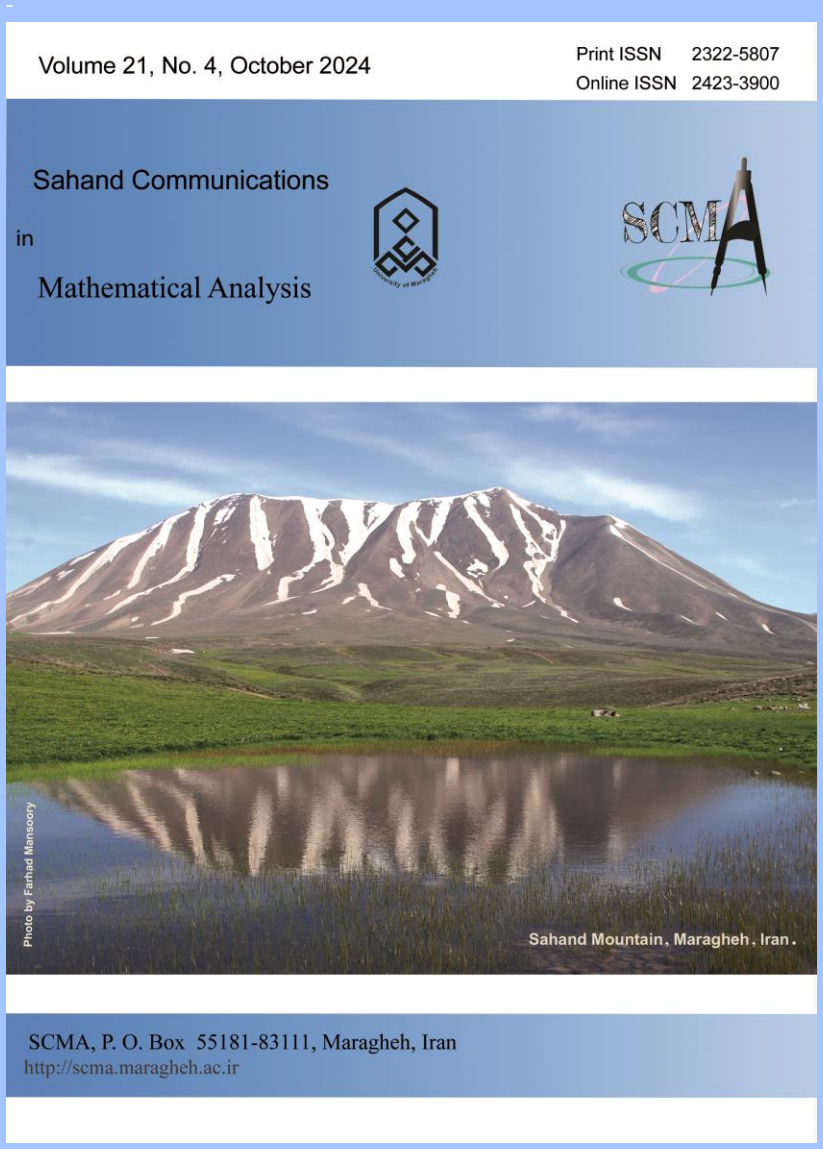
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A Novel Generalization of Strong Probabilistic b-Metric Spaces and Fixed Point Theorems

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Abstract. The controlled strong probabilistic (fuzzy) b-metric space is a novel concept we introduce in this study. We also demonstrate some of its fundamental topological properties and provide examples. In this context, a new result of fixed point property was proved for the probabilistic ω-contraction mapping on these kinds of spaces. Many prior findings in the literature are generalized and unified by our findings. Some related results are also offered to illuminate further the fundamental theorem in ordinary controlled strong b-metric spaces.

1. Introduction

The theory of metric fixed point was initiated by the celebrated result of Banach contraction [4], which is known as a crucial instrument for providing the existence and discussing the uniqueness of solution for many analytics problems [2, 5, 16]. Many mathematicians have generalized this result by modifying the contraction conditions [6, 7] or by changing the structure of the metric space [10, 13, 15, 20]. Bakhtin [3] has weakened the triangle inequality and developed the idea of b-metric spaces as a new generalization of ordinary metric spaces. In this regard, vastly fixed point theorems results in favor of multivalued and single-valued functions have appeared in these types of spaces [10, 14].

In this direction, recently, Kirk and Shahzad [15] modified the relaxed triangle inequality in the definition of b-metric space to introduce the

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idea of strong b-metric space. Santina et al. [21] later added a controlled function to the triangle inequality, and they also proved various fixed point theorems on this space, which is defined by the following.

For a non empty set Ω and $\mu : \Omega \times \Omega \to [1, +\infty)$. If a function $\alpha_\mu : \Omega \times \Omega \to [0, +\infty)$ verified the following criteria for all $a, b, c \in \Omega$,

(i) $\alpha_\mu(a, b) = 0$ iff $a = b$;
(ii) $\alpha_\mu(a, b) = \alpha_\mu(b, a)$;
(iii) $\alpha_\mu(a, b) \leq \mu(x, y)\alpha_\mu(a, c) + \alpha_\mu(c, b)$.

Then the pair (Ω, α_μ) referred to as controlled strong b-metric space, in short CSbMS.

Clearly, the concept of a controlled strong b-metric space is a generalization of the idea of a strong b-metric space. More precisely, if we take $\mu(x, y) = s$ for $s \geq 1$ in the above definition, then it becomes the definition of a strong b-metric space [15].

Menger [19] presented the idea of probabilistic metric space in 1942 as an extension of metric spaces by substituting a random variable that only accepted non-negative real values for non-negative numeric values. It means that a distribution function gives the distance between two points in this kind of space, the idea was developed later by Schweizer and Sklar [23]. Sehgal and Bharucha-Reid [24] proved the first result of the fixed point theory in these kinds of spaces in 1972. Sherwood [25] later enhanced their findings, and since then, fixed point theory in probabilistic metrics has developed [1, 8, 11, 12]. Recently, probabilistic b-metric spaces were investigated in [18] as an extension of probabilistic metric spaces.

This work aims to introduce a new generalization of the metrics we mentioned before, which we called the controlled strong probabilistic b-metric space (in short CSPbMS). Also, we examine some of topological proprieties and prove a fixed point theorem for the probabilistic nonlinear Banach contraction (ω-contraction) in the controlled strong probabilistic b-metric space with an extend to the ordinary controlled strong b-metric spaces.

