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Quotient Banach Lattices Under Unbounded Norm Topology

Leila Hasanzadeh[1] and Asghar Ranjbari[2*]

Abstract. In this paper, we verify some extra properties in quotient Banach lattices. Especially, we characterize the zero neighborhoods under unbounded norm topology (un-topology) in quotient Banach lattices. We verify some types of convergence in a quotient Banach lattice. Also, we show that if a Banach lattice is atomic, then the quotient Banach lattice so is. Finally, we show that the quotient preserves Schur and Banach-Saks properties.

1. Introduction

A Riesz space (or a vector lattice) is an ordered vector space X with the additional property that for each pair of vectors $x, y \in X$ the supremum and the infimum of the set $\{x, y\}$ both exist in X. A vector x in a vector lattice X is called positive whenever $x \geq 0$ holds. The set of all positive vectors of X will be denoted by X^+, i.e., $X^+ = \{x \in X : \ x \geq 0\}$. The set X^+ of positive vectors is called the positive cone of X.

A net (x_α) in a vector lattice X is order convergent to $x \in X$, written as $x_\alpha \xrightarrow{o} x$, if there exists a net (y_β) possibly over a different index set, such that $y_\beta \downarrow 0$ and there exists α_0 such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_0$. A subset A of a Riesz space is said to be order closed whenever $(x_\alpha) \subseteq A$ and $x_\alpha \xrightarrow{o} x$ imply $x \in A$. A net (x_α) in a vector lattice X is unbounded order convergent to $x \in X$, written as $x_\alpha \xrightarrow{uo} x$, if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in X^+$.

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A subset I of a Riesz space is called solid whenever $|x| \leq |y|$ and $y \in I$ imply $x \in I$. A solid vector subspace of a Riesz space is referred to as an ideal. An order closed ideal is referred to as a band.

For every element x of a vector lattice X, the principal ideal generated by x, X_x is defined as the following

$$X_x = \{y \in X : \exists \lambda > 0 \text{ with } |y| \leq \lambda |x|\}.$$

Moreover, the principal band generated by a vector x, B_x is given by

$$B_x = \{y \in X : \ |y| \wedge n|x| \uparrow |y|\}.$$

A vector lattice X is called Archimedean whenever $\frac{1}{n}x \downarrow 0$ holds in X for each $x \in X^+$.

Recall that a vector $x > 0$ ($x \in X^+, x \neq 0$) in a vector lattice X is an atom if for any $u, v \in [0, x]$ with $u \wedge v = 0$, either $u = 0$ or $v = 0$. If X is an Archimedean vector lattice, then $x > 0$ in X is an atom if and only if X_x is the one-dimensional subspace generated by x. An Archimedean vector lattice X is called atomic if it is the band generated by its atoms ([5, Chapter 2]).

Recall that a norm $\|.\|$ on a Riesz space is said to be a lattice norm whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A Riesz space equipped with a lattice norm is known as a normed Riesz space. If a normed Riesz space is also norm complete, then it is referred to as a Banach lattice. In this paper, X^* denotes the norm dual of Banach lattice X. A net (x_α) in a Banach lattice X is weak convergent to $x \in X$, written as $x_\alpha \xrightarrow{w} x$, if $x^*(x_\alpha) \to x^*(x)$ in $\mathbb{R}$ for all $x^* \in X^*$. A net (x_α) in a Banach lattice X is unbounded norm (resp. unbounded absolute weak) convergent to $x \in X$, written as $x_\alpha \xrightarrow{un} x$ (resp. $x_\alpha \xrightarrow{uaw} x$), if $|x_\alpha - x| \wedge u$ converges to zero in norm (resp. weak) for all $u \in X^+$.

Let X be a Banach lattice and let I be a normed closed ideal of X. Let $\pi : X \to X/I$ that $\pi(x) = x + I$ for all $x \in X$, be the canonical mapping of X onto quotient space X/I. Recall that π is linear, continuous and open. We denote:

$$\dot{x} = \pi(x) = x + I,$$

for all $x \in X$.

