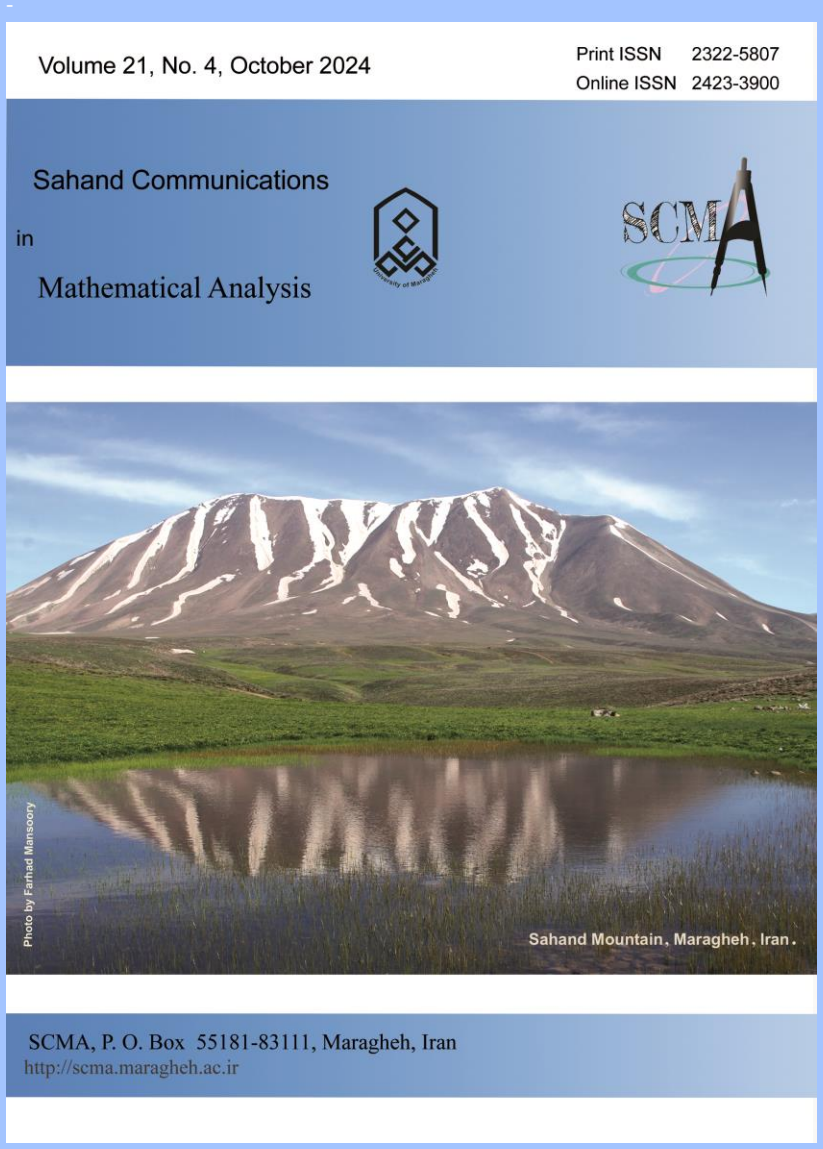
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On Cluster Points of Sequences via Fuzzy Topology

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Abstract. The notion of fuzzy convergence, a prominent tool in fuzzy analysis, has wide applications in fuzzy inference. The present paper examines the distinctive features of the cluster points of fuzzy sequences in various fuzzy topological spaces. A characterization of the fuzzy indiscrete space regarding the cluster points is obtained. Moreover, a detailed study of the cluster points of fuzzy sequences in fuzzy indiscrete, fuzzy discrete, fuzzy co-countable and fuzzy co-finite spaces is presented and characterizations of the cluster points in each space are established. Furthermore, the concepts of maximal cluster points, c-sets, c_m-sets and fuzzy c-sets of fuzzy sequences are proposed and their properties are investigated.

1. Introduction

In his classical paper, published in 1965, L. A. Zadeh [18] coined the idea of a fuzzy set by replacing the binary membership of an element in a classical set with a graded membership ranging between 0 and 1. Since then, plenty of research has been conducted in fuzzy set theory and its applications. In 1968, C. L. Chang [1] integrated fuzziness into topology and extended some basic definitions and results from traditional topology to fuzzy topology. Later, numerous studies were conducted, developing fuzzy topology as an outright generalization of classical topology. In 1976, R. Lowen [5] proposed a modifying of the definition of Chang's fuzzy topology.

Convergence of nets and, in particular, sequences play a vital role in general topology. To accommodate this into a fuzzy context, C. L.

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Chang [1] defined neighbourhoods of fuzzy sets and proposed the convergence of sequences of fuzzy sets. His definitions were exact reflections of the related concepts in classical topology, except that he considered neighbourhoods and sequences of fuzzy sets instead of points. In 1974, C. K. Wong [15] put forth the idea of fuzzy points to create a more natural outlook. However, his definition of fuzzy points was inadequate because it did not consider a crisp point as a special case of a fuzzy point. Also, his definition of the belongingness of a fuzzy point in a fuzzy set needed to be revised, which was later modified by R. Srivastava et al. [13].

In order to develop a satisfactory theory of convergence in fuzzy topology, in 1980, P. P. Ming and L. Y. Ming [7] came up with a novel and efficacious concept, namely the quasi-coincidence of fuzzy sets and using this new idea, they redefined the notion of neighbourhoods of a fuzzy point as Q-neighbourhoods. They also modified the definition of fuzzy points so that crisp points are always a particular case of fuzzy points. This made the study of convergence in fuzzy topology more meaningful and it also acted as a remedy for almost all the existing problems during that period, in developing fuzzy topology as an adequate generalization of classical topology.

In 1994, M. Ying [17] initiated the idea of the degree to which a fuzzy point belongs to a fuzzy set and proposed the notion of graded neighbourhoods in Chang's fuzzy topology. In 1981, M. Sarkar [11] introduced the concept of L-fuzzy topological spaces. Later, in [6], M. Luo et al. provided a necessary and sufficient condition for the Bolzano-Weierstrass property, which does not hold, in general, in L-fuzzy topological spaces.