This article has the following format. Section 2 recalls numerous fundamental notions and known results in probabilistic metric spaces. We introduce controlled strong probabilistic and fuzzy b-metric spaces in section 3 and go through some of the fundamental topological concepts related to these types of spaces. Several fixed point theorems for nonlinear contraction in controlled strong probabilistic and fuzzy b-metric spaces are presented in section 4. As an application, in section 5, we substantiate the validity of our results by obtaining some well-known pre-existing result in ordinary controlled strong b-metric spaces.

2. Preliminaries

Here we recall several basics definitions and lemmas from probabilistic metric space.

Definition 2.1 ([23])**.** Let Λ^+ be the space of all distance distribution functions $\zeta : [0, +\infty] \to [0, 1]$ such that:

(i) ζ is left continuous on $[0, +\infty)$,
(ii) ζ is non-decreasing,
(iii) $\zeta(0) = 0$ and $\zeta(+\infty) = 1$.

We indicate by $\triangle^+ \subset \Lambda^+$ the set $\triangle^+ = \left\{\zeta \in \Lambda^+ : \lim_{x\to+\infty} \zeta(x) = 1\right\}$.

One of the element of $\triangle^+$ is ϵ_a which defined as

$$\epsilon_a(x) = \begin{cases} 0 & \text{if} \quad x \in [0, a], \\ 1 & \text{if} \quad x \in (a, +\infty]. \end{cases}$$

Definition 2.2. Let $\zeta, \zeta' \in \Lambda^+$, $h \in [0, 1)$ and let $(\zeta, \zeta'; h)$ denote the condition

$$0 \le \zeta'(x) \le \zeta(x + h) + h, \quad \text{for all } x \in \left(0, \frac{1}{h}\right).$$

The function α_L defined on $\Lambda^+ \times \Lambda^+$ by

$$\alpha_L(\zeta, \zeta') = \inf\left\{h : \text{ both } \left(\zeta, \zeta'; h\right) \text{ and } \left(\zeta', \zeta; h\right) \text{ hold}\right\},$$

is called the modified Lévy distance.

Recall that $(\zeta, \zeta'; 1)$ and $(\zeta', \zeta; 1)$ are hold for any ζ and ζ' in Λ^+, which means that α_L is well-defined and $\alpha_L(\zeta, \zeta') \le 1$.

Lemma 2.3 ([23])**.** *For each $\zeta, \zeta' \in \Lambda^+$ with $\zeta \le \zeta'$, we have $\alpha_L(\zeta', \epsilon_0) \le \alpha_L(\zeta, \epsilon_0)$.*

Definition 2.4. A binary operation π on Λ^+ is called a triangle function if for all $\zeta, \zeta', \zeta'' \in \Lambda^+$ the conditions listed below are verified:

(i) $\pi(\zeta, \zeta') = \pi(\zeta', \zeta)$,
(ii) $\pi(\zeta, \pi(\zeta', \zeta'')) = \pi(\pi(\zeta, \zeta'), \zeta'')$,
(iii) $\zeta \le \zeta'' \Rightarrow \pi(\zeta, \zeta') \le \pi(\zeta'', \zeta')$,
(iv) $\pi(\zeta, \epsilon_0) = \pi(\epsilon_0, \zeta) = \zeta$.

A mapping $\mathfrak{H} : [0, 1] \times [0, 1] \to [0, 1]$ is referred to as a t-norm if and only if it has the properties of commutativity, associativity, non-decreasing at each location and has 1 neutral element.
Some of the fundamental t-norms are

$$\mathfrak{H}_M(x, y) = \min(x, y), \quad \mathfrak{H}_P(x, y) = x \cdot y, \quad \mathfrak{H}_L(x, y) = \max(x + y - 1, 0).$$

A special class of triangle functions will be presented by assuming the form: for any $f, g \in \Lambda^+$ and for all $t \in [0, +\infty)$,

$$\pi_{\mathfrak{H}}(f,g)(t) = \sup\left\{\mathfrak{H}\left(f(s), g(u)\right) : s+u=t\right\},$$

with $\mathfrak{H}$ is a left-continuous t-norm. Additionally, if $\mathfrak{H}$ is continuous, $\pi_{\mathfrak{H}}$ must likewise be continuous.

Definition 2.5 ([12])**.** We say that a t-norm $\mathfrak{H}$ is of H-type if the family $\left(\mathfrak{H}^n(x)\right)_{n\in\mathbb{N}}$ is equicontinuous at the point $x = 1$, that is for all $\tau \in (0,1)$, there exists $\gamma \in (0,1)$ such that $t > 1-\gamma$ implies $\mathfrak{H}^n(t) > 1-\tau$ for all $n \geq 1$, where for all $x \in [0,1]$ and $n \in \mathbb{N}$ we write

$$\mathfrak{H}^n(x) = \begin{cases} 1, & \text{if } n = 0, \\ \mathfrak{H}(\mathfrak{H}^{n-1}(x), x), & \text{otherwise.} \end{cases}$$

Definition 2.6. The triplet (Ω, Γ, π) is told a probabilistic metric space where Ω is a nonempty set, Γ is a function from $\Omega \times \Omega$ into Λ^+ and π is a triangle function if for all $p, q, r \in \Omega$ and $x, y > 0$, the conditions listed below are verified:

(i) $\Gamma_{p,p} = \epsilon_0$,
(ii) $\Gamma_{p,q} = \epsilon_0$ implies $p = q$,
(iii) $\Gamma_{p,q} = \Gamma_{q,p}$,
(iv) $\Gamma_{p,q} \geq \pi\left(\Gamma_{p,r}, \Gamma_{r,q}\right)$.