Recall that the quotient norm on X/I defined as

$$\|\pi(x)\| = \inf\{\|x - z\| : z \in I\}, \quad (\pi(x) \in X/I).$$

A relation $\leq$ is defined on X/I by letting $\dot{x} \leq \dot{y}$ whenever there exist $x_1 \in \dot{x}$ and $y_1 \in \dot{y}$ with $x_1 \leq y_1$.

By the above norm and order relation, the quotient vector space X/I is a Banach lattice and the canonical projection π of X onto X/I is a lattice homomorphism. It follows that $\dot{x^+} = (\dot{x})^+$, $\dot{x^-} = (\dot{x})^-$, $|\dot{x}| = |\dot{x}|$.

Recall that a Banach lattice X is order continuous if $\|x_\alpha\| \to 0$ for any net (x_α) in X that order converges to 0. Also a Banach lattice X is σ-order continuous if $\|x_n\| \to 0$ for any sequence (x_n) in X that order converges to 0. The canonical projection of X onto X/I is not necessarily order continuous. It is order continuous if and only if I is a band. The canonical projection is σ-order continuous if and only if I is a σ-ideal (i.e. $(x_n) \subseteq I$ and $0 \leq x_n \uparrow x$ imply $x \in I$). For details, see e.g. [1, Chapter 2].

We recall some assertions which are generally for locally convex topological vector spaces (see e.g. [7, Ch. V, Propositions 2, 3]).

Proposition 1.1. *If $t : X \to Y$ is a linear mapping between two normed vector lattices (in generally locally convex spaces) which vanishing on the subspace I of X, we can write $t = u \circ \pi$, where u maps X/I into Y and π is the canonical mapping of X onto X/I. Now, t is continuous if and only if u so is.*

We note that t is continuous if and only if for each zero neighborhood V in Y,

$$t^{-1}(V) = (u \circ \pi)^{-1}(V) = \pi^{-1}(u^{-1}(V))$$

is a zero neighborhood in X, i.e. $u^{-1}(V)$ is a zero neighborhood in X/I. Thus t is continuous if and only if u so is.

Let X be a Banach lattice and I be a normed closed ideal of X. For every $\varepsilon > 0$ and non-zero $u \in X^+$, put

$$V_{\varepsilon,u} = \{x \in X : \||x| \wedge u\| < \varepsilon\}.$$

Recall that the collection of all sets of this form is a base of zero neighborhoods for a topology, and the convergence in this topology agrees with un-convergence. This topology is said un-topology which is a Hausdorff linear topology. Recall that a linear topology τ on a vector lattice X is referred to as locally solid if it has a local basis at zero consisting of solid sets. Since each $V_{\varepsilon,u}$, zero neighborhood under un-topology is solid, then the un-topology on X is a locally solid topology. For details, see e.g. [2, 3].

We refer the reader to [3] for a further review of properties of un-topology. For unexplained terminology on vector and Banach lattices we refer the reader to [1]. It is shown in [4] that the quotient space of sequential relative uniform complete vector lattice is sequential relative uniform complete. It is shown in ([1, Theorem 2.23]) that if X is a vector lattice and I is a uniformly closed ideal of X, then the quotient space X/I is Archimedean vector space. Also, it is proven that if X is an AM-space (resp. an AL-space), then X/I is likewise an AM-space (resp. an AL-space) (see e.g. [1, Ch. 4]).

2. Main results

Let X be a Banach lattice with un-topology. If

$$\mathfrak{B} = \left\{V_{\varepsilon,u} : 0 \neq u \in X^+, \varepsilon > 0\right\}$$

is a base of zero neighborhoods for un-topology, then

$$\begin{aligned}\mathfrak{B}_{X/I} &= \{\pi(V_{\varepsilon,u});\ V_{\varepsilon,u} \in \mathfrak{B}\} \\ &= \{x + I\ ;\ x \in V_{\varepsilon,u}\} \\ &= \{x + I\ ;\ \||x| \wedge u\| < \varepsilon\},\end{aligned}$$

is a base of zero neighborhoods for un-topology on X/I.