In [10], A. A. Nouh put forward the notion of adherent points and studied convergence based on the idea of Q-neighbourhoods of fuzzy points. In 2005, K. K. Mondal and S. K. Samanta [9] introduced graded neighbourhoods in L-fuzzy topology and studied the convergence of L-fuzzy nets. Later, in 2006, they developed the idea of a graded fuzzy net and its convergence [8]. In 2011, W. Yao and L. X. Lu [16] established Moore-Smith convergence in L-fuzzifying topology. Some other generalizations of fuzzy convergence can be seen in [2, 4, 12, 14].

The study of fuzzy nets, particularly fuzzy sequences, is complete with cluster points. In [7], P. P. Ming and L. Y. Ming established the concepts of cluster points of fuzzy nets by using the concept of quasi-coincidence and Q-neighbourhoods. In the present paper, we focus on the cluster points of sequences of fuzzy points and their characteristic aspects in various fuzzy topological spaces. Throughout the paper, we use the definition of fuzzy topology given by Chang [1] and the concepts

of fuzzy points, fuzzy nets, convergence and cluster points proposed by P. P. Ming and L. Y. Ming [7]. Also, in the present paper, by fuzzy sequence, we mean a sequence of fuzzy points and N denotes the set of natural numbers.

Section 2 presents some of the preliminary definitions and results needed for the sequel. Section 3 includes characterizations of the cluster points in fuzzy indiscrete and discrete topological spaces. Section 4 encompasses characterizations of the cluster points in fuzzy co-countable and co-finite topological spaces. In Section 5, the concepts of maximal cluster points, c-sets, c_m-sets and fuzzy c-sets of fuzzy sequences are introduced and some properties based on the new concepts are discussed.

2. Preliminaries

This section contains a few basic definitions of fuzzy topology that serve as a base for the subsequent sections. For the sake of uniformity, we have changed certain notations in some of the definitions. The basic concepts in fuzzy set theory can be found in [3].

Definition 2.1 ([1])**.** A fuzzy topology is a family δ of fuzzy sets in X which satisfies the following conditions:

(i) $\phi, X \in \delta$
(ii) If $A, B \in \delta$, then $A \cap B \in \delta$
(iii) If $A_i \in \delta$ for each $i \in I, then \cup_I A_i \in \delta$.

The pair (X, δ) is called a fuzzy topological space, or fts for short. The elements of δ are called open fuzzy sets and their complements are called closed fuzzy sets with respect to δ.

Definition 2.2 ([1])**.** A fuzzy topological space (X, δ) is said to be

(i) fuzzy indiscrete, if δ contains only the fuzzy sets X and ϕ.
(ii) fuzzy discrete, if δ contains all the fuzzy subsets of X.
(iii) fuzzy co-finite, if $\delta = \phi \cup \{A \in I^X : Supp(A^c)\ is\ finite\}$.
(iv) fuzzy co-countable, if $\delta = \phi \cup \{A \in I^X : Supp(A^c)\ is\ countable\}$.

Definition 2.3 ([7, 13])**.** A fuzzy point x_λ in X with support x and value λ is defined by

$$x_\lambda(y) = \begin{cases} \lambda, & \text{if } y = x \\ 0, & \text{if } y \neq x, \end{cases}$$

where $0 < \lambda \leq 1$. A fuzzy point x_λ belongs to a fuzzy set B if and only if $\lambda \leq B(x)$. Two fuzzy points x_λ and y_γ are said to be distinct if and only if $x \neq y$.

Definition 2.4 ([7])**.** A fuzzy set A in (X, δ) is said to be quasi-coincident with B, denoted by AqB, if and only if there exists $x \in X$ such that

$A(x) > B^c(x)$ *or* $A(x) + B(x) > 1$. In particular, a fuzzy point x_λ is said to be quasi-coincident with A if and only if $\lambda > A^c(x)$. If A is not quasi-coincident with B, we write it as $A\bar{q}B$.

Definition 2.5 ([7])**.** A fuzzy set A in (X, δ) is called a Q-neighbourhood (Q-nbhd) of x_λ if and only if there exists a $B \in \delta$ such that $x_\lambda qB$ and $B \leq A$. A Q-neighbourhood of a fuzzy point may not contain the point itself.

Definition 2.6 ([7])**.** A fuzzy topological space (X, δ) is fuzzy T_2 (Hausdorff) space if and only if for any two distinct fuzzy points x_λ and y_γ, there exist Q-neighbourhoods B and C of x_λ and y_γ, respectively, such that $B \cap C = \phi$.

Definition 2.7 ([7])**.** Let χ be the collection of all the fuzzy points in X. A function $S : N \to \chi$ is called a fuzzy sequence in X. A fuzzy sequence S is usually denoted by $S = \{S_n : n \in N\}$.

Definition 2.8 ([7])**.** Let $S = \{S_n : n \in N\}$ be a fuzzy sequence in X and A be a fuzzy set in X. Then, S is said to be

(i) quasi-coincident with A if and only if $S_n qA$, $\forall n \in N$.
(ii) eventually quasi-coincident with A if and only if $\exists m \in N$ such that $S_n qA$, $\forall n \geq m$.
(iii) frequently quasi-coincident with A if and only if for each $m \in N$, $\exists n \geq m$ such that $S_n qA$.

Definition 2.9 ([7])**.** A sequence S in (X, δ) is said to converge to a point x_λ in X relative to δ if and only if S is eventually quasi-coincident with each Q-neighbourhood of x_λ.

Definition 2.10 ([7])**.** A fuzzy point x_λ in (X, δ) is called a cluster point of a fuzzy sequence S if and only if for every Q-neighbourhood B of x_λ, S is frequently quasi-coincident with B.

