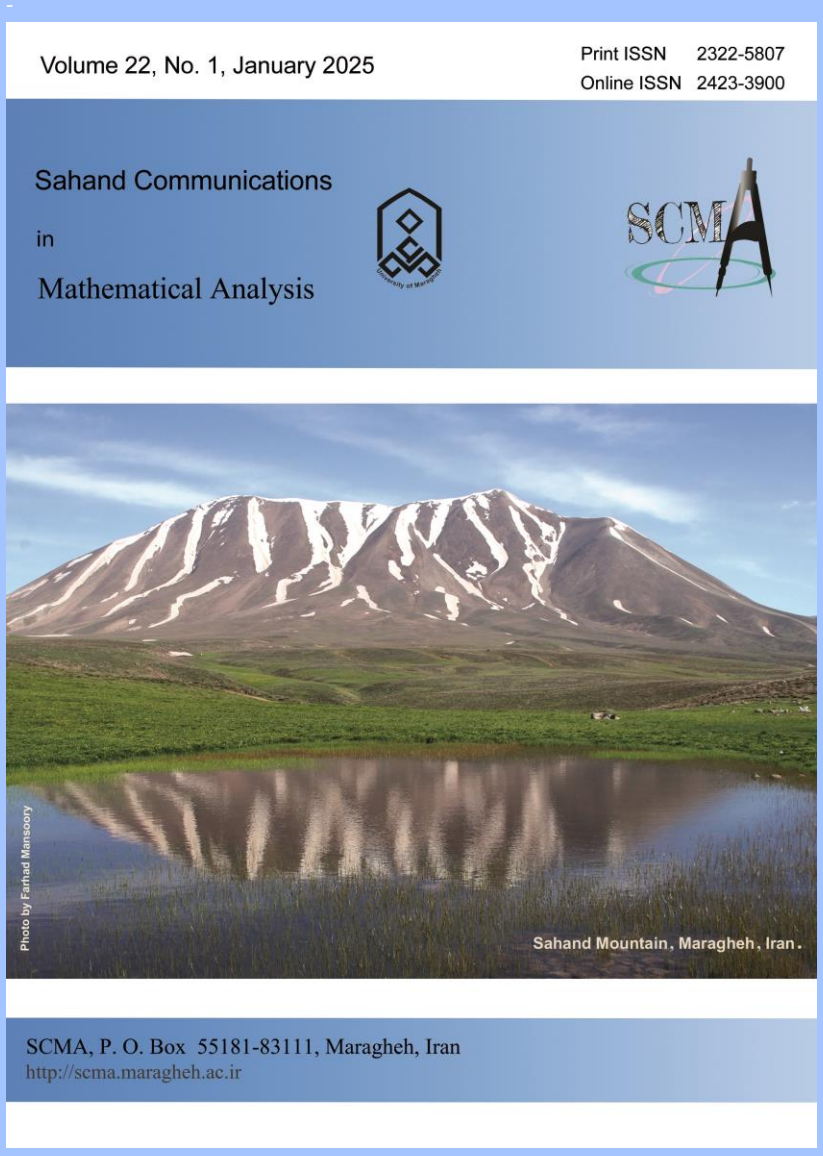
Convergence of Monotone Generalized α-nonexpansive Mappings in Ordered Hyperbolic Metric Spaces

Samir Dashputre and Kavita Sakure

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 22
Number: 1
Pages: 39-57

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2024.2009035.1426



Sahand Communications in Mathematical Analysis (SCMA) Vol. 22 No. 1 (2025), 39-57
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2024.2009035.1426

Convergence of Monotone Generalized α-nonexpansive Mappings in Ordered Hyperbolic Metric Spaces

Samir Dashputre[1] and Kavita Sakure[2*]

Abstract. In this paper, we discuss SPK iterative algorithm for approximating fixed point of monotone generalized α-nonexpansive mapping in the setting of partially ordered hyperbolic spaces. Indeed, we present some convergence results for the SPK iterative algorithm applied to monotone generalized α-nonexpansive mapping. Moreover, we provide an application of our results to nonlinear integral equations.

1. Introduction

Fixed point theory holds a prominent position in pure and applied mathematics due to its wide range of applications in various mathematical fields, including differential equations [15], integral equations [35], variational inequalities and approximation theory. Furthermore, the utility of fixed point results extend beyond mathematics and find relevance in fields such as statistics, computer science, chemistry, physics, economics, biology, medicine, engineering and game theory.

A nonexpansive mapping is one whose Lipschitz constant is equal to one. Non-expansive mapping existence results were provided independently by Browder [8], Göhde [11] and Kirk [18]. Since then, many

2020 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Partial ordered hyperbolic spaces, Monotone nonexpansive mappings, SPK iterative algorithm, Δ-convergence
Received: 2023-08-11, Accepted: 2024-02-18.
* Corresponding author.

researchers have generalized these findings in various ways. The fixed-point theory of non-expansive mapping has various applications in convex feasibility problems, convex optimization problems, monotone inequality problems, image restoration and more. Because of its applicability, many prominent researchers have generalized this theory and extended it to large classes of non-linear mappings.