Now, we characterize the zero neighborhoods of quotient un-topology.

Theorem 2.1. *Let I be a normed closed ideal of a Banach lattice X. Then $\dot{V}_{\varepsilon,u} \subseteq V_{\varepsilon,\dot{u}}$. Moreover, if $\|u\| < \varepsilon$, then $V_{\varepsilon,\dot{u}} = \dot{V}_{\varepsilon,u}$.*

Proof. Let $\dot{x} \in \dot{V}_{\varepsilon,u} = \{\dot{x} :\ x \in V_{\varepsilon,u}\}$. Therefore $\||x| \wedge u\| < \varepsilon$. We must show that $\||\dot{x}| \wedge \dot{u}\| < \varepsilon$. Observe that

$$\begin{aligned}|\dot{x}| \wedge \dot{u} &= |x + I| \wedge (u + I) \\ &= |x|^{\cdot} \wedge \dot{u} \\ &= (|x| \wedge u)^{\cdot}.\end{aligned}$$

Now, we show that $\|(|x| \wedge u)^{\cdot}\| < \varepsilon$. By the assumption, we have $\||x| \wedge u\| < \varepsilon$. Since $0 \in I$, then

$$\inf\{\||x| \wedge u - i\| : i \in I\} < \varepsilon.$$

It follows that $\||\dot{x}| \wedge \dot{u}\| < \varepsilon$. Therefore $\dot{x} \in V_{\varepsilon,\dot{u}}$ and $\dot{V}_{\varepsilon,u} \subseteq V_{\varepsilon,\dot{u}}$. Now, suppose that $\|u\| < \varepsilon$. Let $\dot{x} \in V_{\varepsilon,\dot{u}}$ but $\dot{x} \notin \dot{V}_{\varepsilon,u}$. Then $(x + i)^{\cdot} \notin \dot{V}_{\varepsilon,u}$ for all $i \in I$. So $x + i \notin V_{\varepsilon,u}$ for all $i \in I$. We have $\||x + i| \wedge u\| \geq \varepsilon$ for all $i \in I$. Since $|x + i| \wedge u \leq u$, then $\||x + i| \wedge u\| \leq \|u\|$ for all $i \in I$. Therefore $\varepsilon \leq \|u\|$. This is a contradiction. So $\dot{x} \in \dot{V}_{\varepsilon,u}$ and $V_{\varepsilon,\dot{u}} \subseteq \dot{V}_{\varepsilon,u}$. □

Remark 2.2. The first part of Theorem 2.1 shows that the quotient map is un-continuous.

In the following example we show that the assumption $\|u\| < \varepsilon$ in the last part of Theorem 2.1 is necessary.

Example 2.3. Let $X = \mathbb{R}^2$. By coordinatewise order and Euclidean norm, X is a Banach lattice. Consider $I = \{(x, 0) :\ x \in \mathbb{R}\}$. It is easy to see that I is a normed closed ideal of $\mathbb{R}^2$. Consider $u = (1, 0)$ and $\varepsilon = \frac{1}{2}$. Note that $\|u\| = 1$ and $\|u\| \not\leq \varepsilon = \frac{1}{2}$. We have