3. Characterization of Cluster Points in Fuzzy Indiscrete and Discrete Topological Spaces

For convenience, we define a property P in a fuzzy topological space (X, δ) as follows: for a fuzzy set A and a fuzzy point x_λ in X, we say that A satisfies the property P with respect to x_λ if and only if A is quasi-coincident with all the Q-neighbourhods of x_λ and we denote it by APx_λ.

Now, we give a characterization of fuzzy indiscrete topological spaces in terms of the *cluster points* of fuzzy sequences in such spaces.

Theorem 3.1. *Let (X, δ) be a fuzzy topological space. Then, (X, δ) is fuzzy indiscrete if and only if every fuzzy point is a cluster point of each fuzzy sequence in (X, δ).*

Proof. Let $S = \{S_n : n \in N\}$ be a fuzzy sequence in a fuzzy indiscrete topological space (X, δ). Let x_λ be an arbitrary fuzzy point in X. Since, (X, δ) is fuzzy indiscrete, X is the only Q-neighbourhood of x_λ. Obviously, $S_n q X, \forall n \in N$. So, in particular, S is frequently quasi-coincident with X. Therefore, x_λ is a cluster point of S. Since, S and x_λ were arbitrary, every fuzzy point is a cluster point of each fuzzy sequence in (X, δ). This proves the necessary part.

Conversely, suppose that every fuzzy point is a cluster point of each fuzzy sequence in (X, δ). Let $A \in \delta$ such that $A \neq \phi$. Then, clearly, $\exists x \in X$ such that $A(x) = \lambda$, where $\lambda \in (0, 1]$.

Choose $\gamma \in (1 - \lambda, 1]$ so that $\lambda + \gamma > 1$ and consider the fuzzy point x_γ. Then, since $A \in \delta$ and $A(x) + \gamma > 1$, A is a Q-neighbourhood of x_γ. Now, consider the fuzzy sequence $S = \{x_{1/n}, n \in N\}$. Then, by our assumption, x_γ is a cluster point of S. Therefore, S is frequently quasi-coincident with all the Q-neighbourhoods of x_γ. So, in particular, S is frequently quasi-coincident with A. That is, for each $m \in N$, $\exists n \geq m$ such that $S_n P x_\gamma$. Therefore, $S_n q A$. Now, $S_n q A$ means that $x_{1/n} q A$. That is, $\frac{1}{n} > 1 - A(x)$. Since, $n \geq m$, we get, $\frac{1}{m} \geq \frac{1}{n}$ and so $\frac{1}{m} > 1 - A(x)$. Thus, $\frac{1}{m} > 1 - A(x)$, for each $m \in N$. Therefore, as $m \to \infty$, this implies that $A(x) = 1$.

Now, let $y \in X$ such that $y \neq x$ and consider the fuzzy sequence $S' = \{y_{1/n} : n \in N\}$. Then, x_γ is a cluster point of S'. Since, A is a Q-neighbourhood of x_γ, we get, $A(y) = 1$, as above. Since, $y \neq x$ was arbitrary, $A(y) = 1, \forall y \neq x$. Therefore, $A(x) = 1, \forall x \in X$, and hence $A = X$. Thus, $\delta = \{X, \phi\}$, and hence (X, δ) is fuzzy indiscrete. □

The following theorem provides a characterization of the cluster points of fuzzy sequences in a fuzzy discrete topological space.

Theorem 3.2. *In a fuzzy discrete topological space (X, δ), a fuzzy point x_λ is a cluster point of a fuzzy sequence S if and only if the terms of S are frequently fuzzy points of the form x_μ, where $\mu \geq \lambda$.*

Proof. Let $S = \{S_n : n \in N\}$ be a fuzzy sequence in a fuzzy discrete topological space (X, δ). Suppose that x_λ is a cluster point of S and let $m \in N$. Then, by definition of cluster points, $\exists n \geq m$ such that $S_n P x_\lambda$. Suppose that $S_n = y_\mu$, where $y \in X$.

First, we claim that $y = x$. For, choose $\gamma \in (1-\lambda, 1]$. Then, $\gamma > 1-\lambda$ or $\lambda > 1-\gamma$. Now, consider the fuzzy point $A = x_\gamma$. Then, A is an open set in (X, δ) because δ is the fuzzy discrete topology on X. In addition, we have $\lambda > 1 - \gamma$. That is, $\lambda > 1 - A(x)$. Therefore, $A = x_\gamma$ is a Q-neighbourhood of x_λ. Since, $S_n P x_\lambda$, $S_n q A$. That is, $y_\mu q A$. Therefore,

$$\mu > 1 - A(y) \quad \text{or} \quad \mu + A(y) > 1.$$

This implies that $A(y) > 0$. But, the only point at which A has membership value strictly greater than zero is x. Hence, we get, $y = x$ and so $S_n = x_\mu$.

Now, we claim that $\mu \geq \lambda$. If possible suppose $\mu < \lambda$. Now, choose $\alpha = 1 - \mu$ and consider the fuzzy point $B = x_\alpha$. Then, obviously,

$$\alpha + \lambda = 1 - \mu + \lambda = 1 + \lambda - \mu > 1, \quad \text{since } \lambda - \mu > 0.$$

Therefore, $B = x_\alpha$ is a Q-neighbourhood of x_λ. So, $S_n = x_\mu$ must be quasi-coincident with B. But,

$$\mu + \alpha = \mu + 1 - \mu = 1.$$

Therefore, $S_n \bar{q} B$, which is a contradiction. So, $\mu \geq \lambda$. Thus, $S_n = x_\mu$, where $\mu \geq \lambda$. Now, since $m \in N$ was arbitrary, we can conclude that for each $m \in N$, $\exists n \geq m$ such that $S_n = x_\mu$, for some $\mu \geq \lambda$. That is, the terms of S are frequently fuzzy points of the form x_μ, where $\mu \geq \lambda$. This proves the necessary part.