Takahashi began the study of fixed point theory for nonexpansive mappings in a hyperbolic metric space setting [41]. Members of this class were referred to as convex metric spaces by him. Goebel and Kirk [10] used hyperbolic type spaces, which include spaces with hyperbolic metric (see also [12]). Reich and Shafrir [34], in 1990, proposed hyperbolic metric spaces on a general infinite dimensional manifold and examined iteration processes for nonexpansive mappings in these spaces using an additional hyperbolic metric condition. In 2004, Kohlenbach [22] proposed broader hyperbolic metric spaces to accommodate prior definitions of hyperbolic metric spaces. Busemann spaces [7] are well-known hyperbolic metric spaces. According to Leuştean [24], CAT(0) spaces are uniformly convex hyperbolic metric spaces. Bin Dehaish and Khamsi [2] recently proved a fixed point theorem for a monotone nonexpansive mapping in partially ordered hyperbolic metric spaces. Overall, these developments have seen contributions from various researchers [4, 27, 31, 38] and has led to the exploration of different types of hyperbolic metric spaces, opening up new avenues for studying fixed point properties in such settings.

There are some iterative algorithms in the literature that are used to approximate fixed points for nonexpansive mappings via Mann [26], Ishikawa [13], Noor [29], S-iteration process [3], normal S-iteration [36], Picard normal S-iteration process [14] iterative algorithms.

In 2022, Kim *et al.* [21] introduced the SPK iterative algorithm, an iterative approach outlined as follows.

Consider a complete uniformly convex hyperbolic space (Y, d, W), where M is a nonempty, convex and closed subset of Y. Let $\psi : M \to P(M)$ be a multivalued mapping. Define the sequence $\{\tau_m\}$ by the recursive scheme:

$$\begin{cases} \tau_1 \in M \ v_m = \psi((1-\beta_m)\tau_m + \beta_m\psi(\tau_m)), \\ u_m = \psi(v_m), \\ \tau_{m+1} = \psi((1-\alpha_m)\psi(v_m) + \alpha_m\psi(u_m)), \qquad \forall m \in \mathbb{N}. \end{cases} \tag{1.1}$$

Here, $\{\alpha_m\}$ and $\{\beta_m\}$ are real sequences in the interval $(0, 1)$. Kim *et al.* [21] proved that SPK (1.1) iterative algorithm is faster than all of Picard, Mann [26], Ishikawa [13], Noor [29], Agarwal [3], Abbas [1], Thakur [42] and Piri [33] iteration processes for contractive mappings in a Banach space. Also, Kim *et al.* [21] proved that SPK (1.1) iterative

algorithm is faster than all of Mann [26], Ishikawa [13], Noor [29], Normal S-iteration [36], Thakur [42], Piri [33] iteration processes for generalized α-nonexpansive mapping in hyperbolic spaces.

In this paper, we discuss SPK iterative algorithm, introduced by Kim *et al.* [21], for approximating fixed point of monotone generalized α-nonexpansive mapping, which contains nonexpansive and Suzuki type nonexpansive mappings in partially ordered hyperbolic spaces (POHS). This paper is divided into six sections. Section 2 presents some fundamental definitions and results that are utilized to demonstrate the primary results. In Section 3, we discuss the monotone generalized α-nonexpansive mappings in POHS. Section 4 shows the Strong and $\Delta-$convergence results for the SPK iterative algorithm for monotone generalized α-nonexpansive mappings. Finally, in Section 5, we show how our conclusions can be applied to nonlinear integral equations. This paper is summarized in Section 6 as the conclusion.

2. Preliminaries

Let us recollect the following definition provided by Kohlenbach [22].

Definition 2.1. Assume that (Y, d) be a metric space. Then triplet (Y, d, W) is referred to hyperbolic metric space, if there is a mapping $W : Y \times Y \times [0, 1] \to Y$ such that, for all $s, t, u, v \in Y$ and $\beta, \gamma \in [0, 1]$ the following hold:

(i) $d(u, W(s, t, \beta)) \leq \beta d(u, s) + (1 - \beta)\, d(u, t)$
(ii) $d(W(s, t, \beta), W(s, t, \gamma)) = |\beta - \gamma|\, d(s, t)$
(iii) $W(s, t, \beta) = W(t, s, (1 - \beta))$
(iv) $d(W(s, t, \beta), W(u, v, \beta)) \leq \beta d(s, t) + (1 - \beta)\, d(u, v)$.

Definition 2.2 ([23])**.** Assume that (Y, d, W) is hyperbolic metric space. Then Y is uniformly convex, if there exists a $\delta \in (0, 1]$ such that $d\left(\frac{1}{2}s \oplus \frac{1}{2}t, u\right) \leq (1 - \delta)r$, if for any $s, t, u \in Y$, $\varepsilon \in (0, 2]$ and $\rho > 0$, whenever $d(s, u) \leq \rho, d(t, u) \leq \rho$ and $d(s, t) \geq \varepsilon\rho$.

Definition 2.3. Assume that (Y, d) is metric space. Then Y is said to satisfy Property (R), if $\{F_m\}$ is decreasing sequence of nonempty, bounded, convex and closed subsets of Y, then $\bigcap_{m=1}^{\infty} F_m \neq \emptyset$.

Remark 2.4 ([6])**.** Every uniformly convex hyperbolic spaces enjoy the Property (R).