$$\dot{V}_{\varepsilon,u} = \{\dot{x} :\ x \in V_{\varepsilon,u}\}$$

$$
\begin{aligned}
&= \left\{(a,b)+I:\ \||(a,b)|\wedge(1,0)\|<\frac{1}{2}\right\}\\
&= \left\{(a,b)+I:\ \|(|a|\wedge 1,|b|\wedge 0)\|<\frac{1}{2}\right\}\\
&= \left\{(a,b)+I:\ \|(|a|\wedge 1,0)\|<\frac{1}{2}\right\}\\
&= \left\{(a,b)+I:\ |a|\wedge 1<\frac{1}{2}\right\}\\
&= \left\{(a,b)+I:\ |a|<\frac{1}{2}\right\}\\
&\subsetneqq \mathbb{R}^2/I.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
V_{\varepsilon,\dot{u}} &= \left\{\dot{x}:\ \||\dot{x}|\wedge\dot{u}\|<\frac{1}{2}\right\}\\
&= \left\{(a,b)+I:\ \||(a,b)+I|\wedge(u+I)\|<\frac{1}{2}\right\}\\
&= \left\{(a,b)+I:\ \||(a,b)+I|\wedge I\|<\frac{1}{2}\right\}\\
&= \left\{(a,b)+I:\ \|I\|<\frac{1}{2}\right\}\quad (\text{note that } \|I\|=0)\\
&= \{(a,b)+I:\ a,b\in\mathbb{R}\}\\
&= \mathbb{R}^2/I.
\end{aligned}
$$

Therefore $\dot{V}_{\varepsilon,u}\neq V_{\varepsilon,\dot{u}}$.

A net (x_α) in an Archimedean vector lattice X is uniform convergent to $x\in X$, written as $x_\alpha \xrightarrow{u} x$, if there exists $e\in X^+$ such that for every $\varepsilon>0$ there exists α_0 such that $|x_\alpha - x|\leq \varepsilon e$ whenever $\alpha\geq\alpha_0$.

The next proposition shows that the quotient mapping $\pi: X\to X/I$ is unbounded norm continuous, weakly continuous, unbounded absolutely weakly continuous and uniform continuous.

Proposition 2.4. *Let X be a Banach lattice, I be a normed closed ideal of X and (x_α) be a net in X. Then*

(a) *If $x_\alpha \xrightarrow{un} x$ in X, then $\dot{x}_\alpha \xrightarrow{un} \dot{x}$ in X/I.*
(b) *If $x_\alpha \xrightarrow{w} x$ in X, then $\dot{x}_\alpha \xrightarrow{w} \dot{x}$ in X/I.*
(c) *If $x_\alpha \xrightarrow{uaw} x$ in X, then $\dot{x}_\alpha \xrightarrow{uaw} \dot{x}$ in X/I.*
(d) *If $x_\alpha \xrightarrow{u} x$ in X, then $\dot{x}_\alpha \xrightarrow{u} \dot{x}$ in X/I.*

Proof. (a) Without loss of generality, let $x = 0$. Suppose that $x_\alpha \xrightarrow{un} 0$ in X. Take $v \in (X/I)^+$. Then $v = \pi(u)$ for some $u \in X^+$. We have

$$|\dot{x}_\alpha| \wedge v = |\pi(x_\alpha)| \wedge v = |\pi(x_\alpha)| \wedge \pi(u) = \pi(|x_\alpha| \wedge u),$$

because π is a lattice homomorphism. Since $|x_\alpha| \wedge u \to 0$ in norm and π is norm continuous, then we have $\pi(|x_\alpha| \wedge u) \to 0$ in norm. Therefore $\dot{x}_\alpha \xrightarrow{un} \dot{0}$ in X/I.

(b) By [1, Theorem 5.22], since π is a norm continuous operator, then π is weakly continuous. Hence $x_\alpha \xrightarrow{w} x$ implies $\pi(x_\alpha) \xrightarrow{w} \pi(x)$.