Conversely, suppose that the terms of S are frequently fuzzy points of the form x_μ, where $\mu \geq \lambda$. Let A be a Q-neighbourhood of x_λ. Then, evidently, any fuzzy point with support x and membership value at least λ is quasi-coincident with A. Since, A was arbitrary, any fuzzy point of the form x_μ, where $\mu \geq \lambda$ is quasi-coincident with all the Q-neighbourhoods of x_λ. Then, by our assumption, this implies that, S is frequently quasi-coincident with all the Q-neighbourhoods of x_λ. Hence, x_λ is a cluster point of S. □

Example 3.3. Consider the fuzzy sequence $S = \{x_{1/2}, x_{2/3}, x_{3/4}, \ldots\}$ in a fuzzy topological space (X, δ). If (X, δ) is fuzzy indiscrete, then, by Theorem 3.1, each fuzzy point in X is a cluster point of S. Now, if (X, δ) is fuzzy discrete, then by Theorem 3.2, any fuzzy point x_λ, $0 < \lambda < 1$ is a cluster point of S and no fuzzy point with support distinct from x is a cluster point of S.

4. Characterization of Cluster Points in Fuzzy Co-countable and Co-Finite Spaces

In this section, for brevity, we define some sets associated with a fuzzy sequence $S = \{S_n : n \in N\}$ in a fuzzy topological space (X, δ). By range set of S we mean the crisp set of all supports of the terms in S. Corresponding to a fuzzy point x_λ and $m \in N$, we define the sets V_m, V and S_V in such a way that $V_m = \{n \in N : n \geq m \text{ and } S_n P x_\lambda\}$, $V = \bigcup_{m=1}^{\infty} V_m$ and $S_V = \{S_n : n \in V\}$. It is clear that if x_λ is a cluster point of S, then V_m is non-empty, for each $m \in N$.

Now, we present a characterization theorem for cluster points of fuzzy sequences in fuzzy co-countable topological spaces.

Theorem 4.1. *In a fuzzy co-countable topological space (X, δ), a fuzzy point x_λ is a cluster point of a fuzzy sequence S if and only if the terms of S are frequently of the form x_μ, where $\mu \geq \lambda$.*

Proof. Let $S = \{S_n : n \in N\}$ be a fuzzy sequence in a fuzzy co-countable topological space (X, δ). Suppose that x_λ is a cluster point of S. Then, by definition of cluster points, S is frequently quasi-coincident with all the Q-neighbourhoods of x_λ. That is, for each $m \in N$, $\exists n \geq m$ such that $S_n P x_\lambda$. Consider the sets V_m, V and S_V, corresponding to x_λ and m, associated with S. Then, it is enough to prove that all the elements in S_V are of the form x_μ, where $\mu \geq \lambda$.

First, we claim that each element in S_V has support x. For, let A be the range set of S. Then, evidently, A is a countable set. Now, choose $\alpha \in (1 - \lambda, 1]$ so that $\alpha + \lambda > 1$ and define the fuzzy set, B, in such a way that

$$B(x) = \alpha, B(y) = 0, \quad \forall y \in A - \{x\}$$
$$B(z) = 1, \quad \forall z \in X - (A \cup \{x\})$$

Then, the support of B^c is a subset of $A \cup \{x\}$ and hence countable. So B is a fuzzy co-countable set. Therefore, B is fuzzy open in (X, δ). Moreover, since $\lambda + \alpha > 1$, $x_\lambda q B$. So, B is a Q-neighbourhood of x_λ. But, by the construction of B, for any $\gamma \in (0, 1]$, $y_\gamma \bar{q} B, \forall y \in A - \{x\}$. Therefore, $S_n \in S_V$ implies that $S_n = x_\mu$, for some μ. That is, all the elements in S_V has support x.

Now, we claim that $\mu \geq \lambda$. For, choose $\gamma < \lambda$ and let $\beta = 1 - \gamma$. Construct the fuzzy set C such that

$$C(x) = \beta, C(y) = 0, \quad \forall y \in A - \{x\}$$
$$C(z) = 1, \quad \forall z \in X - (A \cup \{x\}).$$

Then, C is fuzzy open and since, $\lambda + \beta > 1$, $x_\lambda q C$. Therefore, C is a Q-neighbourhood of x_λ.

But, since $\gamma + \beta = 1$, $x_\gamma \bar{q} C$. Therefore, for each $\gamma < \lambda$ we can find a fuzzy open set C such that C is a Q-neighbourhood of x_λ, but $x_\gamma \bar{q} C$. This implies that $x_\gamma \notin S_V$. Therefore, for each $S_n \in S_V$, $S_n = x_\mu$, for some $\mu \geq \lambda$. So, for each $n \in V$, $S_n = x_\mu$, for some $\mu \geq \lambda$. Then, since, V_m is non empty for all m and $V = \bigcup_{m=1}^{\infty} V_m$, for each $m \in N$, $\exists n \geq m$ such that $S_n = x_\mu$, for some $\mu \geq \lambda$. Hence, the terms of S are frequently fuzzy points of the form x_μ, where $\mu \geq \lambda$.

The converse is similar to that of Theorem 3.2 and can be proved analogously. □

Next, we state and prove a characterization theorem for the cluster points of fuzzy sequences in fuzzy co-finite topological space.

Theorem 4.2. *In a fuzzy cofinite topological space (X, δ), a fuzzy point x_λ is a cluster point of a fuzzy sequence S if and only if the terms of S are frequently fuzzy points of the form x_μ, where $\mu \geq \lambda$.*

Proof. Let $S = \{S_n : n \in N\}$ be a fuzzy sequence in a fuzzy co-finite topological space (X, δ). Suppose that x_λ is a cluster point of S. Then, by definition, S is frequently quasi-coincident with all the Q-neighbourhoods of x_λ. That is, for each $m \in N$, $\exists n \geq m$ such that $S_n P x_\lambda$. Consider the sets V_m, V and S_V, corresponding to x_λ and m, associated with S. Then, it is enough to prove that all the elements in S_V are of the form x_μ, where $\mu \geq \lambda$.

