t)) = \frac{1}{C_g}\overline{\Phi^\alpha_g(g_{a,x,t})(\cdot,\cdot,\cdot)},$$

the relation (2.21) give the desired result. On the other hand by the relations (2.11),(2.16) and (2.19) we find that

$$\begin{aligned}|\mathrm{K}_g\left((b,y,s),(a,x,t)\right)| &\leq \frac{1}{C_g}\|g\|_{2,\mu_\alpha}\|g_{a,x,t}\|_{2,\mu_\alpha} \\ &\leq \frac{\|g\|^2_{2,\mu_\alpha}}{C_g},\end{aligned}$$

which gives the result. $\square$

3. Extremal Function Associated with the Laguerre-Bessel Wavelet Transform

3.1. **Introduction.** Let

$$L : H_k \longrightarrow H$$

be a bounded linear operator of Hilbert spaces, the following problem is a classical and fundamental problem which is known as best approximate mean square norm problems

$$\inf_{f \in H_k} \|Lf - g\|_H, \tag{3.1}$$

where $g \in H$ is given. If there exists $f^* \in H_k$ which attains this infimum, the problem (3.1) is called solvable otherwise it is called unsolvable.

If H_k is a reproducing kernel Hilbert space admitting a reproducing kernel $K(p, q)$ on a set E then whether the problem (3.1) is solvable or not, the following problem

$$\inf_{f \in H_k} \left\{ r\|f\|^2_{H_k} + \|Lf - g\|^2_H \right\}, \tag{3.2}$$

is always solvable for all $(r > 0)$ and we obtain a method for obtain the extremal function $f^*_{r,g}$ of the smallest norm in H_k wich attains the infimum (3.2).

The problem (3.2) is called the Tikhonov regularization for the problem (3.1) and if the problem (3.1) is solvable then we have

$$f^*_{r,g} \longrightarrow f^*_g \text{ as } r \longrightarrow 0^+,$$

in H_k and f^*_g is the element of minimum norm which attaints the infimum (3.1). We start with the following fundamental theorem. For the proof we refer the reader to [9] and the references therein.

Theorem 3.1. *Let H_K be a Hilbert space admitting the reproducing kernel $K(p, q)$ on a set E and H an Hilbert space. Let $L : H_K \to H$ be a bounded linear operator. For $r > 0$, we introduce the inner product in H_K and we call it H_{K_r} as*

$$\langle f, g \rangle_{H_{K_r}} = r \langle f, g \rangle_{H_K} + \langle Lf, Lg \rangle_H .$$

Then we have

- *(i) H_{K_r} is a Hilbert space with the reproducing* kernel $K_r(p, q)$ *on E and satisfying the equation*

$$K_r(., q) = (rI + L^*L)\, K(., q), \tag{3.3}$$

where L^* is the adjoint operator of L and I is the identity operator.

- *(ii) For any $r > 0$ and for any g in H, the infimum*

$$\inf_{f \in H_K} \left\{ r\|f\|_{H_K}^2 + \|Lf - g\|_H^2 \right\},$$

is attained by a unique function $f_{r,g}^*$ in H_K and this extremal function is given by

$$f_{r,g}^*(q) = \langle g, LK_r(.,q) \rangle_H . \tag{3.4}$$

3.2. **Sobolev Type Spaces Associated with the Laguerre-Bessel Transform.** Let $s > 0$, we define the Sobolev spaces associated with the Laguerre-Bessel transform as

$$H_\alpha^s(\mathbb{K}) := \left\{ f \in L_\alpha^2(\mathbb{K}) / \left[1 + \lambda^2 \left(1 + m^2\right)\right]^{s/2} \mathcal{F}_\alpha(f) \in L_\alpha^2\left(\hat{\mathbb{K}}\right) \right\}.$$

The space $H_\alpha^s(\mathbb{K})$ endowed with the inner product

$$\langle f, g \rangle_{H_\alpha^s} := \int_{\hat{\mathbb{K}}} \left[1 + \lambda^2 \left(1 + m^2\right)\right]^s \mathcal{F}_\alpha(f)(\lambda, m) \overline{\mathcal{F}_\alpha(g)(\lambda, m)} d\gamma_\alpha(\lambda, m), \tag{3.5}$$

and the norm

$$\begin{aligned} \|f\|_{H_\alpha^s}^2 &:= \langle f, f \rangle_{H_\alpha^s} \\ &= \int_{\hat{\mathbb{K}}} \left[1 + \lambda^2 \left(1 + m^2\right)\right]^s |\mathcal{F}_\alpha(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m), \end{aligned} \tag{3.6}$$

is a Hilbert space.