(c) Let $x_\alpha \xrightarrow{uaw} x$ in X. Let $\dot{y} \in (X/I)^+$. Then there exists $i \in I$ such that $y + i \geq 0$. By the assumption, $|x_\alpha - x| \wedge (y + i) \xrightarrow{w} 0$. By (b), $(|x_\alpha - x| \wedge (y+i))^{\cdot} \xrightarrow{w} \dot{0}$ in X/I. So $|\dot{x}_\alpha - \dot{x}| \wedge (y+i)^{\cdot} \xrightarrow{w} \dot{0}$. Therefore $|\dot{x}_\alpha - \dot{x}| \wedge \dot{y} \xrightarrow{w} \dot{0}$. Then $\dot{x}_\alpha \xrightarrow{uaw} \dot{x}$ in X/I.

(d) Let $x_\alpha \xrightarrow{u} x$ in X. Then there exists $e \in X^+$ such that for every $\varepsilon > 0$ there exists α_0 such that $|x_\alpha - x| \leq \varepsilon e$ whenever $\alpha \geq \alpha_0$. We have $\dot{e} \geq \dot{0}$ in X/I and

$$|\dot{x}_\alpha - \dot{x}| = |x_\alpha - x|^{\cdot} \leq (\varepsilon e)^{\cdot} = \varepsilon \dot{e}.$$

whenever $\alpha \geq \alpha_0$. It follows that $\dot{x}_\alpha \xrightarrow{u} \dot{x}$ in X/I. □

In the following example we show that the quotient mapping $\pi : X \to X/I$ is not generally order continuous:

Example 2.5. Consider the Banach lattice ℓ_∞ and its normed closed ideal c_0. Consider $u = (1, 1, 1, \dots)$ and the sequence $(x_n) \subseteq \ell^\infty$ which $x_n = (0, 0, \dots, 0, 1, 1, \dots)$ with n zeros at the head for all $n \in \mathbb{N}$. This is a decreasing sequence in ℓ_∞ with infimum 0, and then it converges to zero in order in ℓ_∞. On the other hand, $x_n - u \in c_0$ for all $n \in \mathbb{N}$. This yields that $\pi(x_n) = \pi(u)$ is a (non-zero) constant sequence in the quotient Banach lattice ℓ^∞/c_0 which does not converge to $\dot{0}$ in order.

We can also see that the quotient mapping is not uo-continuous. Indeed considering the sequence (x_n) in Example 2.5, since (x_n) is order convergent, then it is unbounded order convergent. On the other hand, $\pi(x_n)$ is order bounded and we know that for order bounded sequences, order convergent and unbounded order convergent are equivalent (see e.g. [6]). It follows that $\pi(x_n)$ is not unbounded order continuous.

Proposition 2.6. *Let X be a Banach lattice and I be a normed closed ideal in X. Let $x \in X$ be an atom in X such that $x \notin I$. Then $x + I$ is an atom in X/I.*

Proof. Suppose that $0 \leq v \leq \pi(x)$. Then $v = \pi(u)$ for some $u \in [0, x]$. Since every Banach lattice is Archimedean and x is an atom, then u is a scalar multiple of x. It follows that v, which is equal to $\pi(u)$, is a scalar multiple of $\pi(x)$. It follows that $\pi(x)$ is an atom in X/I, since X/I is an Archimedean vector lattice. □

Remark 2.7. If $x \in I$, then $x + I = I$ is the zero element of quotient space which is not atom in X/I, even x is an atom in X.

Theorem 2.8. *If X is an atomic Banach lattice and I is a normed closed ideal of X, then X/I is an atomic Banach lattice.*

Proof. Let X be atomic and $\dot{x} \in X/I$ be arbitrary. Since X is generated by its atoms, then for every $x \in X$ we can write $x = \sum_{i=1}^{n} \lambda_i x_i$ for some atoms $x_i \in X$, $\lambda_i \in \mathbb{R}$ and $n \in \mathbb{N}$. We have $\dot{x} = \sum_{i=1}^{n} \lambda_i \dot{x_i}$ which $\dot{x_i}$ (i= 1, 2, ..., n) are atoms in X/I by Proposition 2.6. This shows that X/I is atomic. □