First, we claim that no fuzzy points with support different from x can belong to S_V. For, let y_γ be a fuzzy point such that $y \neq x$. Choose $\alpha \in (1 - \lambda, 1]$ so that $\alpha + \lambda > 1$. Consider the fuzzy set A such that $A(x) = \alpha$, $A(y) = 0$ and $A(z) = 1, \forall z \in X - \{x, y\}$. Then, A^c has finite support and hence A is fuzzy open in (X, δ). Also, $x_\lambda q A$. Therefore, A is a Q-neighbourhood of x_λ. Since, $A(y) = 0$, $y_\gamma \bar{q} A$ for any $\gamma \in (0, 1]$. So, no fuzzy points y_γ, where $y \neq x$ can be quasi-coincident with all the Q-neighbourhoods of x_λ. Therefore, no fuzzy points with support distinct from x can belong to S_V.

Now, we claim that no fuzzy points with support x and membership value strictly less than λ can belong to S_V. For, let $\gamma < \lambda$ and let $\beta = 1 - \gamma$. Construct the fuzzy set B such that

$$B(x) = \beta \quad \text{and} \quad B(y) = 1, \quad \forall y \in X - \{x\}.$$

Then, B is fuzzy open and $x_\lambda q B$. Therefore, B is a Q-neighbourhood of x_λ. But, $x_\gamma \bar{q} B$. So, $x_\gamma \notin S_V$. Since, γ was arbitrary, no fuzzy points with support x and membership value strictly less than λ can be a member of S_V. Therefore, all the elements in S_V have support x and membership value greater than or equal to λ.

It is obvious that any fuzzy point with membership value greater than or equal to λ and support x is quasi-coincident with all the Q-neighbourhoods of x_λ. Therefore, all the elements in S_V are quasi-coincident with each and every Q-neighbourhoods of x_λ. Now, by the construction of the set S_V, for each $m \in N$, $\exists n \geq m$ such that $S_n P x_\lambda$. This implies that for each $m \in N$, $\exists n \geq m$ such that S_n has support x and membership value at least λ. Hence, frequently, the terms of S are of the form x_μ, where $\mu \geq \lambda$.

The converse is similar to that of Theorem 3.2 and can be proved analogously. □

Example 4.3. Let (X, δ) be a fuzzy topological space. Let $x \in X$ and choose the fuzzy sequence $S = \{x_1, x_{1/2}, x_{1/3}, \ldots\}$ in (X, δ). Suppose (X, δ) is fuzzy co-countable or fuzzy co-finite. There does not exist a fuzzy point y_λ in X such that the terms of S are frequently fuzzy points of the form y_μ, $\mu \geq \lambda$. Therefore, by Theorems 4.1 and 4.2, no fuzzy point is a cluster point of S.

Note 4.4. By Theorems 3.2, 4.1 and 4.2, we can conclude that the characterization of the cluster points in fuzzy co-finite, fuzzy co-countable and fuzzy discrete toplogical spaces are identical. Indeed, the proof of Theorem 4.2 guarantees that in any fuzzy topological space finer than fuzzy co-finite space we can characterize cluster points of fuzzy sequences in a similar way.

Now, by Theorem 4.2, we have the following proposition.

Proposition 4.5. *In a fuzzy co-finite (or finer) topological space (X, δ), if a fuzzy point x_λ is a cluster point of the fuzzy sequence S, then x belongs to range set of S.*

Proof. Let x_λ be a fuzzy point in the fuzzy co-finite (or finer) topological space (X, δ). Then, by Theorem 4.2 and its proof, we have, if x_λ is a cluster point of a fuzzy sequence S in a fuzzy co-finite (or finer) toplogical space, then frequently the terms of S are fuzzy points with support x. That is, the sequence S has terms with support x and hence x belongs to range set of S. □

The next theorem and its corollary points out a speciality of the cluster points of convergent fuzzy sequences in fuzzy co-finite and finer topological spaces.

Theorem 4.6. *In a fuzzy co-finite topological space, if a fuzzy sequence S converges to x_λ, then all the cluster points of S have the support x.*

Proof. Let S be a convergent fuzzy sequence in a fuzzy co-finite topological space (X, δ). Suppose that S converges to the fuzzy point x_λ. Then, S is eventually quasi-coincident with all the Q-neighbourhoods of x_λ. That is, $\exists m \in N$ such that, $\forall n \geq m$, $S_n P x_\lambda$. Then, for each $n \geq m$, S_n must has support x because, otherwise, if the support of S_n is $z \neq x$, then S_n will not be quasi-coincident with the Q-neighbourhood A of x_λ for which $A(x) \in (1 - \lambda, 1], A(z) = 0$ and $A(w) = 1, \forall w \in X - \{x, z\}$. So, eventually the terms of S have support x. Hence, by Theorem 4.2, y_γ cannot be a cluster point of S for any $y \neq x$ and $\gamma \in (0, 1]$. Thus, all the cluster points of S have support x. Since, S was arbitrary, all the cluster points of any convergent fuzzy sequence S in a fuzzy cofinite (or finer) topological space have the same support. □

Corollary 4.7. *In any fuzzy topological space, finer than fuzzy co-finite space, all the cluster points of a convergent fuzzy sequence have the same support.*

The next theorem guarantees that Theorem 4.6 also remains valid for any fuzzy Hausdorff topological spaces.