Let Y be a partially ordered set with a partial order '$\preceq$' and let $(Y, d, \preceq)$ be a partially ordered hyperbolic metric space. The elements $s, t \in Y$ are said to be comparable if $s \preceq t$ or $t \preceq s$. Denote order interval (cf. [6]) for $s, t \in Y$, as follows:

$$[t, \to) = \{x \in Y : t \preceq x\}$$

$$(\leftarrow, s] = \{x \in Y : x \preceq s\}.$$

Assume that (Y, d, W) is hyperbolic metric space and M is a nonempty subset of Y. Assume that $\{\theta_m\}$ bounded sequence in Y and $\theta \in M$. Then

(i) $r(\{\theta_m\}, \theta) = \limsup_{m\to\infty} d(\theta_m, \theta)$ is known as asymptotic radius of $\{\theta_m\}$ at θ;

(ii)
$$r(\{\theta_m\}, M) = \inf\{r(\{\theta_m\}, \theta) : \theta \in Y\}$$
is known as asymptotic radius of $\{\theta_m\}$ relative to M; and

(iii)
$$A(\{\theta_m\}, M) = \{\theta \in Y : r(\{\theta_m\}, M) = r(\{\theta_m\}, \theta)\}$$
is known as asymptotic center of $\{\theta_m\}$ relative to M.

Lim [25] developed the concept of Δ-convergence in a metric space. The idea of Δ-convergence is applied in CAT(0) space by Kirk and Panyanak [20] who demonstrated numerous Banach space results involving weak convergence have precise analogs in this environment.

Definition 2.5 ([20])**.** Assume that (Y, d, W) is hyperbolic metric space and $\{\theta_m\} \in Y$ is bounded sequence. Then $\{\theta_m\}$ is said to Δ-converge to $\theta \in Y$, if θ is unique asymptotic centre of every $\{\theta_{m_k}\}$, where $\{\theta_{m_k}\}$ is subsequence of $\{\theta_m\}$.

Definition 2.6 ([5])**.** Assume that M is a nonempty subset of a hyperbolic metric space (Y, d). Then a function $\Gamma : M \to [0, \infty)$ is called a type function, if there is a bounded sequence $\{\theta_m\}$ in M such that

$$\Gamma(\theta) = \limsup_{m\to\infty} d(\theta_m, \theta)$$

for any $\theta \in M$.

Note that every bounded sequence generates a unique type function. Now, we present the concept of Δ-convergence in hyperbolic metric space.

Definition 2.7. A bounded sequence $\{\theta_m\}$ in M is said to be Δ-convergent at a point $\theta \in M$, if θ is the unique and a type function generated by every subsequence $\{\theta_{m_k}\}$ of $\{\theta_m\}$ attains its infimum at θ.

Definition 2.8 ([6])**.** Let (Y, d, W) denote a metric space with the partial order $\preceq$. A self mapping ψ on Y is referred as monotone if

$$\psi(\tau) \preceq \psi(u) \quad \text{whenever } \tau \preceq u$$

for all $\tau, u \in Y$.

Definition 2.9 ([6])**.** Let (Y, d, W) denotes a metric space with the partial order $\preceq$. A self mapping ψ on Y is referred as monotone nonexpansive if S is monotone and

$$d(\psi(\tau_1), \psi(\tau_2)) \leq d(\tau_1, \tau_2)$$

for all $\tau, u \in Y$, where τ_1 and τ_2 are comparable.

In 2011, Aoyama *et al.* [5] defined α-nonexpansive mapping, as follow:

Definition 2.10. Assume that M is a nonempty subset of a Banach space Y. Then a mapping $\psi : M \to M$ is said to be $\alpha-$nonexpansive, if there is $\alpha \in [0, 1)$ such that

$$\|\psi(\tau_1) - \psi(\tau_2)\|^2 \leq \alpha\|\psi(\tau_1) - \tau_2\|^2 + \alpha\|\psi(\tau_2) - \tau_1\|^2 + (1 - 2\alpha)\|\tau_1 - \tau_2\|^2$$

for all $\tau_1, \tau_2 \in M$.

In 2016, Pant *et al.* [32] introduced the monotone generalized α-nonexpansive mapping in Banach space, as follows:

Definition 2.11. Assume that M is a nonempty subset of a Banach space Y. Then a mapping $\psi : M \to M$ is said to be generalized $\alpha-$nonexpansive, if for $\frac{1}{2}\|\tau_1 - \psi(\tau_1)\| \leq \|\tau_1 - \tau_2\|$, there is $\alpha \in [0, 1)$ such that

$$\|\psi(\tau_1) - \psi(\tau_2)\| \leq \alpha\|\psi(\tau_1) - \tau_2\| + \alpha\|\psi(\tau_2) - \tau_1\| + (1 - 2\alpha)\|\tau_1 - \tau_2\|$$

for all $\tau_1, \tau_2 \in M$, where τ_1 and τ_2 are comparable.

Lemma 2.12 ([16])**.** *Assume that (Y, d, W) is a uniformly convex hyperbolic space with monotone modulus of uniform convexity η and $\tau^* \in Y$ and $\{\alpha_m\}$ is a sequence such that $0 < p \leq \alpha_m \leq q < 1$. If $\{u_m\}$ and $\{v_m\}$ are sequences in Y such that $\limsup_{m\to\infty} d(u_m, \tau^*) \leq \rho$, $\limsup_{m\to\infty} d(v_m, \tau^*) \leq r\rho$ and $\limsup_{m\to\infty} d(\alpha_m v_m \oplus (1-\alpha_m)u_m, \tau^*) = \rho$, for $\rho \geq 0$. Then, $\lim_{m\to\infty} d(u_m, v_m) = 0$.*

3. Monotone Generalized α-Nonexpansive Mappings in Partially Ordered Hyperbolic Spaces

In this section, we present monotone generalized α-nonexpansive mappings in POHS and some results which will be used while proving the main results.