When $\pi = \pi_{\mathfrak{H}}$ for a given t-norm $\mathfrak{H}$, then the space $(\Omega, \Gamma, \pi_{\mathfrak{H}})$ is referred to as Menger space and we write $(\Omega, \Gamma, \mathfrak{H})$ which implies that the condition iv equivalent to

$$\Gamma_{p,r}(x+y) \geq \mathfrak{H}\left(\Gamma_{p,q}(x), \Gamma_{q,r}(y)\right). \tag{2.1}$$

We consider (Ω, Γ) as a probabilistic semi-metric space (i.e. i, ii and iii of definition 2.6 are validated). The (τ,γ)-topology in (Ω, Γ) is induced by the neighboring family

$$\mathcal{N} = \left\{N_p(\tau,\gamma) : \ p \in \Omega, \ \tau > 0 \text{ and } \gamma > 0\right\},$$

where

$$N_p(\tau,\gamma) = \left\{q \in \Omega : \ \Gamma_{p,q}(\tau) > 1-\gamma\right\}.$$

We define The function D_A on $[0, +\infty]$ by

$$D_A(x) = \begin{cases} \lim\limits_{t\to x^-} \theta_A(t) & \text{if} \quad 0 \leq x < +\infty, \\ 1 & \text{if} \quad x = +\infty, \end{cases}$$

where A is a nonempty set of Ω and

$$\theta_A(t) = \inf\left\{\Gamma_{p,q}(t) \ \mid \ p, q \in A\right\},$$

then D_A is called the probabilistic diameter of A.

It is clear that $D_A \in \Lambda^+$ for any $A \subset \Omega$. Also A will be bounded if D_A is in $\triangle^+$.

Definition 2.7. Let (Ω, Γ) be a probabilistic semi-metric space, a sequence $\{x_n\}$ in Ω is:

(i) Convergent to $x \in \Omega$, if for each $\tau > 0$ and $\gamma > 0$ there is $N \in \mathbb{N}$ such that $\Gamma_{x_n,x}(\gamma) > 1 - \tau$ whenever $n \geq N$.
(ii) Cauchy sequence, if for every $\tau > 0$ and $\gamma > 0$ there is $N \in \mathbb{N}$ where $\Gamma_{x_n,x_{n'}}(\gamma) > 1 - \tau$ when $n, n' \geq N$.

A probabilistic semi-metric space (Ω, Γ) is told to be complete if each Cauchy sequence of elements of Ω convergent to some point in Ω.

3. The Controlled Strong Probabilistic b-Metric Space

The concept of controlled strong probabilistic b-metric space, which is broader than the notion of probabilistic metric spaces, is introduced in this section.

Definition 3.1. A controlled strong probabilistic b-metric space is a quadruple $(\Omega, \Gamma, \pi, \mu)$ where Ω is nonempty set, Γ is a mapping from $\Omega \times \Omega$ into Λ^+, π is a triangle function, μ is a function from $\Omega \times \Omega$ into $[1, +\infty)$, and the conditions listed below are verified for all $p, q, r \in \Omega$ and $x, y > 0$,

(i) $\Gamma_{p,p} = \epsilon_0$,
(ii) $\Gamma_{p,q} = \epsilon_0 \Rightarrow p = q$,
(iii) $\Gamma_{p,q} = \Gamma_{q,p}$,
(iv) $\Gamma_{p,q}(\mu(p,q)x + y) \geq \pi(\Gamma_{p,r}, \Gamma_{r,q})(x + y)$.

The case where $\pi = \pi_{\mathfrak{H}}$ for some t-norm $\mathfrak{H}$, the quadruple $(\Omega, \Gamma, \pi_{\mathfrak{H}}, \mu)$ is called controlled strong b-Menger space noted $(\Omega, \Gamma, \mathfrak{H}, \mu)$ which implies that the condition iv become

$$\Gamma_{p,r}(\mu(p,r)x + y) \geq \mathfrak{H}(\Gamma_{p,q}(x), \Gamma_{q,r}(y)). \tag{3.1}$$

It's obvious that if $\mu(p,q) = 1$ then $(\Omega, \Gamma, \pi, 1)$ is a probabilistic metric space.

Example 3.2. Let $\Omega = \{1, 2, 3\}$ and define $\Gamma : \Omega \times \Omega \to \Lambda^+$ by

$$\Gamma_{p,q}(t) = \begin{cases} \frac{t}{t+|p-q|^2} & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t = 0. \end{cases}$$

Let define the mapping $\mu : \Omega \times \Omega \to [1, +\infty)$ by $\mu(p,q) = 1 + p + q$.

We show that $(\Omega, \Gamma, \pi_{\mathfrak{H}_P}, \mu)$ is a controlled strong b-Menger space. Indeed, The controlled strong probabilistic triangle inequality must actually be demonstrated, since that i, ii and iii of definition 3.1 are trivially satisfied.

For $p = 1$ and $q = 2$, we have

$$\Gamma_{1,2}(\mu(1,2)t + t') = \frac{4t + t'}{4t + t' + 1}, \quad \Gamma_{1,3}(t) = \frac{t}{t+4}, \quad \Gamma_{3,2}(t') = \frac{t'}{t'+1}.$$

$$\begin{aligned}\Gamma_{1,3}(t).\Gamma_{3,2}(t') &= \frac{1}{1+\frac{4}{t}}.\frac{1}{1+\frac{1}{t'}}\\ &\leq \frac{1}{1+\frac{4}{4tt'y}}.\frac{1}{1+\frac{1}{4t+t'}}\\ &\leq \frac{1}{1+\frac{1}{4t+t'}}\\ &= \frac{4t+t'}{4t+t'+1}.\end{aligned}$$

Hence,

$$\Gamma_{1,2}(\mu(1,2)t+t') \geq \mathfrak{H}_P(\Gamma_{1,3}(t),\Gamma_{3,2}(t')).$$

Similarly, we get

$$\Gamma_{1,3}(\mu(1,3)t+t') \geq \mathfrak{H}_P(\Gamma_{1,2}(t),\Gamma_{2,3}(t')),$$

and

$$\Gamma_{2,3}(\mu(2,3)t+t') \geq \mathfrak{H}_P(\Gamma_{2,1}(t),\Gamma_{1,3}(t')).$$

Therefore, $(\Omega,\Gamma,\mathfrak{H}_P,\mu)$ is a controlled strong b-Menger space.