Theorem 4.8. *In a fuzzy Hausdorff topological space (X, δ), all the cluster points of a convergent fuzzy sequence have the same support.*

Proof. Let (X, δ) be a fuzzy Hausdorff space and let $S = \{S_n : n \in N\}$ be a convergent fuzzy sequence in (X, δ). Suppose that S converges to the fuzzy point x_λ. We prove that all the cluster points of S have support x. If possible, suppose y_γ is a cluster point of S, where $y \neq x$. Now, since, x_λ and y_γ are distinct fuzzy points in the fuzzy Hausdorff space (X, δ), there exists Q-neighbourhoods U and V of x_λ and y_γ, respectively, such that $U \cap V = \phi$. Since, S converges to x_λ, eventually S is quasi-coincident with U. That is, $\exists m \in N$ such that $S_n q U, \forall n \geq m$. Again, since y_γ is a cluster point of S, for the above mentioned m, $\exists n \geq m$ such that $S_n q V$. Therefore, S_n is quasi-coincident with both U and V. Then, if S_n has support $z \in X$, then $U(z) > 0$ and $V(z) > 0$. So, $(U \cap V)(z) \neq 0$ and hence $U \cap V \neq \phi$, which is a contradiction. Therefore, all the cluster points of S have support x. Since, S was arbitrary, all the cluster points of a convergent fuzzy sequence in a fuzzy Hausdorff space have the same support. □

5. Maximal Cluster Points, c-sets and c_m-sets of Fuzzy Sequences

Firstly, we define, the *maximal cluster point* of a fuzzy sequence in a fuzzy topological space.

Definition 5.1. A fuzzy point x_λ is called a maximal cluster point of a fuzzy sequence S in a fuzzy topological space (X, δ), if λ is the greatest member in $(0, 1]$ such that x_λ is a cluster point of S.

It should be noted that a fuzzy sequence in a fuzzy topological space need not have a maximal cluster point even though it has cluster points. For example, consider the fuzzy co-finite topological space (X, δ). Let $x \in X$ and consider the fuzzy sequence $S = \{x_{1/2}, x_{2/3}, x_{3/4}, \ldots\}$. Then, as we can see, the terms of S have support x and, for each $n \in N$, the n^{th} term of S has membership value $\frac{n}{n+1}$, which increases to 1 and never attains 1. So, S has no maximal cluster points. But, by Theorem 4.2, any fuzzy point x_λ, where $0 < \lambda < 1$ is a cluster point of S.

Now, we have the following theorem for cluster points.

Theorem 5.2. *If x_λ is a cluster point of a fuzzy sequence S in a fuzzy topological space (X, δ), then x_μ is also a cluster point for any $\mu \in (0, \lambda]$.*

Proof. Let (X, δ) be a fuzzy topological space and let S be a fuzzy sequence in (X, δ). Suppose that x_λ is a cluster point of S. Then, by definition of cluster points, S is frequently quasi-coincident with all the Q-neighbourhoods of x_λ. Now, if $\mu \leq \lambda$, then it is evident that all the Q-neighbourhoods of x_μ are also Q-neighbourhoods of x_λ. So, since, S is frequently quasi-coincident with all the Q-neighbourhoods of x_λ, we can say that S is frequently quasi-coincident with all the Q-neighbourhoods of x_μ, where $\mu \in (0, \lambda]$. Hence, x_μ is also a cluster point of S for any $\mu \in (0, \lambda]$. □

Corollary 5.3. *If x_λ is a maximal cluster point of a fuzzy sequence S, then x_μ is a cluster point of S if and only if $\mu \leq \lambda$.*

The following theorem shows the uniqueness of maximal cluster points of convergent fuzzy sequences in fuzzy co-finite and finer topological spaces.

Theorem 5.4. *In a fuzzy co-finite (or finer) topological space (X, δ), the maximal cluster point, if exists, of any convergent fuzzy sequence is unique.*

Proof. Let $S = \{S_n : n \in N\}$ be a convergent fuzzy sequence in the fuzzy cofinite (or finer) topological space (X, δ). Suppose that S converges to the fuzzy point x_λ. Then, by definition of convergence of fuzzy sequences, S is eventually quasi-coincident with all the Q-neighbourhoods of x_λ. So, it is obvious that x_λ is a cluster point of S. Therefore, by Corollary 4.7, all the cluster points of S have support x. Hence, the maximal cluster point of S, if exists, is unique. Since, S was arbitrary, the maximal cluster point, if exists, of any convergent fuzzy sequence is unique. □

Now, we define c-sets and c_m-sets of fuzzy sequences in a fuzzy topological space.

Definition 5.5. The set of all cluster points of a fuzzy sequence S is called the c-set of the fuzzy sequence and is denoted by S_c. That is,
$S_c = \{x_\lambda : S$ is frequently quasi-coincident with all the $Q-$nbhds of $x_\lambda\}$.

Definition 5.6. The set of all maximal cluster points of a fuzzy sequence S is called the c_m-set of the fuzzy sequence and is denoted by S_{c_m}. So,

$$S_{c_m} = \{x_\lambda : x_\lambda \text{ is a maximal cluster point of } S\}.$$

It should be noted that both c-set and c_m-set of a fuzzy sequence are crisp sets of fuzzy points. The following are some examples for c-sets and c_m-sets of fuzzy sequences.

Example 5.7. Consider the fuzzy sequence $S = \{x_1, x_{1/2}, x_{1/3}, \ldots\}$ in a fuzzy topological space (X, δ), where $x \in X$. If (X, δ) is fuzzy indiscrete space, then by Theorem 3.1, $S_c = \chi$ and $S_{c_m} = \{x_1 : x \in X\}$. If (X, δ) is fuzzy cofinite topological space, then $S_c = S_{c_m} = \phi$, by Theorem 4.2.

Example 5.8. Consider the fuzzy sequence $S = \{x_{1/2}, x_{2/3}, x_{3/4}, \ldots\}$ in a fuzzy topological space (X, δ). If (X, δ) is fuzzy indiscrete, then $S_c = \chi$ and $S_{c_m} = \{x_1 : x \in X\}$. If (X, δ) is fuzzy cofinite space, then $S_c = \{x_\lambda : 0 < \lambda < 1\}$, whereas $S_{c_m} = \phi$.