Definition 3.2. Let h be a Laguerre-Bessel wavelet on $\mathbb{K}$ in $L_\alpha^2(\mathbb{K})$. We introduce the inner product in the Hilbert space $H_\alpha^s(\mathbb{K})$ for any fixed $r > 0$ by

$$\langle f, g \rangle_{H_{r,h}^s} := r \langle f, g \rangle_{H_\alpha^s} + \langle \Phi_h^\alpha(f), \Phi_h^\alpha(g) \rangle_{\theta_\alpha} . \tag{3.7}$$

The norm associated to this inner product is defined by

$$\|f\|_{H_{r,h}^s}^2 := r\|f\|_{H_\alpha^s}^2 + \|\Phi_h^\alpha(f)\|_{2,\theta_\alpha}^2. \tag{3.8}$$

The space $H_{r,h}^s(\mathbb{K})$ is the Hilbert space $H_\alpha^s(\mathbb{K})$ provided with the inner product given by the relation (3.7).

Remark 3.3. Simple calculations show that the norms $\|.\|_{H_\alpha^s}$ and $\|.\|_{H_{r,h}^s}$ are equivalent for $r > 0$.

We have the following result

Proposition 3.4. *Let $s > \frac{3\alpha+2}{2}$, h be a Laguerre-Bessel wavelet on $\mathbb{K}$ in $L_\alpha^2(\mathbb{K})$ and $r > 0$ then we have*

$$f \in H_{r,h}^s(\mathbb{K}) \quad \Rightarrow \quad \mathcal{F}_\alpha(f) \in L_\alpha^1\left(\hat{\mathbb{K}}\right). \tag{3.9}$$

Proof. Let $f \in H^s_{r,h}(\mathbb{K})$, by the relations (3.6)),(3.8),(2.25) and (2.6) we find that

(3.10)

$$\|f\|^2_{H^s_{r,h}} = \int_{\hat{\mathbb{K}}} \left[r(1+\lambda^2\left(1+m^2\right)\right)^s + C_h]|\mathcal{F}_\alpha(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)$$

by Hölder's inequality and the fact that $s > \frac{3\alpha+2}{2}$ we find that

$$\|\mathcal{F}_\alpha(f)\|_{1,\gamma_\alpha} \leq \|f\|_{H^s_{r,h}} \left(\int_{\hat{\mathbb{K}}} \frac{d\gamma_\alpha(\lambda,m)}{[r(1+\lambda^2\left(1+m^2\right))^s] + C_h}\right)^{\frac{1}{2}} < \infty$$

which give the result. □

Theorem 3.5. *Let $s > \frac{3\alpha+2}{2}$, h be a Laguerre-Bessel wavelet on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$ and $r > 0$ then the space $(H^s_{r,h}(\mathbb{K})), \langle , \rangle_{H^s_{r,h}})$ is a reproducing kernel Hilbert space with kernel given by*

$$\mathscr{K}_{r,h}[(x,t),(y,z)] = \int_{\hat{\mathbb{K}}} \frac{\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,z)}{r\left[1+\lambda^2\left(1+m^2\right)\right]^s + C_h} d\gamma_\alpha(\lambda,m) \tag{3.11}$$

that is for every $(y,z) \in \mathbb{K}$

(i) the function $(x,t) \to \mathscr{K}_{r,h}[(x,t),(y,z)] \in H^s_{r,h}(\mathbb{K})$.

(ii) For every $f \in H^s_{r,h}(\mathbb{K})$ and $(y,z) \in \mathbb{K}$ we have

$$f(y,z) = \langle f, \mathscr{K}_{r,h}[\cdot,(y,z)]\rangle_{H^s_{r,h}}.$$

Proof. Let $(y,z) \in \mathbb{K}$, by the fact $s > \frac{3\alpha+2}{2}$ and the relation (2.1) the function

$$(\lambda,m) \to \frac{\varphi_{\lambda,m}(y,z)}{r\left[1+\lambda^2\left(1+m^2\right)\right]^s + C_h}$$

belongs to $L^1_\alpha\left(\hat{\mathbb{K}}\right) \cap L^2_\alpha\left(\hat{\mathbb{K}}\right)$, by Plancherel's theorem for the Laguerre-Bessel transform there exist a unique function in $L^2_\alpha(\mathbb{K})$, which we denote by $\mathscr{K}_{r,h}[\cdot,(y,z)]$ such that