Example 3.3. Let $\Omega=\{u,v,s\}$ and define $\Gamma:\Omega\times\Omega\to\Lambda^+$ by

$$\Gamma_{p,q}(t)=\begin{cases}\epsilon_0(t) & \text{if } \ p=q,\\ \epsilon_0(t-\frac{1}{2}) & \text{if } \ p=u \text{ and } \ q=v \text{ or } p=v \text{ and } q=u\\ \epsilon_0(t-6) & \text{if } \ p=s \text{ and } q=u \text{ or } p=u \text{ and } q=s\\ \epsilon_0(t-5) & \text{if } \ p=v \text{ and } q=s \text{ or } p=s \text{ and } q=v.\end{cases}$$

Let $\mu:\Omega\times\Omega\to[1,+\infty)$ be a mapping that satisfy

$$\begin{aligned}&\mu(u,u)=\mu(v,v)=\mu(s,s)=1,\\ &\mu(u,s)=\mu(s,u)=2,\\ &\mu(v,s)=\mu(s,v)=\frac{5}{4},\\ &\mu(u,v)=\mu(u,v)=\frac{11}{10}.\end{aligned}$$

It is easy to verify that $(\Omega,\Gamma,\mathfrak{H}_M,\mu)$ is a controlled strong b-Menger space.

By taking $\Gamma_{p,q}(t)=\mathcal{Q}(p,q,t)$ for all $t>0$ with $\Gamma_{p,q}(0)=0$, the controlled strong fuzzy b-metric space will be defined as the following

Definition 3.4. A controlled strong fuzzy b-metric space is a quadruple $(\Omega,\mathcal{Q},\mathfrak{H},\mu)$ where Ω is an arbitrary nonempty set, $\mathfrak{H}$ is a continuous t-norm, $\mathcal{Q}$ is a fuzzy set on $\Omega\times\Omega\times(0,+\infty)$ and μ is a function from $\Omega\times\Omega$ into $[1,+\infty)$ satisfy for all $p,q,r\in\Omega$ and $t,v>0$ the conditions listed below:

(i) $\mathcal{Q}(p,q,0)=0$,

(ii) $\mathcal{Q}(p,q,t) = 1$ if and only if $p = q$,
(iii) $\mathcal{Q}(p,q,t) = \mathcal{Q}(q,p,t)$,
(iv) $\mathcal{Q}(p,r,\mu(p,r)t + v)) \geq \mathfrak{H}\left(\mathcal{Q}(p,q,t), \mathcal{Q}(q,r,v)\right)$,
(v) $\mathcal{Q}(p,q,.) : [0,+\infty) \to [0,1]$ is left-continuous and nondecreasing.

When $\mu(p,r) = s$ then $(\Omega, \mathcal{Q}, \mathfrak{H}, s)$ is a strong fuzzy b-metric space [22] in the form of Kramosil and Michalek [9].

Lemma 3.5. *Let $(\Omega, \Gamma, \pi, \mu)$ be a controlled strong probabilistic b-metric space. The (τ,γ)-topology is a Hausdorff topology If the triangle function π is continuous.*

Proof. Let $p, q \in \Omega$ such that $p \neq q$. By putting $\Gamma^{\mu}_{p,q}(t) = \Gamma_{p,q}(\mu(p,q)\frac{t}{2} + \frac{t}{2})$ we obtain that $\Gamma^{\mu}_{p,q} \neq \epsilon_0$, which implies that $\alpha_L(\Gamma^{\mu}_{p,q}, \epsilon_0) > 0$.
From the continuity of π, there exist $t > 0$ such that

$$\alpha_L(\pi(\Omega_1, \Omega_2), \epsilon_0) < \alpha_L(\Gamma^{\mu}_{p,q}, \epsilon_0) \tag{3.2}$$

whenever $\alpha_L(\Omega_1, \epsilon_0) < t$ and $\alpha_L(\Omega_2, \epsilon_0) < t$ with $\Omega_1, \Omega_2 \in \Lambda^+$. Suppose that there exist $r \in N_p(t,t) \cap N_q(t,t)$. Then $\alpha_L(\Gamma_{p,r}, \epsilon_0) < t$ and $\alpha_L(\Gamma_{r,q}, \epsilon_0) < t$. By using (3.2) and Lemma 2.3 we obtain

$$\alpha_L\left(\Gamma^{\mu}_{p,q}, \epsilon_0\right) \leq \alpha_L\left(\pi\left(\Gamma_{p,r}, \Gamma_{r,q}\right), \epsilon_0\right) < \alpha_L\left(\Gamma^{\mu}_{p,q}, \epsilon_0\right),$$

which contradicts itself, therefore $N_p(t,t) \cap N_q(t,t) = \emptyset$. Hence (τ,γ)-topology is a Hausdorff topology. □

The next result shows that every ordinary controlled strong b-metric space is an controlled strong b-Menger space.

Lemma 3.6. *Let (Ω, α_μ) be a controlled strong b-metric space, and let define $\Gamma : \Omega \times \Omega \to \Lambda^+$ by*

$$\Gamma_{p,q}(t) = \epsilon_0(t - \alpha_\mu(p,q)),$$

then $(\Omega, \Gamma, \mathfrak{H}_M, \mu)$ is a controlled strong b-Menger space.

Proof. It's clear that Γ satisfies the conditions i, ii and iii of definition 3.1, it left to prove the following inequality, for each $p, q, r \in \Omega$ and $x, y \in [0, +\infty)$

$$\Gamma_{p,q}(\mu(p,q)x + y) \geq \min(\Gamma_{p,r}(x), \Gamma_{r,q}(y)). \tag{3.3}$$

Since $\Gamma_{p,q}(\mu(p,q)x + y) \in \{0,1\}$, then (3.3) trivially holds if $\min(\Gamma_{p,r}(x), \Gamma_{r,q}(y)) = 0$. Assume that $\min(\Gamma_{p,r}(x), \Gamma_{r,q}(y)) = 1$, then $\alpha_\mu(p,r) < x$ and $\alpha_\mu(r,q) < y$.

Since that (Ω, α_μ) is a controlled strong b-metric space, we obtain

$$\begin{aligned} \alpha_\mu(p,q) &\leq \mu(p,q)\alpha_\mu(p,r) + \alpha_\mu(r,q) \\ &< \mu(p,q)(x+y). \end{aligned}$$

Which implies that $\Gamma_{p,q}(\mu(p,q)x+y)=1$, and thus demonstrates that (3.3) is likewise true. Hence, $(\Omega,\Gamma,\mathfrak{H}_M,\mu)$ is a controlled strong b-Menger space. □

As an other result, It is simple to confirm that $(\Omega,\Gamma,\pi_{\mathfrak{H}_M},\mu)$ is complete if and only if (Ω,α_μ) is complete.