Example 2.9. Suppose that Banach lattice $I = \{(x, 0) : x \in \mathbb{R}\}$ and $X = \mathbb{R}^2$ are as in Example 2.3. Also, suppose that $J = \{(0, y) : \ y \in \mathbb{R}\}$ which is a normed closed ideal in $\mathbb{R}^2$. Also, it is obvious that the elements of sets by the form $\{(a, 0) : \ a > 0\}$ and $\{(0, a) : \ a > 0\}$ are atoms in X. So X is an atomic Banach lattice ($X = \mathbb{R}^2$ generated by (1,0) , (0,1)). With considering I, since $(a, 0) \in I$, then $(a, 0)+I$ is the zero of quotient space X/I. Therefore $(a, 0)+I$ is not an atom in X/I but $(0, a)+I$ is an atom in X/I. Similarly, $(0, a) + J$ is not an atom in X/J but $(a, 0) + J$ is an atom in X/J.

Proposition 2.10. *If X is an atomic order continuous Banach lattice and I is a normed closed ideal of X, then un-topology on X/I is a locally convex topology.*

Proof. Since X is an order continuous Banach lattice, then X/I is also order continuous [1]. On the other hand, by Theorem 2.8, X/I is atomic. We know that un-topology on X is a locally convex topology iff X be an atomic order continuous Banach lattice ([3, Theorem 5.2]). It follows that un-topology on X/I is a locally convex topology. □

Theorem 2.11. *Let X and Y be order continuous Banach lattices and let t and u be the mappings defined as in Proposition 1.1. Then t is un-continuous if and only if u so is.*

Proof. If Y is not atomic, then by Proposition 5.1 of [3], the only convex neighborhood of zero for un-topology is Y itself. Therefore for any

neighborhood of zero V in X, $t(V) \subseteq Y$ and it follows that t is continuous. Similarly u is also continuous. Now, suppose that Y is atomic. Then Y is a locally convex space under un-topology. Therefore the proof is obtained by Proposition 1.1. □

An operator $T : X \to Y$ between two Banach spaces is a Dunford-Pettis operator whenever $x_n \xrightarrow{w} 0$ in X implies $\|Tx_n\| \to 0$ in Y. Note that every Dunford-Pettis operator is continuous. Recall that a Banach space is said to have the Schur property whenever every weak convergent sequence is norm convergent, i.e., whenever $x_n \xrightarrow{w} 0$ implies $x_n \xrightarrow{\|.\|} 0$. For details, see e.g. [1, Chapter 5].

Theorem 2.12. *Let X and Y be Banach lattices and I be a normed closed ideal of X. Let t and u be the mappings defined as in Proposition 1.1. Then u has Dunford-pettis property if and only if t has Dunford-pettis property.*

Proof. Let $u : X/I \to Y$ have the Dunford-pettis property and (x_n) be a sequence in X such that $x_n \xrightarrow{w} 0$. Since $\pi : X \to X/I$ is norm continuous, then π is weak continuous. Therefore $\dot{x_n} \xrightarrow{w} \dot{0}$. Then by the assumption, $\|u(\pi(x_n))\| \to 0$. This yields that $\|t(x_n)\| \to 0$. So t has the Dunford-pettis property. Conversely, let t have the Dunford-pettis property and $\dot{x_n} \xrightarrow{w} \dot{0}$ in X/I. Therefore $f(\dot{x}_n) \to f(\dot{0})$ for every $f \in (X/I)^*$. We conclude that $f \circ \pi(x_n) \to 0$. Suppose that $g \in X^*$ be arbitrary. We define the mapping f as follows:

$$f : X/I \to \mathbb{R}; \quad f(x + I) = g(x).$$

So $f \in (X/I)^*$ such that $g = fo\pi$. Then $g(x_n) \to 0$. It follows that $x_n \xrightarrow{w} 0$. By the assumption, $\|t(x_n)\| \to 0$. Since $t = u \circ \pi$, then $\|u(\dot{x}_n)\| \to 0$. So u has the Dunford-pettis property.