Definition 3.1. Assume that M is a nonempty subset of a POHS Y. Then a mapping $\psi : M \to M$ is said to be monotone generalized $\alpha-$nonexpansive, if $\psi : M \to M$ is monotone and for $\frac{1}{2}d(\tau_1, \psi(\tau_1)) \leq d(\tau_1, \tau_2)$, there is $\alpha \in [0, 1)$ such that

$$d(\psi(\tau_1), \psi(\tau_2)) \leq \alpha d(\psi(\tau_1), \tau_2) + \alpha d(\psi(\tau_2), \tau_1) + (1 - 2\alpha)d(\tau_1, \tau_2)$$

for all $\tau_1, \tau_2 \in M$, where τ_1 and τ_2 are comparable.

The following proposition is analogous to the result of Suzuki [40, Proposition 2].

Proposition 3.2. *Assume that M is nonempty subset of POHS $(Y, d, \preceq)$ and $\psi : M \to M$ is a monotone generalized α-nonexpansive mapping with a fixed point $\tau^* \in M$. Then ψ is quasi-nonexpansive.*

The following lemma shows the property of monotone generalized α-nonexpansive mappings, which can be proved similarly to [32, Lemma 3.8].

Lemma 3.3. *Assume that M is nonempty subset of POHS $(Y, d, \preceq)$ and $\psi : M \to M$ is a monotone generalized α-nonexpansive mapping. Then, for all $\tau_1, \tau_2 \in M$ such that τ_1 and τ_2 are comparable, we have*

$$d(\tau_1, \psi(\tau_2)) \leq \left(\frac{3+\alpha}{1-\alpha}\right) d(\tau_1, \psi(\tau_1)) + d(\tau_1, \tau_2).$$

Theorem 3.4. *Assume that M is nonempty, closed and convex subset of uniformly convex POHS $(Y, d, \preceq)$ and $\psi : M \to M$ is a monotone generalized α-nonexpansive mapping. Assume that there exists $\tau_1 \in M$ such that τ_1 and $\psi(\tau_1)$ are comparable and $\{\tau_m\}$ is a sequence defined by SPK iterative algorithm* (1.1)*. Then ψ has a fixed point.*

Proof. Without the loss of generality, we may assume that $\tau_1 \preceq \psi(\tau_1)$. Due to monotonicity of ψ, one get

$$\psi(\tau_1) \preceq \psi^2(\tau_1).$$

In similar way, one get

$$\psi(\tau_1) \preceq \psi^2(\tau_1) \preceq \psi^3(\tau_1) \preceq \psi^4(\tau_1) \cdots .$$

Define $\tau_m = \psi^m(\tau)$ for all $m \in \mathbb{N}$. Since Y is uniformly convex, therefore by Remark 2.4, one get

$$\begin{aligned} M_\infty &= \bigcap_{m=1}^{\infty} [\tau_m, \to) \bigcap M \\ &= \bigcap_{m=1}^{\infty} \{\tau \in M : \tau_m \preceq \tau\} \\ &\neq \emptyset. \end{aligned}$$

Assume that $\tau \in M_\infty$, then $\tau_m \preceq \tau$. By monotonicity of ψ, one get

$$\tau_m \preceq \psi(\tau_{m_r}) \preceq \psi(\tau).$$

Again, $\Gamma : M_\infty \to [0, \infty)$ is type function generated by the sequence $\{\tau_{m_r}\}$,

$$\Gamma(\tau) = \limsup_{r \to \infty} d(\tau_{m_r}, \tau).$$

Therefore, there is a unique element $t \in M_\infty$ such that

$$\Gamma(t) = \inf\{\Gamma(\tau) : \tau \in M_\infty\}.$$

By the definition of type function Γ,

$$\Gamma(\psi(\tau)) = \limsup_{r\to\infty} d(\tau_{m_r}, \psi(t)).$$

From Lemma 3.3, one get

$$\begin{aligned}\Gamma(\psi(\tau)) &= \limsup_{r\to\infty} d(\tau_{m_r}, \psi(t)) \\ &\leq \left(\frac{3+\alpha}{1-\alpha}\right) \limsup_{r\to\infty} d(\tau_{m_r}, \psi(\tau_{m_r})) + \limsup_{r\to\infty} d(\tau_{m_r}, \psi(t)) \\ &= \Gamma(\tau).\end{aligned}$$

Therefore, due to uniqueness of minimal element, we have, $\psi(\tau) = \tau$. □

4. Convergence Results for Monotone Generalized α-Nonexpansive Mapping

Lemma 4.1. *Assume that M is nonempty, closed and convex subset of uniformly convex POHS $(Y, d, \preceq)$ and $\psi : M \to M$ is a monotone mapping. Let $\tau_1 \in M$ be such that $\tau_1 \preceq \psi(\tau_1)$(or $\psi(\tau_1) \preceq \tau_1$). Then, for sequence $\{\tau_m\}$ defined by SPK iterative algorithm* (1.1)*, we have*

(i) $\tau_m \preceq \psi(\tau_m) \preceq \tau_{m+1}$(*or* $\tau_{m+1} \preceq \psi(\tau_m) \preceq \tau_m$);
(ii) $\tau_m \leq \tau^*$(*or* $\tau^* \preceq \tau_m$)*, provided* $\{\tau_m\}$ Δ*-converges to a point* $\tau^* \in M$*, for all* $m \in \mathbb{N}$.