4. Fixed Point Theorem in Controlled Strong Probabilistic b-Metric Space

Throughout this section, (Ω,Γ,π,μ) be a controlled strong probabilistic b-metric space with π is a continuous triangular function and h is a self map on Ω. The definition of power of h is $h^0x=x$ and $h^{n+1}x=hh^nx$, $n\geq 0$. We will use the notation $h^nx=x_n$, and for $x_0\in\Omega$, $\wp_h(x_0)=\{x_0,hx_0,h^2x_0,h^3x_0,\ldots\}$ represent the orbit of h starting at x_0.
We denote by ξ the class of all function $\omega:[0,+\infty)\to[0,+\infty)$ satisfying

$$\omega(t)<t \text{ and } \lim_{n\to+\infty}\omega^n(t)=0 \text{ for all } t>0.$$

Now, we define the ω-contraction in the strong extended probabilistic b-metric space.

Definition 4.1. Let (Ω,Γ,π,μ) be a controlled strong probabilistic b-metric space and $\omega:[0,+\infty)\to[0,+\infty)$. We say that a mapping $h:\Omega\to\Omega$ is a probabilistic ω-contraction if for every $p,q\in\Omega$ and $t>0$ we have

$$\Gamma_{hp,hq}(\omega(t))\geq\Gamma_{p,q}(\mu(p,q)t). \tag{4.1}$$

The fuzzy type of ω-contraction in controlled strong fuzzy b-metric space can be introduced as the following

Definition 4.2. Let $(\Omega,\mathcal{Q},\mathfrak{H},\mu)$ be a controlled strong fuzzy b-metric space and $\omega:[0,+\infty)\to[0,+\infty)$. A mapping $h:\Omega\to\Omega$ is called a fuzzy ω-contraction if for every $p,q\in\Omega$ and $t>0$ we have

$$\mathcal{Q}(hp,hq,\omega(t))\geq\mathcal{Q}(p,q,\mu(p,q)t). \tag{4.2}$$

We require the next lemma in order to obtain our primary result.

Lemma 4.3. *Let (Ω,Γ,π,μ) be a controlled strong probabilistic b-metric space whith $Ran\Gamma\subset\triangle^+$ and h is a probabilistic ω-contraction on Ω with $\omega\in\xi$. Then $\{x_n\}$ is a Cauchy sequence if the orbit $\wp_h(x)$ is bounded for some $x\in\Omega$.*

Proof. Let $n,m\in\mathbb{N}$ which $m>n$ and $t>0$. From (4.1) we obtain

$$\begin{aligned}\Gamma_{x_n,x_m}(\omega^n(t))&\geq\Gamma_{x_{n-1},x_{m-1}}\left(\mu(x_{n-1},x_{m-1})\omega^{n-1}(t)\right)\\&\geq\Gamma_{x_{n-1},x_{m-1}}(\omega^{n-1}(t))\end{aligned}$$

$$\begin{aligned}
&\geq \Gamma_{x_{n-2},x_{m-2}}\left(\mu(x_{n-2},x_{m-2})\omega^{n-2}(t)\right)\\
&\geq \Gamma_{x_{n-2},x_{m-2}}\left(\omega^{n-2}(t)\right)\\
&\ \ \vdots\\
&\geq \Gamma_{x_0,x_{m-n}}(\mu(x_0,x_{m-n})t)\\
&\geq \Gamma_{x_0,x_{m-n}}(t)\\
&\geq D_{\wp_h(x)}(t).
\end{aligned}$$

On the other hand, let $\tau > 0$ and $\delta \in (0,1)$, since $D_{\wp_h(x)}(t) \to 1$ as $t \to +\infty$, then there exist $t_0 > 0$ satisfy

$$D_{\wp_h(x)}(t_0) > 1 - \delta.$$

Since $\omega^n(t_0) \to 0$ when $n \to +\infty$, there is $n_0 \in \mathbb{N}$ where

$$\omega^n(t_0) < \tau \text{ whenever } n \geq n_0.$$

We obtain by the monotonicity of Γ that for all $n, m \geq n_0$

$$\begin{aligned}
\Gamma_{x_n,x_m}(\tau) &\geq \Gamma_{x_n,x_m}(\omega^n(t_0))\\
&\geq D_{\wp_h(x)}(t_0)\\
&> 1 - \delta.
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence. □

Theorem 4.4. *Let $(\Omega, \Gamma, \pi, \mu)$ be a completely controlled strong probabilistic b-metric space with Ran$\Gamma \subset \triangle^+$ and h is a probabilistic ω-contraction mapping on Ω where $\omega \in \xi$. If the orbit $\wp_{h(x)}$ is bounded for some $x \in \Omega$, then h admits a unique fixed point b. Furthermore, $\{h^n x\}$ converge to b.*

Proof. Taking $x \in \Omega$ such that $\wp_{h(x)}$ is a bounded orbit, then from lemma 4.3, $\{x_n\}$ is a Cauchy sequence, and since that $(\Omega, \Gamma, \pi, \mu)$ is complete, then $\{x_n\}$ converge to some $b \in \Omega$.

We will prove that b is a fixed point of h. Let $t > 0$, then from (4.1) we have

$$\begin{aligned}
\Gamma_{x_n,hb}(\omega(t)) &\geq \Gamma_{x_{n-1},b}(\mu(x_{n-1},b)t)\\
&\geq \Gamma_{x_{n-1},b}(t).
\end{aligned}$$

And since that $\omega \in \xi$, we obtain

$$\Gamma_{x_n,hb}(t) \geq \Gamma_{x_{n-1},b}(t),$$

Therefore

$$\Gamma_{b,hb}(t) \geq \pi(\Gamma_{b,x_n}, \Gamma_{x_n,hb})\left(\frac{t}{2\mu(b,hb)} + \frac{t}{2}\right)$$

$$\geq \pi(\Gamma_{b,x_n}, \Gamma_{x_{n-1},b}) \left(\frac{t}{2\mu(b,hb)} + \frac{t}{2} \right).$$

Letting $n \to +\infty$, we get

$$\Gamma_{b,hb}(t) \geq 1, \quad \text{for all } t > 0.$$

Which implies that $b = hb$. Hence, b is a fixed point of h.