$$\mathcal{F}_\alpha\left(\mathscr{K}_{r,h}[\cdot,(y,z)]\right) = \frac{\varphi_{\lambda,m}(y,z)}{r\left[1+\lambda^2\left(1+m^2\right)\right]^s + C_h}, \tag{3.12}$$

by the relation (2.4) we find that

$$\mathscr{K}_{r,h}[(x,t),(y,z)] = \int_{\hat{\mathbb{K}}} \frac{\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,z)}{r\left[1+\lambda^2\left(1+m^2\right)\right]^s + C_h} d\gamma_\alpha(\lambda,m),$$

furthermore by the relations (2.1), (3.9), (3.12) we find that

$$\begin{aligned}\|\mathscr{K}_{r,h}[\cdot,(y,z)]\|^2_{H^s_{r,h}} &\leq \int_{\hat{\mathbb{K}}} \frac{d\gamma_\alpha(\lambda,m)}{r\left[1+\lambda^2\left(1+m^2\right)\right]^s + C_h} \\ &< \infty.\end{aligned}$$

which prove that $\mathscr{K}_{r,h}[\cdot,(y,z)] \in H^s_{r,h}(\mathbb{K})$. Let $f \in H^s_{r,h}(\mathbb{K})$, by the relations (3.2),(3.5),(3.7),(3.12),(2.8),(2.5) we find that

$$\langle f, \mathscr{K}_{r,h}[\cdot,(y,z)]\rangle_{H^s_{r,h}} = \int_{\hat{\mathbb{K}}} \varphi_{\lambda,m}(x,t)\mathcal{F}_\alpha(f)(\lambda,m)d\gamma_\alpha(\lambda,m),$$

the relation (3.9) and inversion formula (2.4) give the result. □

In the following we give the main result of this section.

Theorem 3.6. *Let h be a Laguerre-Bessel wavelet on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$, $g \in L^2_\alpha(\mathbb{R}_+ \times \mathbb{K})$ and $r > 0$ then the infimum*

$$\inf_{f\in H^s(\mathbb{K})} \left\{ r\|f\|^2_{H^s} + \|\Phi^\alpha_h(f) - g\|^2_{2,\theta_\alpha} \right\} \tag{3.13}$$

is attained by a unique function $f^*_{g,h,r}$ given explicitly by

$$f^*_{g,h,r}(x,t) = \int_{\mathbb{R}_+\times\mathbb{K}} g(a,y,z)\psi_{h,r}\left(a,(y,z),(x,t)\right) \mathrm{d}\theta_\alpha(a,y,z),$$

where $\psi_{h,r}$ is given by

$$\begin{aligned}&\psi_{h,r}\left(a,(y,z),(x,t)\right) \mathrm{d}\theta_\alpha(a,y,z)\\ &\quad= \int_{\hat{\mathbb{K}}} \frac{\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,z)\mathcal{F}_\alpha(h)(a\lambda,m)}{r\left[1+\lambda^2\left(1+m^2\right)\right]^s + C_h} d\gamma_\alpha(\lambda,m).\end{aligned}$$

Proof. The existence of the solution of the problem (3.13) is assured by Theorem 3.1, on the other hand by (3.4) we have

$$f^*_{g,h,r}(x,t) = \left\langle g, \Phi^\alpha_h\left(\mathscr{K}_{r,h}[\cdot,(x,t)]\right)\right\rangle_{\theta_\alpha}, \tag{3.14}$$

where $\mathscr{K}_{r,h}$ is the kernel given by (3.11), by the relation (2.24) Parseval's (2.5) we find that

$$\Phi^\alpha_h\left(\mathscr{K}_{r,h}[\cdot,(x,t)]\right)(a,y,z) = \left\langle \mathcal{F}_\alpha\left(\mathscr{K}_{r,h}\right), \mathcal{F}_\alpha\left(h_{a,y,z}\right)\right\rangle_\gamma,$$

by the relations (3.12),(2.20),(2.10) we find that

$$\Phi^\alpha_h(\mathscr{K}_{r,h}[\cdot,(x,t)]) = \int_{\hat{\mathbb{K}}} \frac{\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,z)\mathcal{F}_\alpha(\bar{h})(a\lambda,m)}{r\left[1+\lambda^2\left(1+m^2\right)\right]^s + C_h} d\gamma_\alpha(\lambda,m), \tag{3.15}$$

by the relations (3.14),(3.15) we find the desired result. □

Corollary 3.7. *Let h be a Laguerre-Bessel wavelet on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$, for $g = \Phi_h(f)$ we have*

$$f^*_{g,h,r} \longrightarrow f \text{ as } r \longrightarrow 0^+ \text{ uniformly.}$$