□

Theorem 2.13. *Let X and Y be Banach lattices and I be a normed closed ideal of X. Let u and t defined as in the Proposition 1.1. Then t is lattice homomorphism if and only if u so is.*

Proof. Let $t : X \to Y$ be a lattice homomorphism. Then by [1, Theorem 2.14], $t(|x|) = |t(x)|$ for all $x \in X$. We have $t = u \circ \pi$ which π is the quotient mapping. Suppose that $\dot{x} \in X/I$ be arbitrary. Then, we have

$$u(|\dot{x}|) = u(|x|\dot{}) = u \circ \pi(|x|) = |u \circ \pi(x)| = |u(\dot{x})|.$$

It follows that u is a lattice homomorphism. Conversely, Let $u : X/I \to Y$ be a lattice homomorphism and $x \in X$ be arbitrary. We have

$$t(|x|) = u \circ \pi(|x|) = u(|\pi(x)|) = u(|\dot{x}|) = |u(\dot{x})| = |u \circ \pi(x)| = |t(x)|.$$

Then t is a lattice homomorphism. □

Corollary 2.14. *For these u and t, t is a positive operator if and only if u so is.*

Proposition 2.15. *If the Banach lattice X has the Schur property and I is a normed closed ideal of X, then X/I has also the Schur property.*

Proof. Let $\dot{x}_n \xrightarrow{w} \dot{0}$. Then for every $f \in (X/I)^*$, $f \circ \pi(x_n) \to 0$. Let $g \in X^*$ be arbitrary. Similar to the proof of the Theorem 2.13, we have $g(x_n) \to 0$. Therefore $x_n \xrightarrow{w} 0$. Since X has the Schur property, then $\|x_n\| \to 0$. By the definition of quotient norm, we have $\|\dot{x}_n\| \leq \|x_n\|$. Then $\|\dot{x}_n\| \to 0$. Therefore X/I has the Schur property. □

We recall that a locally solid vector lattice (X, τ) has the Banach-Saks property if every τ-bounded sequence $(x_n) \subseteq X$ has a subsequence (x_{n_k}) which $\left(\frac{1}{n}\sum_{k=1}^{n} x_{n_k}\right)$ is τ-convergent.

Proposition 2.16. *Suppose that the Banach lattice X has the Banach-Saks property and I is a normed closed ideal of X. Then X/I has also the Banach-Saks property.*

Proof. Let $(\dot{x}_n)$ be a norm bounded sequence in X/I. Then there exists $M > 0$ such that $\|\dot{x}_n\| \leq M$, for all $n \in \mathbb{N}$. Then $\inf\{\|x_n + i\| : i \in I\} < M$. So there exists $i_n \in I$ such that $\|x_n + i_n\| < M$, for all n. It follows that $(x_n + i_n)$ is a norm bounded sequence in X. Since X has the Banach-Saks property, there exists a subsequence $(x_{n_k} + i_{n_k})$ of the above sequence such that $\left(\frac{1}{n}\sum_{k=1}^{n}(x_{n_k} + i_{n_k})\right)$ is norm convergent to some $x \in X$. Since the quotient mapping $\pi : X \to X/I$ is linear and continuous, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \pi(x_{n_k} + i_{n_k}) = \pi(x),$$

i.e. $\frac{1}{n}\sum_{k=1}^{n}(x_{n_k} + i_{n_k})^{\cdot} \xrightarrow{\|\cdot\|} \dot{x}$ in X/I. Then for the subsequence $(\dot{x}_{n_k})$ of $(\dot{x}_n)$, $\left(\frac{1}{n}\sum_{k=1}^{n}\dot{x}_{n_k}\right)$ is convergent in X/I under quotient norm. This yields that X/I has the Banach-Saks property. □

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