To show the uniqueness, we consider the existence of another fixed point u in Ω of h. Indeed, let $t > 0$, we have

$$\begin{aligned} \Gamma_{b,u}(\omega(t)) &= \Gamma_{hb,hu}(\omega(t)) \\ &\geq \Gamma_{b,u}(\mu(b,u)t) \\ &\geq \Gamma_{b,u}(t). \end{aligned}$$

Also from that $\omega \in \xi$, we have

$$\Gamma_{b,u}(t) \geq \Gamma_{b,u}(\omega(t))$$

Therefore,

$$\Gamma_{b,u}(\omega(t)) = \Gamma_{b,u}(t). \tag{4.3}$$

Next, we shall prove that $\Gamma_{b,u}(t) = 1$ for each $t > 0$. Assuming that there is $t_1 > 0$ such that $\Gamma_{b,u}(t_1) < 1$. Since that $\Gamma_{b,u} \in \triangle^+$, then $\lim_{t \to +\infty} \Gamma_{b,u}(t) = 1$, which means that there exist $t_2 > t_1$ such that

$$\Gamma_{b,u}(t_2) > \Gamma_{b,u}(t_1). \tag{4.4}$$

Induction and (4.3) makes it simple to prove that

$$\Gamma_{b,u}(\omega^n(t)) = \Gamma_{b,u}(t), \quad \text{for all } n \geq 1.$$

And since $\omega^n(t_2) \to 0$ as $n \to +\infty$, then there exists a positive integer $n > 1$ that verify $\omega^n(t_2) < t_1$. Which implies by the monotonicity of Γ that

$$\Gamma_{b,u}(\omega^n(t_2)) \leq \Gamma_{b,u}(t_1).$$

Hence

$$\Gamma_{b,u}(t_2) = \Gamma_{b,u}(\omega^n(t_2)) \leq \Gamma_{b,u}(t_1),$$

which contradicts itself. Therefore, $\Gamma_{b,u}(t) = 1$ for all $t > 0$, and from that $\Gamma_{b,u} \in \triangle^+$, then we have $\Gamma_{b,u}(t) = \epsilon_0(t)$.

Finally we conclude that $b = u$. □

Example 4.5. Let $\Omega = [0,1]$ and define $\Gamma : \Omega \times \Omega \to \Lambda^+$ by

$$\Gamma_{p,q}(t) = \begin{cases} \frac{t}{t+|p-q|} & \text{if } \ t > 0, \\ 0 & \text{if } \ t = 0. \end{cases}$$

We claim that $(\Omega, \Gamma, \pi_{\mathfrak{H}_M}, \mu)$ is a complete controlled strong b-Menger space with $\mu(p,q) = 2$ for all $p, q \in \Omega$. Define $h : \Omega \to \Omega$ by $h(p) = \frac{p+7}{8}$.

We will prove that h is a probabilistic ω-contraction, where $\omega(t) = \frac{1}{2}t$ for all $t > 0$. In fact, for all $p, q \in \Omega$ and $t > 0$ we have

$$\begin{aligned}\Gamma_{h(p),h(q)}(\omega(t)) &= \frac{\frac{1}{2}t}{\frac{1}{2}t + |h(p) - h(q)|} \\ &= \frac{\frac{1}{2}t}{\frac{1}{2}t + \frac{1}{8}|p - q|} \\ &= \frac{t}{t + \frac{1}{4}|p - q|} \\ &\geq \frac{2t}{2t + |p - q|} \\ &= F_{p,q}(2t).\end{aligned}$$

Hence, h is a probabilistic ω-contraction. Since $\wp_{h(1)}$ is bounded, then all the conditions of our Theorem are satisfied. Therefore, h has a unique fixed point in Ω, which is 1.

We will apply the above result to obtain the associated fixed point theorems for probabilistic ω-contraction in controlled strong b-Menger spaces. Before starting our next main result, we will need the following lemma.

Lemma 4.6. *Let $(\Omega, \Gamma, \pi, \mu)$ be a controlled strong probabilistic b-metric space such that Ran$\Gamma \subset \triangle^+$, so each Cauchy sequence is a bounded sequence.*

Proof. Even in a controlled strong probabilistic b-metric space, the arguments of proof in ([17], Lemma 2.4) are still the same, because the triangle inequality will not be used. □

Theorem 4.7. *Let $(\Omega, \Gamma, \mathfrak{H}, \mu)$ be a completely controlled strong b-Menger space with Ran$\Gamma \subset \triangle^+$ such that $\mathfrak{H}$ is a continuous t-norm of* H-type *and h is a probabilistic ω-contraction mapping on Ω where $\omega \in \xi$. If the function μ is bounded from the right, then h has a unique fixed point. Furthermore, the sequence $\{h^n x\}$ converges to b.*

Proof. Let $x_0 \in \Omega$ be arbitrary, by the contractivity condition, we have

$$\Gamma_{x_n,x_{n+1}}(\omega(t)) \geq \Gamma_{x_{n-1},x_n}(\mu(x_{n-1}, x_n)t), \quad \text{for all } n \in \mathbb{N} \text{ and } t > 0.$$

as in the lemma's 4.3 proof we get

$$\Gamma_{x_n,x_{n+1}}(\omega^n(t)) \geq \Gamma_{x_0,x_1}(t), \quad \text{for all } n \in \mathbb{N} \text{ and } t > 0.$$

Next we show that $\lim_{n\to+\infty} \Gamma_{x_n,x_{n+1}}(t) = 1$ for all $t > 0$.

Since $\lim_{t\to+\infty} \Gamma_{x_0,x_1}(t) = 1$. Then for each $\tau \in (0, 1]$ there exists $t_0 > 0$ satisfy $\Gamma_{x_0,x_1}(t_0) > 1 - \tau$. Let $t_1 \geq t_0$, as $\omega \in \xi$, then $\lim_{n\to+\infty} \omega^n(t_1) = 0$.

So for each $t > 0$ there exists $n_0 \in \mathbb{N}$ satisfy $\omega^n(t_1) < t$ for each $n \geq n_0$. By the monotonicity of Γ we have

$$\begin{aligned}\Gamma_{x_n,x_{n+1}}(t) &\geq \Gamma_{x_n,x_{n+1}}(\omega^n(t_1)) \\ &\geq \Gamma_{x_0,x_1}(t_1) \\ &\geq \Gamma_{x_0,x_1}(t_0) > 1-\tau, \quad \text{for all } n \geq n_0\end{aligned}$$

which is implied that

$$\lim_{n\to+\infty} \Gamma_{x_n,x_{n+1}}(t) = 1, \quad \text{for all } t > 0. \tag{4.5}$$

Let $n \in \mathbb{N}$, now we demonstrate by induction that for any $k \in \mathbb{N}$,

$$\Gamma_{x_n,x_{n+k}}(\mu(x_n,x_{n+k})t) \geq \mathfrak{H}^k\left(\Gamma_{x_n,x_{n+1}}(t-\omega(t))\right). \tag{4.6}$$

Verifying the validity of the inequality (4.6) for $k = 0, 1, 2$ is straightforward.