Proof. By assumption, we have $\tau_1 \preceq \psi(\tau_1)$ and due to convexity of the ordered interval $[\tau_1, \psi(\tau_1)]$ and (1.1), one get

$$\tau_1 \preceq u_1 \preceq \psi(\tau_1).$$

By monotonocity of ψ, one get

$$\psi(\tau_1) \preceq \psi(u_1)$$

and due to convexity of the ordered interval $[\psi(u_1), \psi(v_1)]$ and (1.1), one get

$$\psi(u_1) \preceq v_1 \preceq \psi(u_1).$$

Hence, we have

$$\tau_1 \preceq u_1 \preceq \psi(\tau_1) \preceq v_1.$$

By monotonocity of ψ, one get

$$v_1 \preceq \psi(u_1) \preceq \psi(v_1)$$

and due to convexity of the ordered interval $[v_1, \psi(v_1)]$ and (1.1),

$$v_1 \preceq \tau_2 \preceq \psi(v_1).$$

Therefore, we have

$$\tau_1 \preceq u_1 \preceq \psi(\tau_1) \preceq v_1 \preceq \tau_2 \preceq \psi(v_1).$$

This result is true for $m = 1$. Similarly, for $m = k - 1$, one get

$$\tau_{m-1} \preceq \psi(\tau_{k-1}) \preceq \tau_m,$$

by mathematical induction, for each $m \in \mathbb{N}$,

$$\tau_m \preceq u_m \preceq \psi(\tau_m) \preceq v_m \preceq \tau_{m+1} \preceq \psi(u_m),$$

hence

$$\tau_m \preceq \psi(\tau_m) \preceq \tau_{m+1}.$$

Suppose that τ^* is $\Delta-$limit of τ_m. The sequence $\{\tau_m\}$ is monotone increasing and order interval $[\tau_m, \to)$ is convex and closed. We claim that $\tau^* \in [\tau_m, \to)$ for each $m \in \mathbb{N}$. If possible, suppose that $\tau^* \notin [\tau_m, \to)$, then type function generated by the subsequence $\{\tau_{m_r}\}$ of $\{\tau_m\}$ defined by leaving first (m-1) elements of sequence $\{\tau_m\}$ will not attain infimum at τ^*, so we get contradiction. □

Now, we will look at some strong convergence and Δ-convergence results for the SPK iterative algorithm (1.1) in POHS.

Theorem 4.2. *Assume that M is nonempty, closed and convex subset of uniformly convex POHS $(Y, d, \preceq)$ and $\psi : M \to M$ is a monotone generalized α-nonexpansive mapping. Suppose $F(\psi)$ is nonempty and τ_1 and τ^* are comparable for every $\tau^* \in F(\psi)$. Assume that there exists $\tau_1 \in M$ such that τ_1 and $\psi(\tau_1)$ are comparable and $\{\tau_m\}$ is a sequence defined by SPK iterative algorithm* (1.1)*. Then the following statements are correct.*

(i) $\{\tau_m\}$ *is bounded and* $\lim_{m\to\infty} d(\tau_m, \tau^*)$ *exists.*

(ii) $\max\{d(\tau_{m+1}, \tau^*), d(u_m, \tau^*), d(v_m, \tau^*) \leq d(u_m, \tau^*)\}$ *for all* $m \in \mathbb{N}$.

Proof. Without sacrificing generality, we assume that $\tau_1 \preceq \tau^*$. Since ψ is monotone, $\psi(\tau_1) \preceq \psi(\tau^*) = \tau^*$. By (1.1), we have

$$\tau_1 \preceq v_1 \preceq \psi(\tau_1) \preceq u_1 \leq \psi(v_1). \tag{4.1}$$

So, $v_1 \leq \tau^*$. By monotonicity of ψ,

$$\psi(v_1) \preceq \psi(\tau^*) = \tau^*,$$

and also, from (4.1), we have $w_1 \preceq \tau^*$. By monotonicity of ψ,

$$\psi(u_1) \preceq \psi(\tau^*) = \tau^*, \tag{4.2}$$

By (1.1)

$$\psi(v_1) \preceq \tau_2 \preceq \psi(u_1) \tag{4.3}$$

From (4.2) and (4.3), we have

$$\psi(\tau_2) \preceq \psi(\tau^*) = \tau^*. \tag{4.4}$$

Again from Lemma 4.1 and (4.4), for $m = 2$, we have

$$\tau_2 \preceq \psi(\tau_2) \preceq \tau^*.$$

Applying the same argument, we get

$$\tau_m \preceq \psi(\tau_m) \preceq \tau^*.$$

Now, by (1.1) and Proposition 3.2, we have

$$\begin{aligned} d(v_m, \tau^*) &= d(\psi((1-\beta_m)\tau_m \oplus \beta_m\psi(\tau_m)), \tau^*) \\ &\leq (1-\beta_m)d(\tau_m, \tau^*) + \beta_m d(\psi(\tau_m), \tau^*) \\ &\leq (1-\beta_m)d(\tau_m, \tau^*) + d(\tau_m, \tau^*) \\ &= d(\tau_m, \tau^*), \end{aligned} \tag{4.5}$$

also

$$\begin{aligned} d(u_m, \tau^*) &= d(\psi(v_m), \tau^*) \\ &\leq d(v_m, \tau^*) \\ &= d(\tau_m, \tau^*) \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} d(\tau_{m+1}, \tau^*) &= d((1-\alpha_m)\psi(u_m) \oplus \alpha_m\psi(v_m), \tau^*) \\ &\leq (1-\alpha_m)d(u_m, \tau^*) + \alpha_m d(v_m, \tau^*) \\ &\leq (1-\alpha_m)d(v_m, \tau^*) + \alpha_m d(v_m, \tau^*) \\ &\leq d(v_m, \tau^*) \\ &= d(\tau_m, \tau^*). \end{aligned}$$