Example 5.9. Let (X, δ) be a fuzzy topological space and let $x, y \in X$ such that $x \neq y$. Consider the fuzzy sequence $S = \{S_n, n \in N\}$ in (X, δ), where

$$S_n = \begin{cases} x_{n/n+1}, & \text{if } n \text{ is odd} \\ y_{n/n+1}, & \text{if } n \text{ is even.} \end{cases}$$

Then, if (X, δ) is fuzzy indiscrete, then $S_c = \chi$ and $S_{c_m} = \{x_1 : x \in X\}$. Now, if (X, δ) is a fuzzy co-finite space, then

$$S_c = \{x_\lambda : \lambda \in (0, 1)\} \cup \{y_\gamma : \gamma \in (0, 1)\}, \text{ whereas } S_{c_m} = \phi.$$

Theorem 5.10. *Let (X, δ) be a fuzzy topological space. Then, (X, δ) is fuzzy indiscrete space if and only if the c_m-set of any fuzzy sequence in (X, δ) is $\{x_1 : x \in X\}$.*

Proof. Let $S = \{S_n : n \in N\}$ be a fuzzy sequence in an indiscrete topological space (X, δ). We prove that $S_{c_m} = \{x_1 : x \in X\}$. Now, by the necessary part of Theorem 3.1, every fuzzy point is a cluster point of S. So, x_1 is a maximal cluster point of S, for each $x \in X$. Hence, $S_{c_m} = \{x_1 : x \in X\}$. Since S was arbitrary, the c_m-set of any fuzzy sequence in an indiscrete fuzzy topological space is $\{x_1 : x \in X\}$.

Conversely, suppose that the c_m-set of any fuzzy sequence in the fuzzy topological space (X, δ) is $\{x_1 : x \in X\}$. Then, by Theorem 5.2, every fuzzy point is a cluster point of each fuzzy sequence in (X, δ). So, by the converse of Theorem 3.1, (X, δ) is fuzzy indiscrete. □

The following theorem shows that the c_m-set of a fuzzy sequence in a fuzzy co-finite or finer topological space is countable.

Theorem 5.11. *If (X, δ) is a fuzzy co-finite (or finer) topological space, then the c_m-set of any fuzzy sequence in (X, δ) is a countable set.*

Proof. Let $S = \{S_n : n \in N\}$ be a fuzzy sequence in the fuzzy co-finite (or finer) topological space (X, δ). Let x_λ be a maximal cluster point of S. Then by Proposition 4.5, x belongs to the range set of S. So, if A is the range set of S, then the set of supports of elements in S_{c_m} is a subset of A. It is evident that the range set of any fuzzy sequence is countable.

Also, two different maximal cluster points have distinct supports, and hence S_{c_m} is countable. Since, the fuzzy sequence S was arbitrary, the c_m-set of any fuzzy sequence in (X, δ) is countable. □

Now, we define the fuzzy c-set of a sequence of fuzzy points in a fuzzy topological space.

Definition 5.12. The smallest fuzzy set containing all the cluster points of a fuzzy sequence in a fuzzy topological space (X, δ) is called the fuzzy c-set of the sequence. In other words, the fuzzy c-set of a fuzzy sequence S in (X, δ) is the set

$$\cap\{A \in I^X : A \textit{ contains all the cluster points of } S\}.$$

Example 5.13. Consider Example 5.7. If (X, δ) is fuzzy indiscrete, then fuzzy c-set of S is X. If (X, δ) is fuzzy co-finite space, then the fuzzy c-set of S is ϕ.

Example 5.14. Consider Example 5.8. If (X, δ) is fuzzy indiscrete, then the fuzzy c-set of S is X and if (X, δ) is fuzzy co-finte, then the fuzzy c-set of S is the fuzzy set A such that $A(x) = 1$ and $A(y) = 0, \forall y \neq x$ in X. That is, the fuzzy c-set of S is the fuzzy point x_1.

Example 5.15. Consider Example 5.9. If (X, δ) is fuzzy indiscrete, then the fuzzy c-set of S is the fuzzy set X. Now, if (X, δ) is fuzzy co-finite space, then the fuzzy c-set of S is the fuzzy set A defined by $A(x) = A(y) = 1$ and $A(z) = 0, \forall z \in X - \{x, y\}$.

The following theorem characterizes fuzzy indiscrete spaces in terms of fuzzy c-sets of fuzzy sequences.

Theorem 5.16. *Let (X, δ) be a fuzzy topological space. Then, (X, δ) is fuzzy indiscrete if and only if the fuzzy c-set of any fuzzy sequence in (X, δ) is the fuzzy set X.*

Proof. Let (X, δ) be a fuzzy indiscrete space and let $S = \{S_n : n \in N\}$ be a fuzzy sequence in (X, δ). Now, by Theorem 3.1, every fuzzy point is a cluster point of S. Therefore, x_1 is a cluster point of S, for every $x \in X$. So, X is the smallest fuzzy set containing all the cluster points of S. That is, the fuzzy c-set of the fuzzy sequence S is X. Since, S was arbitrary, the fuzzy c-set of any fuzzy sequence in (X, δ) is the fuzzy set X.

Conversely, assume that the fuzzy c-set of any fuzzy sequence in (X, δ) is the fuzzy set X. Then, X is the smallest fuzzy set containing all the cluster points of any fuzzy sequence in (X, δ). By Theorem 3.1, in order to prove that (X, δ) is fuzzy indiscrete, it is enough to prove that every fuzzy point in X is a cluster point of every fuzzy sequence in (X, δ).

Consider a fuzzy sequence S in (X, δ). Then, by our assumption, the fuzzy c-set of S is X. So, for each $x \in X$,

$$Sup\{\lambda : x_\lambda \text{ is a cluster point of } S\} = 1.$$

This implies that x_λ is a cluster point of S for any $\lambda \in (0, 1)$.