Let's say that (4.6) holds for $k > 2$. Using the contraction condition and the monotonicity of $\mathfrak{H}$ we have

$$\begin{aligned}&\Gamma_{x_n,x_{n+k+1}}(\mu(x_n,x_{n+k})t) \\ &\quad= \Gamma_{x_n,x_{n+k+1}}\left(\mu(x_n,x_{n+k})(t-\omega(t)) + \mu(x_n,x_{n+k})\omega(t)\right) \\ &\quad\geq \Gamma_{x_n,x_{n+k+1}}\left(\mu(x_n,x_{n+k})(t-\omega(t)) + \omega(t)\right) \\ &\quad\geq \mathfrak{H}(\Gamma_{x_n,x_{n+1}}(t-\omega(t)), \Gamma_{x_{n+1},x_{n+1+k}}(\omega(t)) \\ &\quad\geq \mathfrak{H}(\Gamma_{x_n,x_{n+1}}(t-\omega(t)), \Gamma_{x_n,x_{n+k}}(\mu(x_n,x_{n+k})t)) \\ &\quad\geq \mathfrak{H}(\Gamma_{x_n,x_{n+1}}(t-\omega(t)), \mathfrak{H}^k(\Gamma_{x_n,x_{n+1}}(t-\omega(t))) \\ &\quad= \mathfrak{H}^{k+1}(\Gamma_{x_n,x_{n+1}}(t-\omega(t))).\end{aligned}$$

Hence (4.6) is proved for all $k \in \mathbb{N}$.

As μ is bounded from the rigth, then a real constant $c \geq 1$ exist such that for each $x, y \in \Omega$ we have $\mu(x,y) \leq c$.

Then, by the inequality (4.6), for each $n, k \in \mathbb{N}$ and $t > 0$ we get

$$\Gamma_{x_n,x_{n+k}}(ct) \geq \mathfrak{H}^k\left(\Gamma_{x_n,x_{n+1}}(t-\omega(t))\right). \tag{4.7}$$

Finally we show that $\{x_n\}$ is a Cauchy sequence.
Let $\tau \in (0,1)$, from that $\mathfrak{H}$ is of *H-type*, there exists $\delta > 0$ satisfy

$$\mathfrak{H}^n(t) > 1-\tau, \quad \text{for all } t \in (1-\delta, 1] \text{ and } n \in \mathbb{N}. \tag{4.8}$$

By (4.5) we have

$$\lim_{n\to+\infty} \Gamma_{x_n,x_{n+1}}(t-\omega(t)) = 1.$$

So, there exists $N \in \mathbb{N}$ that verify $\Gamma_{x_n,x_{n+1}}(t-\omega(t)) > 1-\delta$ for all $n \geq N$.

From (4.8) and (4.7) we get

$$\Gamma_{x_n,x_{n+k}}(ct) > 1-\tau, \quad \text{for all } n \geq N \text{ and } k \in \mathbb{N}.$$

Hence, $\{x_n\}$ is a Cauchy sequence. It follows by lemma 4.6 that $\wp_h(x)$ is bounded, then from Theorem 4.4, we conclude that h has a unique fixed point. □

Since we didn't use the condition "$\Gamma_{p,q}(+\infty) = 1$" in the proof of theorem 4.7, then we arrive at the corresponding theorem in controlled strong fuzzy b-metric spaces where $\Gamma_{x,y}(t) = \mathcal{Q}(x, y, t)$ for all $t > 0$.

Theorem 4.8. *Let $(\Omega, \mathcal{Q}, \mathfrak{H}, \mu)$ be a completely controlled strong fuzzy b-metric space under a continuous t-norm $\mathfrak{H}$ of* H-type *with $\lim_{t\to+\infty} \mathcal{Q}(x, y, t) = 1$ for all $x, y \in \Omega$ and h is a probabilistic ω-contraction mapping on Ω such that $\omega \in \xi$. If the function μ is bounded from the rigth, then h admits a unique fixed point. Furthermore, the sequence $\{h^n x\}$ converges to b.*

Note that the condition of boundedness on μ is necessary to ensure the existence of the fixed point. On the other hand, if we set $\mu(x, y) = 1 \geq 1$, we get the result obtained by Ćirić in [8].

Corollary 4.9 ([8])**.** *Let $(\Omega, \Gamma, \mathfrak{H})$ be a complete Menger space where $Ran\Gamma \subset \triangle^+$ under a continuous t-norm $\mathfrak{H}$ of* H-type *and h is a probabilistic ω-contraction mapping on Ω with $\omega \in \xi$. Then h admits a unique fixed point. Furthermore, the sequence $\{h^n x\}$ converges to b.*

5. Related Results in Controlled Strong b-Metric Space

We prove the equivalent fixed point theorem in ordinary controlled strong b-metric spaces as an application of the main result.

Theorem 5.1. *Let (Ω, α_μ) be a complete controlled strong b-metric space such that μ is bounded from the rigth and $\omega \in \xi$ is a strictly non-decreasing function. Let h be a self mapping of (Ω, α_μ) that satisfy for all $x, y \in \Omega$*

$$\alpha_\mu(hx, hy) \leq \omega\left(\frac{\alpha_\mu(x, y)}{\mu(x, y)}\right). \tag{5.1}$$

Then h admit a unique fixed point b, and $h^n(x) \to b$ for all $x \in \Omega$.

Proof. First we define $\Gamma : \Omega \times \Omega \to \triangle^+$ by

$$\Gamma_{x,y}(t) = \epsilon_0(t - \alpha_\mu(x, y)). \tag{5.2}$$

From lemma 3.6, $(\Omega, \Gamma, \mathfrak{H}_M, \mu)$ is a complete controlled strong b-Menger space.
We need to demonstrate that (5.1) lead to the fact that h is a probabilistic ω-contraction mapping in $(\Omega, \Gamma, \mathfrak{H}_M, \mu)$.