Proof. By the relation (3.14) we have

$$\begin{aligned} f^*_{g,h,r}(x,t) &= \left\langle g, \Phi^\alpha_h\left(\mathscr{K}_{r,h}[\cdot,(x,t)]\right)\right\rangle_{\theta_\alpha}\\ &= \left\langle \Phi^\alpha_h(g), \Phi^\alpha_h\left(\mathscr{K}_{r,h}[\cdot,(x,t)]\right)\right\rangle_{\theta_\alpha},\end{aligned}$$

by the relations (2.5),(2.22),(3.12) we find that

$$f^*_{g,h,r}(x,t) = C_h \int_{\hat{\mathbb{K}}} \frac{\varphi_{\lambda,m}(x,t)\mathcal{F}_\alpha(f)(\lambda,m)}{r\left[1+\lambda^2\left(1+m^2\right)\right]^s + C_h} d\gamma_\alpha(\lambda,m)$$

let $r \longrightarrow 0^+$, the inversion formula (2.4) gives the desired result. □

4. Calderón's Reproducing Formula for the Laguerre-Bessel Two-Wavelet Transform

4.1. **Generalized Parseval's and Inversion Formulas for the Laguerre-Bessel Two-Wavelet Transform.** The main purpose of this subsection is to extend the notion of the wavelet to the two-wavelet in the Laguerre-Bessel setting as follows

Definition 4.1. Let u, v be two Laguerre-Bessel wavelet functions in $L^2_\alpha(\mathbb{K})$ we say that the pair (u,v) is a Laguerre-Bessel two-wavelet on $\mathbb{K}$ if the following integral

$$C_{u,v} := \int_0^{+\infty} \overline{\mathcal{F}_\alpha(u)(a\lambda,m)}\mathcal{F}_\alpha(v)(a\lambda,m)\frac{\mathrm{d}a}{a} \tag{4.1}$$

is constant for almost all $(\lambda,m) \in \hat{\mathbb{K}}$ and we call the number $C_{u,v}$ the Laguerre-Bessel two-wavelet constant associated to the functions u, v.

Remark 4.2. It is easy to see that if u is a Laguerre-Bessel two-wavelet then the pair (u,u) is a Laguerre-Bessel two-wavelet and $C_{u,u}$ coincides with C_u given by (2.14).

The following theorem generalizes the Plancherel's formula for the Laguerre-Bessel wavelet transform (2.22) proved in [5].

Theorem 4.3. *Let (u,v) be a Laguerre-Bessel two-wavelet in $L^2_\alpha(\mathbb{K})$ then for all $f, g \in L^2_\alpha(\mathbb{K})$ we have*

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{K}} \Phi^\alpha_u(f)(a,x,t)\overline{\Phi^\alpha_v(g)(a,x,t)}\mathrm{d}\theta_\alpha(a,x,t) \\ = C_{u,v}\int_{\mathbb{K}} f(y,s)\overline{g(y,s)}d\mu_\alpha(y,s),\end{aligned} \tag{4.2}$$

where $C_{u,v}$ is given by the relation (4.1).

Proof. Using (2.5), (2.21), (2.17) and Fubini's theorem we get

$$\begin{aligned}&\int_{\mathbb{R}_+\times\mathbb{K}} \Phi^\alpha_u(f)(a,x,t)\overline{\Phi^\alpha_v(g)(a,x,t)}\mathrm{d}\theta_\alpha(a,x,t)\\ &= \int_0^{+\infty}\left[\int_{\mathbb{K}} (\overline{D_a(u)} *_\alpha f)(x,t)\overline{(\overline{D_a(v)} *_\alpha g)(x,t)}d\mu_\alpha(x,t)\right]\frac{\mathrm{d}a}{a}\\ &= \int_0^{+\infty}\left[\int_{\hat{\mathbb{K}}} \mathcal{F}_\alpha(\overline{D_a(u)})(\lambda,m)\mathcal{F}_\alpha(f)(\lambda,m)\overline{\mathcal{F}_\alpha(g)(\lambda,m)\mathcal{F}_\alpha(\overline{D_a(v)})}d\gamma_\alpha(\lambda,m)\right]\frac{\mathrm{d}a}{a}\end{aligned}$$

$$= \int_0^{+\infty} \overline{\mathcal{F}_\alpha(u)(a\lambda,m)}\mathcal{F}_\alpha(v)(a\lambda,m)\left[\int_{\hat{\mathbb{K}}} \mathcal{F}_\alpha(f)(\lambda,m)\overline{\mathcal{F}_\alpha(g)(\lambda,m)}d\gamma_\alpha(\lambda,m)\right]\frac{\mathrm{d}a}{a}.$$