Thus, $\{d(\tau_m, \tau^*)\}$ is bounded and decreasing sequence, so, $\lim_{m\to\infty} d(\tau_m, \tau^*)$ exists and

$$\max\{d(\tau_{m+1}, \tau^*), d(u_m, \tau^*), d(v_m, \tau^*) \leq d(u_m, \tau^*)\}$$

for all $m \in \mathbb{N}$. □

Theorem 4.3. *Assume that M is a nonempty, closed and convex subset of uniformly convex POHS $(Y, d, \preceq)$ and $\psi : M \to M$ is a monotone generalized α-nonexpansive mapping. Suppose $F(\psi)$ is nonempty and τ_1 and τ^* are comparable for every $\tau^* \in F(\psi)$. Assume that there exists $\tau_1 \in M$ such that τ_1 and $\psi(\tau_1)$ are comparable and $\{\tau_m\}$ be a sequence defined by SPK iterative algorithm* (1.1)*. Then the following statements are correct:*

(i) $\lim_{m\to\infty} dist(\tau_m, F(\psi))$ *exists, where* $dist(\tau, F(\psi))$ *denotes the distance from* τ *to* $F(\psi)$.

(ii) $\lim_{m\to\infty} d(\tau_m, \psi(\tau_m)) = 0.$

Proof. For every $\tau^* \in F(\psi)$ and $m \in \mathbb{N}$, one get

$$d(\tau_{m+1}, t^*) \leq d(\tau_m, \tau^*).$$

For $\tau^* \in F(\psi)$, one get

$$\operatorname{dist}(\tau_{m+1}, F(\psi)) \leq \operatorname{dist}(\tau_m, F(\psi))$$

for all $m \in \mathbb{N}$. Therefore the sequence $\operatorname{dist}(\tau_m, F(\psi))$ is bounded and decreasing. Hence $\lim_{m\to\infty} \operatorname{dist}(\tau_m, F(\psi))$ exists. Let

$$\lim_{m\to\infty} d(\tau_m, \tau^*) = l. \tag{4.7}$$

From (4.7) and Proposition 3.2, one get

$$\limsup_{m\to\infty} d(\psi(\tau_m), \tau^*) \leq l. \tag{4.8}$$

By (4.5) and (4.8), one get

$$\limsup_{m\to\infty} d(v_m, \tau^*) \leq l. \tag{4.9}$$

From (4.9) and Proposition 3.2, one get

$$\limsup_{m\to\infty} d(\psi(v_m), \tau^*) \leq l. \tag{4.10}$$

By (4.6) and (4.10), one get

$$\limsup_{m\to\infty} d(u_m, \tau^*) \leq l. \tag{4.11}$$

From (4.11) and Proposition 3.2, one get

$$\limsup_{m\to\infty} d(\psi(u_m), \tau^*) \leq l. \tag{4.12}$$

From (4.7) and (4.12), one get

$$\limsup_{m\to\infty} d(\tau_{m+1}, \tau^*) \leq l.$$

Now,

$$\begin{aligned} l &= \limsup_{m\to\infty} \{\gamma_m d(\tau_m, \tau^*) + (1-\gamma_m) d(\psi(\tau_m), \tau^*)\} \\ &\leq \limsup_{m\to\infty} d(\tau_m, \tau^*). \end{aligned} \tag{4.13}$$

From (4.13) and Lemma 2.12, one get

$$\lim_{m\to\infty} d(\psi(\tau_m), \tau_m) = 0.$$

□

Now, we present Δ-convergence result as follows.

Theorem 4.4. *Assume that M is nonempty, closed and convex subset of uniformly convex POHS $(Y, d, \preceq)$ and $\psi : M \to M$ is a monotone generalized α-nonexpansive mapping. Suppose τ_1 and τ^* are comparable for every $\tau^* \in F(\psi)$. Assume that there exists $\tau_1 \in M$ such that τ_1 and $\psi(\tau_1)$ are comparable and Suppose $F(\psi)$ is nonempty and totally ordered. Then, the sequence $\{\tau_m\}$ defined by SPK iterative algorithm (1.1) Δ-converges to fixed point of ψ.*

Proof. Due to Theorem 4.3, $\{\tau_m\}$ is bounded sequence. Thus, there is a subsequence $\{\tau_{m_k}\}$ of $\{\tau_m\}$ such that $\{\tau_{m_k}\}$ Δ-converges to point $\tau^* \in M$ such that

$$\tau_1 \preceq \tau_{m_k} \preceq \tau^*, \quad \forall m \in \mathbb{N}.$$

Now, we show that there is a unique Δ-limit in $F(\psi)$ corresponding to each Δ-convergent subsequence of $\{\tau_m\}$. Let us consider two subsequences $\{\tau_{m_k}\}$ and $\{\tau_{m_r}\}$ of $\{\tau_m\}$ which are Δ-convergent to l_1 and l_2, respectively. Then, $\{\tau_{m_k}\}$ is bounded and by Theorem 4.3,

$$\lim_{m\to\infty} d(\psi(\tau_{m_k}), \tau_{m_k}) = 0 \tag{4.14}$$

which concludes that $l_1 \in F(\psi)$. Suppose that $\Gamma : M \to [0, \infty)$ is a type mapping generated by $\{\tau_{m_k}\}$. Then,

$$\Gamma(l_1) = \limsup_{m\to\infty} d(\tau_{m_k}, l_1). \tag{4.15}$$

By Lemma 3.3,

$$\begin{aligned} \Gamma(\psi(l_1)) &= \limsup_{m\to\infty} d(\tau_{m_k}, \psi(l_1)) \\ &\leq \frac{3+\alpha}{1-\alpha} \limsup_{m\to\infty} d(\tau_{m_k}, \psi((\tau_{m_k}))) + \limsup_{m\to\infty} d(\tau_{m_k}, l_1) \\ &\leq \Gamma(l_1). \end{aligned} \tag{4.16}$$