Let $t \geq 0$, then

$$\Gamma_{hx,hy}(\omega(t)) = \epsilon_0(\omega(t) - \alpha_\mu(hx, hy))$$

$$= \begin{cases} 0, & \omega(t) \leq \alpha_\mu(hx, hy); \\ 1, & \omega(t) > \alpha_\mu(hx, hy). \end{cases}$$

Case 1: If $\Gamma_{x,y}(\mu(x,y)t) = 0$, then $\Gamma_{hx,hy}(\omega(t)) \geq \Gamma_{x,y}(\mu(x,y)t)$.

Case 2: If $\Gamma_{x,y}(\mu(x,y)t) = 1$, then $t > \dfrac{\alpha_\mu(x,y)}{\mu(x,y)}$.

And from that ω is strictly non-decreasing we obtain

$$\omega(t) > \omega\left(\frac{\alpha_\mu(x,y)}{\mu(x,y)}\right) \geq \alpha_\mu(hx, hy).$$

It follows from (5.1) that $\Gamma_{hx,hy}(\omega(t)) = 1$. Which implies that $\Gamma_{hx,hy}(\omega(t)) \geq \Gamma_{x,y}(\mu(x,y)t)$.

Therefore, we conclude that h is a probabilistic ω-contraction from case 1 and case 2. So, the conclusion of Theorem 5.1 comes from Theorem 4.7. □

Taking $\omega(t) = kt$ in Theorem 5.1, then we get the result shown below.

Corollary 5.2. *Let (Ω, α_μ) be a complete strong b-metric space such that μ is bounded from the rigth. Let h be a self mapping of (Ω, α_μ), if there is $k \in]0,1[$ satisfy for all $x, y \in \Omega$*

$$\alpha_\mu(hx, hy) \leq k\frac{\alpha_\mu(x,y)}{\mu(x,y)}. \tag{5.3}$$

Then h admit a unique fixed point b, and $h^n(x) \to b$ for all $x \in \Omega$.

6. Conclusion

As a probabilistic generalization of controlled strong b-metric space, we introduced the idea of controlled strong probabilistic and fuzzy b-metric space in this article. We also found certain fixed-point findings in these kinds of spaces. As a consequence of our main result, we obtained the corresponding fixed point result in probabilistic metric space by taking $\mu(x,y) = 1 \geq 1$. In addition, we presented the related results on usual controlled strong b-metric spaces to support the results thus obtained.

References

1. Y. Achtoun, M.L. Sefian and I. Tahiri, *(f, g)-φ-contraction mappings in menger*, Results in Nonlinear Analysis, 6 (2023),pp. 97–106.
2. R.P. Agarwal, U. Aksoy, E. Karapinar and I.M. Erhan, *F-contraction mappings on metric-like spaces in connection with integral equations on time scales*, RACSAM, 114 (2020), pp. 147.

3. I.A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal., 30 (1989), pp. 26-37.
4. S. Banach, *Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales*, Fundamenta mathematicae, 3 (1922), pp. 133-181.
5. M.I. Berenguer, M.V.F. Munoz, A.I.G. Guillem and M.R. Galan, *Numerical treatment of fixed point applied to the nonlinear Fredholm integral equation*, Fixed Point Theory Appl., 2009 (2009),pp. 1-8.
6. V. Berinde and M. Păcurar, *Existence and approximation of fixed Points of enriched contractions and enriched φ-contractions*, Symmetry, 13 (2021),pp. 498.
7. D.W. Boyd and J.S.W. Wong, *On nonlinear contractions*, Proc. Am. Math. Soc., 20 (1969), pp. 458-464.
8. L. Ćirić, *Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces*, Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods, 72 (2010), pp. 2009-2018.
9. S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena, 46 (1998), pp. 263-276.
10. S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav., **1** (1993), pp. 5-11.
11. J.X. Fang, *On φ-contractions in probabilistic and fuzzy metric spaces*, Fuzzy Sets Syst., 267 (2015), pp 86-99.
12. O. Hadžić and E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic Publishers, Dordrecht (2001).
13. T. Kamran, M. Samreen and Q. ULAin, *A generalization of b-metric space and some fixed point theorems*, Mathematics, 5 (2017), pp 19.
14. N. Kir and H. Kiziltun, *On Some well known fixed point theorems in b-Metric spaces*, Turkish journal of analysis and number theory, **1** (2013), pp. 13-16.
15. W. Kirk and N. Shahzad, *Fixed Point Theory in Distance Spaces*, Cham, Switzerland: Springer International Publishing, (2014).
16. H. Lakzian, D. Gopal and W. Sintunavarat, *New fixed point results for mappings of contractive type with an application to nonlinear fractional differential equations*, J. Fixed Point Theory Appl., 18 (2016), pp. 251-266.
17. A. Mbarki, A. Ouahab and I. Tahiri, *(f, g)-Boyd-Wong Contraction Mappings in Probabilistic Metric Space*, Appl. Math. Sci., 7 (2013), pp. 623-632.
18. A. Mbarki and R. Oubrahim, *Probabilistic b-metric spaces and nonlinear contractions*, Fixed Point Theory Appl., 2017 (2017), pp. 1-15.

19. K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci. U. S. A., 2 (1942), pp. 535-537.
20. Z. D. Mitrović and S. Radenović, *The Banach and Reich contractions in $b_v(s)$ $bv(s)$-metric spaces*, J. Fixed Point Theory Appl., 19 (2017), pp. 3087-3095.
21. D. Santina, WA. Mior Othman, KB. Wong and N. Mlaiki, *New Generalization of Metric-Type Spaces-Strong Controlled*, Symmetry, 15 (2023), pp. 416.
22. T. Oner, *On topology of fuzzy strong b-metric spaces*, Journal of New Theory, 2018 (2018), pp. 59-67.
23. B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland, New York (1983).
24. V.M. Sehgal and A.T. Bharucha-Reid, *Fixed points of contraction mappings on probabilistic metric spaces*, Math. Syst. Theory, 6 (1972), pp. 97-102.
25. H. Sherwood, *Complete probabilistic metric spaces*, Z. Wahrscheinlichkeitstheor. Verw. Geb., 20 (1971), pp. 117-128.

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