A Parseval's formula for the Laguerre-Bessel transform (2.5) gives the desired result. □

Corollary 4.4. *If $C_{u,v} = 0$, then the spaces $\Phi_u^\alpha(L^2_\alpha(\mathbb{K}))$ and $\Phi_v^\alpha(L^2_\alpha(\mathbb{K}))$ are orthogonal.*

In the following we establish an inversion formula for the Laguerre-Bessel two-wavelet.

Theorem 4.5. *Let (u,v) be a Laguerre-Bessel two-wavelet such that $C_{u,v} \neq 0$ for all $f \in L^1_\alpha(\mathbb{K})$ such that $\mathcal{F}_\alpha(f) \in L^1_\alpha\left(\hat{\mathbb{K}}\right) \cap L^\infty_\alpha\left(\hat{\mathbb{K}}\right)$ we have*

$$f(\cdot) = \frac{1}{C_{u,v}}\int_0^\infty \left(\int_{\mathbb{K}} \Phi_u^\alpha(f)(a,x,t)v_{a,x,t}(\cdot)d\mu_\alpha(x,t)\right)\frac{da}{a}.$$

Weakly in $L^2_\alpha(\mathbb{K})$.

Proof. Let $f, g \in L^2_\alpha(\mathbb{K})$, by the relation (4.2), Fubini's theorem we find that

$$\begin{aligned}
&\frac{1}{C_{u,v}}\int_{\mathbb{R}_+\times\mathbb{K}} \Phi_u^\alpha(f)(a,x,t)\overline{\Phi_v^\alpha(g)(a,x,t)}\mathrm{d}\theta_\alpha(a,x,t)\\
&\quad= \int_{\mathbb{K}} f(y,s)\overline{g(y,s)}d\mu_\alpha(y,s)\\
&\quad= \frac{1}{C_{u,v}}\int_{\mathbb{R}_+\times\mathbb{K}} \Phi_u^\alpha(f)(a,x,t)\left[\int_{\mathbb{K}} \overline{g(y,s)}v_{a,x,t}(y,s)d\mu_\alpha(y,s)\right]\mathrm{d}\theta_\alpha(a,x,t)\\
&\quad= \frac{1}{C_{u,v}}\int_{\mathbb{K}}\left[\int_{\mathbb{R}_+\times\mathbb{K}} \Phi_u^\alpha(f)(a,x,t)v_{a,x,t}(y,s)\mathrm{d}\theta_\alpha(a,x,t)\right]\overline{g(y,s)}d\mu_\alpha(y,s)
\end{aligned}$$

which gives the result. □

Remark 4.6. For $u = v$ we find the relation (2.23).

4.2. **Calderón's Reproducing Formula.** The main purpose of this subsection is to establish a Calderón's reproducing type formula for the Laguerre-Bessel two-wavelet (u,v) under the following condition

$$C_{u,v} \neq 0 \quad \text{and} \quad \mathcal{F}_\alpha(u), \mathcal{F}_\alpha(v) \in L^\infty_\alpha\left(\hat{\mathbb{K}}\right). \tag{4.3}$$

We begin by the following result

Proposition 4.7. *For $0 < \varepsilon < \delta < \infty$, we put*

$$G_{\varepsilon,\delta}(x,t) := \frac{1}{C_{u,v}}\int_\varepsilon^\delta \left(\overline{D_a(u)} *_\alpha D_a(v)\right)(x,t)\frac{da}{a}$$

and

$$K_{\varepsilon,\delta}(\lambda,m):=\frac{1}{C_{u,v}}\int_0^{+\infty}\overline{\mathcal{F}_\alpha(u)(a\lambda,m)}\mathcal{F}_\alpha(v)(a\lambda,m)\frac{\mathrm{d}a}{a}.$$