Now, we prove that, for each $x \in X$, x_1 is also a cluster point of S. Consider $x \in X$ and let A be a Q-neighbourhood of x_1. Then, $\exists B \in \delta$ such that $x_1 q B$ and $B \leq A$. So, $1 > 1 - B(x)$ and, therefore, $B(x) \neq 0$. Then, by using the dense property of real numbers, we can find a real number γ such that $1 > \gamma > 1 - B(x)$. So, $x_\gamma q B$. Hence, A is also a Q-neighbourhood of x_γ. Since, $B(x) \in (0, 1]$, we have, $\gamma \in (0, 1)$. Therefore, x_γ is a cluster point of S. So, S is frequently quasi-coincident with A. Since, A was arbitrary, S is frequently quasi-coincident with all the Q-neighbourhoods of x_1. Hence, x_1 is a cluster point of S, for each $x \in X$. Thus, since, S was arbitrary, every fuzzy point is a cluster point of each fuzzy sequence in (X, δ). Hence, by Theorem 3.1, (X, δ) is fuzzy indiscrete. □

Theorem 5.17. *In a fuzzy co-finite (or finer) topological space (X, δ), the fuzzy c-set of any convergent sequence is closed.*

Proof. Let S be a convergent fuzzy sequence in a fuzzy co-finte (or finer) topological space (X, δ). Then, by Corollary 4.7, all the cluster points of S have support x. So, clearly, the support of the fuzzy c-set of S is $\{x\}$, which is finite. Therefore, the complement of fuzzy c-set of S is fuzzy co-finite and hence is open in (X, δ). Hence, the fuzzy c-set of S is closed. Since, S was arbitrary, the fuzzy c-set of any convergent sequence in a fuzzy co-finite (or finer) topological space is closed. □

Now, by Proposition 4.5 we have the following theorem.

Theorem 5.18. *If (X, δ) is a fuzzy co-countable topological space, then the fuzzy c-set of any fuzzy sequence in (X, δ) is closed.*

Proof. Let (X, δ) be a fuzzy co-countable space. Let $S = \{S_n : n \in N\}$ be a fuzzy sequence in (X, δ). Now, by Proposition 4.5, if x_λ is a cluster point of S, then x belongs to the range set of S. Therefore, the set of supports of all the cluster points of S is a subset of the range set of S, which is always countable. Therefore, the set of supports of elements in the c-set of S is a countable set. Hence, clearly, the fuzzy c-set of S has countable support. Therefore, the fuzzy complement of the fuzzy c-set is fuzzy co-countable and hence open in (X, δ). Hence, the fuzzy c-set of S is closed. Since, the fuzzy sequence S was arbitrary, the fuzzy c-set of any fuzzy sequence in (X, δ) is closed. □

6. Conclusion

The study of fuzzy nets, particularly, fuzzy sequences, and their convergence are inevitable in the fuzzy topology. In the present paper, we focused on cluster points of fuzzy sequences. We derived some characterizations of cluster points in various fuzzy topological spaces such as fuzzy indiscrete, fuzzy co-finite, fuzzy co-countable and fuzzy discrete spaces. Motivated by the study, we introduced concepts such as maximal cluster points, c-sets, c_m-sets and fuzzy c-sets of fuzzy sequences. The results can also be extended to other types of fuzzy topological spaces. Moreover, they provide scope for further advancement of the study of fuzzy sequences.

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References

1. C.L. Chang, *Fuzzy Toplogical Spaces*, J. Math. Anal. Appl., 24 (1968), pp. 182-190.
2. S.L. Chen and S.T. Chen, *A New Extension of Fuzzy Convergence*, Fuzzy Sets Syst., 109 (2) (2000), pp. 199-204.
3. G.J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic Theory and Applications*, Prentice Hall P T R, Upper Saddle River, New Jersey, 1995.
4. C.S. Li and C.J. Shu, *θ-convergence of nets of L-fuzzy sets and its applications*, Fuzzy Sets Syst., 86 (2) (1997), pp. 235-240.
5. R. Lowen, *Fuzzy Topological Spaces and Fuzzy Compactness*, J. Math. Anal. Appl., 56 (1976), pp. 621-633.
6. M. Luo, Y.M. Liu and J.H. Liang, *On Convergence Classes in L-fuzzy Topological Spaces*, J. Math. Anal. Appl., 224 (1998), pp. 316-331.
7. P.P. Ming and L.Y. Ming, *Fuzzy Topology. I. Neighborhood Structure of a Fuzzy point and Moore-Smith Convergence*, J. Math. Anal. Appl., 76 (1980), pp. 571-599.
8. K.K. Mondal and S.K. Samanta, *Convergence of Generalized Nets and Filters in Graded L-Fuzzy Topological Spaces*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math., 13 (1) (2006), pp. 47-69.
9. K.K. Mondal and S.K. Samanta, *Fuzzy Convergence Theory-I*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math., 12 (1) (2005), pp. 75-91.

10. A.A. Nouh, *On Convergence Theory in Fuzzy Topological Spaces and its Applications*, Czechoslovak Math. J., 55 (130) (2005), pp. 295-316.
11. M. Sarkar, *On L-Fuzzy Topological Spaces*, J. Math. Anal. Appl., 84 (1981), pp. 431-442.
12. F.G. Shi and C.Y. Zheng, *O-convergence of fuzzy nets and its applications*, Fuzzy Sets Syst., 140 (3) (2003), pp. 499-507.
13. R. Srivastava, S.N. Lal and A.K. Srivastava, *Fuzzy Hausdorff Topological Spaces*, J. Math. Anal. Appl., 81 (1981), pp. 497-506.
14. M.M. Stadler and M.A. de Prada Vicente, *On N-convergence of FuzzyNets*, Fuzzy Sets Syst., 51 (2) (1992), pp. 203-217.
15. C.K. Wong, *Fuzzy Points and Local Properties of Fuzzy Topology*, J. Math. Anal. Appl., 46 (1974), pp. 316-328.
16. W. Yao and L.X. Lu, *Moore-Smith Convergence in L-Fuzzifying Topologial Spaces*, J. Math. Res. Exposition, 31 (5) (2011), pp. 770-780.
17. M. Ying, *On the Method of Neighbourhood Systems in Fuzzy Topology*, Fuzzy Sets Syst., 68 (2) (1994), pp. 227-238.
18. L.A. Zadeh, *Fuzzy sets*, Inf. Control., 8 (1965), pp. 338-353.

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