Since l_1 is unique, one gets

$$\psi(l_1) = l_1.$$

Similarly,

$$\psi(l_2) = l_2.$$

Since Γ is continuous, one gets

$$\begin{aligned} \limsup_{m\to\infty} d(\tau_m, l_1) &= \limsup_{k\to\infty} d(\tau_{m_k}, l_1) \\ &< \limsup_{k\to\infty} d(\tau_{m_k}, l_2) \\ &= \limsup_{m\to\infty} d(\tau_m, l_2) \\ &= \limsup_{r\to\infty} d(\tau_{m_r}, l_2) \end{aligned}$$

$$\begin{aligned} &< \limsup_{r\to\infty} d(\tau_{m_r}, l_1) \\ &= \limsup_{m\to\infty} d(\tau_m, l_1), \end{aligned}$$

which is a contradiction, unless $l_1 = l_2$. □

Theorem 4.5. *Assume that M is nonempty, closed and convex subset of uniformly convex POHS $(Y, d, \preceq)$ and $\psi : M \to M$ is a monotone generalized α-nonexpansive mapping. Suppose τ_1 and τ^* are comparable for every $\tau^* \in F(\psi)$. Assume that there exists $\tau_1 \in M$ such that τ_1 and $\psi(\tau_1)$ are comparable. Suppose $F(\psi)$ is nonempty. Then, the sequence $\{\tau_m\}$ defined by SPK iterative algorithm (1.1) converges strongly to a fixed point of ψ if and only if $\liminf_{m\to\infty} dist\,(\tau_m, F(\psi)) = 0$, provided $F(\psi)$ is a totally ordered set.*

Proof. Assume that $\liminf_{m\to\infty} \text{dist}\,(\tau_m, F(\psi)) = 0$. From Theorem 4.3, $\lim_{m\to\infty} \text{dist}\,(\tau_m, F(\psi)) = 0$ exists, so

$$\lim_{m\to\infty} \text{dist}\,(\tau_m, F(\psi)) = 0.$$

First, we will prove that $F(\psi)$ is closed. For this, assume that $\{f_m\}$ is sequence in $F(\psi)$ converging to point $f^* \in M$. Since

$$\begin{aligned} \frac{1}{2} d(f_m, \psi(f_m)) &= 0 \\ &\leq d(f_m, f^*), \end{aligned}$$

for all $m \in \mathbb{N}$, we have

(4.17)

$$\begin{aligned} d(\psi(f_m), \psi(f^*)) &= d(f_m, \psi(f^*)) \\ &\leq \alpha d(\psi(f_m), f_m) + \alpha d(\psi(f^*), f^*) + (1 - 2\alpha) d(f_m, f^*) \\ &\leq \alpha d(\psi(f^*), f_m) + \alpha d(\psi(f_m), f^*) + (1 - 2\alpha) d(f_m, f^*). \end{aligned}$$

As $1 - \alpha > 0$,

$$\begin{aligned} \limsup_{m\to\infty} d(f_m, f^*) &\leq \limsup_{m\to\infty} d(f_m, f^*) \\ &= 0, \end{aligned}$$

therefore, $\{f_m\}$ converges strongly to $\psi(f^*)$, which implies that $\psi(f^*) = f^*$. Hence, $F(\psi)$ is closed. Since $\lim_{m\to\infty} \text{dist}\,(\tau_m, F(\psi)) = 0$, assume that $\{\tau_{m_n}\}$ is a subsequence of $\{\tau_m\}$ such that $d(\tau_{m_n}, f_r) \leq \frac{1}{2^r}$ for all $r \geq 1$, where $\{f_r\}$ is a sequence in $F(\psi)$. Due to Theorem 4.3, we get

$$\begin{aligned} d(\tau_{m_{r+1}}, f_r) &\leq d(\tau_{m_r}, f_r) \\ &\leq \frac{1}{2^r}, \end{aligned}$$

hence by triangular inequality,

$$\begin{aligned} d(f_{r+1}, f_r) &\leq d(f_{r+1}, u_{m_{r+1}}) + d(u_{m_{r+1}}, f_r) \\ &< \frac{1}{2^{r-1}}, \end{aligned}$$

which shows that $\{f_r\}$ is Cauchy sequence. Since $F(\psi)$ is closed, so that $\{f_r\}$ converges to some point $f^* \in F(S)$. Now,

$$d(\tau_{m_r}, f^*) \leq d(\tau_{m_r}, f_r) + d(f_r, f^*),$$

as $r \to \infty$, $\{\tau_{m_n}\}$ converges strongly to f^*. Due to Theorem 4.3, $\lim_{m\to\infty} d(\tau_m, f^*)$ exists, so that $\{\tau_m\}$ converges strongly to f^*. The converse part of the Theorem is trivial, hence the proof is completed. □

In the next result, we present the strong convergence of $\{\tau_m\}$ generated by (1.1) using Condition (I).