Under the condition (4.3) we have

$$G_{\varepsilon,\delta}\in L^2_\alpha(\mathbb{K}),\qquad K_{\varepsilon,\delta}\in L^1_\alpha\left(\hat{\mathbb{K}}\right)\cap L^\infty_\alpha\left(\hat{\mathbb{K}}\right)$$

and

$$\mathcal{F}_\alpha(G_{\varepsilon,\delta})(\lambda,m)=K_{\varepsilon,\delta}(\lambda,m). \tag{4.4}$$

Proof. Using Hölder's inequality for the measure $\frac{da}{a}$ we obtain

$$|G_{\varepsilon,\delta}(x,t)|^2\leq\frac{\log(\delta/\varepsilon)}{C^2_{u,v}}\int_\varepsilon^\delta\left|\left(\overline{D_a(u)}*_\alpha D_a(v)\right)(x,t)\right|^2\frac{da}{a},\quad (x,t)\in\mathbb{K}.$$

So

$$\|G_{\varepsilon,\delta}\|^2_{2,\mu_\alpha}\leq\frac{\log(\delta/\varepsilon)}{C^2_{u,v}}\int_\varepsilon^\delta\left(\int_{\mathbb{K}}\left|\left(\overline{D_a(u)}*_\alpha D_a(v)\right)(x,t)\right|^2 d\mu_\alpha(x,t)\right)\frac{da}{a},$$

by using relations (2.6), (2.13) and (2.16) we find that

$$\begin{aligned}\|G_{\varepsilon,\delta}\|^2_{2,\mu_\alpha}&\leq\frac{\log(\delta/\varepsilon)}{C^2_{u,v}}\|\mathcal{F}_\alpha(u)\|^2_{\infty,\gamma_\alpha}\|v\|^2_{2,\alpha}\int_\varepsilon^\delta\frac{da}{a^{\frac{3\alpha+3}{2}}}\\&<\infty,\end{aligned}$$

which prove that $G_{\varepsilon,\delta}\in L^2_\alpha(\mathbb{K})$, the result $K_{\varepsilon,\delta}\in L^1_\alpha\left(\hat{\mathbb{K}}\right)\cap L^\infty_\alpha\left(\hat{\mathbb{K}}\right)$ can be easily checked, on the other hand by relations (2.5) and (2.10) we find that

$$\begin{aligned}&\left(\overline{D_a(u)}*_\alpha D_a(v)\right)(x,t)\\&\quad=\int_{\hat{\mathbb{K}}}\mathcal{F}_\alpha(v)(a\lambda,m)\overline{\mathcal{F}_\alpha(u)(a\lambda,m)}\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m),\end{aligned}$$

hence applying Fubini's theorem we get

$$G_{\varepsilon,\delta}(x,t)=\int_{\hat{\mathbb{K}}}\varphi_{\lambda,m}(x,t)K_{\varepsilon,\delta}(\lambda,m)d\gamma_\alpha(\lambda,m),$$

inversion formula (2.4) gives the relation (4.4). □

We can now state the main result of this section

Theorem 4.8 (Calderón's reproducing formula)**.** *Let (u,v) be a Laguerre-Bessel two-wavelet satisfying the condition (4.3) and let $0<\varepsilon<\delta<\infty$ then for all $f\in L^2_\alpha(\mathbb{K})$, the function $f_{\varepsilon,\delta}$ given by*

$$f_{\varepsilon,\delta}(x,t)=\frac{1}{C_{u,v}}\int_\varepsilon^\delta\left(\int_{\mathbb{K}}\Phi^\alpha_u(f)(a,y,s)v_{a,x,t}(y,s)d\mu_\alpha(y,s)\right)\frac{da}{a},\ (x,t)\in\mathbb{K},$$

belongs to $L^2_\alpha(\mathbb{K})$ *and satisfies*

$$\lim_{\varepsilon\to 0,\delta\to\infty} \|f_{\varepsilon,\delta} - f\|_{2,\mu_\alpha} = 0. \tag{4.5}$$

Proof. It is easy to see that

$$f_{\varepsilon,\delta} = f *_\alpha G_{\varepsilon,\delta}$$

then by the relations (2.12) and (4.4) we find that

$$\|f_{\varepsilon,\delta} - f\|^2_{2,\mu_\alpha} = \int_{\hat{\mathbb{K}}} |\mathcal{F}_\alpha(f)(\lambda,m)|^2(1 - K_{\varepsilon,\delta}(\lambda,m))^2 d\gamma_\alpha(\lambda,m),$$

by the relation (4.1), the relation (4.5) follows from the dominated convergence theorem. □

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