Definition 4.6 ([37])**.** Assume that M is a nonempty, convex and closed subset of hyperbolic space Y. The function $\psi : M \to M$ is referred to satisfy *Condition (I)*, if there is a non-decreasing mapping $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$ such that for all $v \in M$,

$$d(v, \psi(v)) \geq \phi(d(v, F(\psi)), \quad \forall v \in M.$$

Theorem 4.7. *Assume that M is nonempty, closed and convex subset of uniformly convex POHS $(Y, d, \preceq)$ and $\psi : M \to M$ is a monotone generalized α-nonexpansive mapping. Suppose τ_1 and τ^* are comparable for every $\tau^* \in F(\psi)$. Assume that there exists $\tau_1 \in M$ such that τ_1 and $\psi(\tau_1)$ are comparable. Suppose $F(\psi)$ is nonempty. If ψ satisfies Condition (I), then the sequence $\{\tau_m\}$ defined by SPK iterative algorithm* (1.1) *converges strongly to a fixed point of ψ.*

Proof. Due to Theorem 4.3,

$$\liminf_{m\to\infty} d(\psi(\tau_m), \tau_m) = 0.$$

Since ψ satisfies Condition (I), we get

$$d(\psi(\tau_m), \tau_m) \geq \phi(\text{dist } (\tau_m, F(\psi)),$$

so, we have

$$\liminf_{m\to\infty} \phi(\text{dist } (\tau_m, F(\psi)) = 0.$$

Therefore, by assumption, one gets

$$\liminf_{m\to\infty} \text{dist } (\tau_m, F(\psi)) = 0.$$

As all the assumptions of Theorem 4.5 are satisfied, so $\{\tau_m\}$ converges to a fixed point of ψ. □

5. Application

We will discuss about how to apply our findings to nonlinear integral equations in this section. In an ordered Banach space (OBS), which is a special case of POHS, we present the existence theorem.

Theorem 5.1. *Assume that M is nonempty compact subset of OBS Y and $\psi : M \to M$ is a monotone generalized α-nonexpansive mapping. Suppose that there is an AFPS $\{t_m\}$. Then ψ has a fixed point.*

Proof. Due to Theorem 4.3 and [40, Theorem 4], the proof is completed. □

Take the following nonlinear functional integral equation (FIE) as an example.

$$\zeta(\tau) = f(\tau) + \lambda \int_0^\tau \kappa(\tau, u) g(\tau, \zeta(u)) du \tag{5.1}$$

for all $\tau, u \in [0, 1]$, where λ is positive constant and $f : [0, 1] \longrightarrow \mathbb{R}_+$, $g : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}_+$, $\kappa : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}_+$. Assume that $C[0, 1]$ is space of continuous functions on $[0, 1]$. Consider the following assumptions.

- (**K₁**) The function $f : [0, 1] \longrightarrow [0, 1]$ is continuous.
- (**K₂**) The functions $\kappa : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}_+$ is continuous such that for all $\tau, u \in [0, 1]$
$$\int_0^\tau \kappa(\tau, u) du \leq K$$
- (**K₃**) The functions $g : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}_+$ is continuous and there is a constant L such that for all $\tau \in [0, 1]$;
$$|g(\tau, \zeta_1) - g(\tau, \zeta_2)| \leq L|\zeta_1(\tau) - \zeta_2(\tau)|$$
where $\zeta_1, \zeta_2 \in C[0, 1]$.
- (**K₄**) $\lambda LK \leq 1$.

Theorem 5.2. *Assume that $Y = C[0, 1]$ is space of continuous functions on $[0, 1]$ and C is compact subset of Y where supremum norm is defined by $\|\zeta_1 - \zeta_2\| = \sup_{\tau \in J} |\zeta_1(\tau) - \zeta_2(\tau)|$ and assumptions from (**K₁**) to (**K₄**) are true. The mapping $\psi : C \to C$ is defined by*

$$\psi(\zeta(\tau)) = f(\tau) + \lambda_1 \int_0^\tau \kappa(\tau, u) g(u, \zeta(u)) du$$

Then the nonlinear FIE (5.1) has solution in $Y = C[0, 1]$.

Proof. Assume that $\zeta_1, \zeta_2 \in Y$. Then

$$|\psi(\zeta_1(\tau)) - \psi(\zeta_2(\tau))| = |f(\tau) + \lambda \int_0^\tau \kappa(\tau, u) g(u, \zeta_1(u)) du$$

$$\begin{aligned} &- f(\tau) - \lambda \int_0^\tau \kappa(\tau,u) g(u,\zeta_2(u)) du \\ &\leq \lambda \int_0^\tau \kappa(\tau,u) \big| g(u,\zeta_1(u)) - g(u,\zeta_2(u)) \big| du \\ &\leq \lambda \int_0^\tau \kappa(\tau,u) L \big| \zeta_1(u) - \zeta_2(u) \big| du. \end{aligned}$$

On taking supremum of both sides, we get

$$\begin{aligned} \|\psi(\zeta_1) - \psi(\zeta_2)\| &\leq \lambda K L \|\zeta_1 - \zeta_2\| \\ &= \|\zeta_1 - \zeta_2\|. \end{aligned}$$

This shows that the map ψ is monotone generalized α-nonexpansive mapping for $\alpha = 0$ and all the conditions of Theorem 5.1 are satisfied, hence ψ has fixed point. Therefore, nonlinear FIE (5.1) has a solution in C. □

Example 5.3. Let us consider the following nonlinear FIE : For $\tau, s \in [0,1]$,

$$\zeta(\tau) = \left(\tau^2 + 4\right) + \frac{3}{20} \int_0^\tau \left[(\tau^3+1)(2u+4)\right] \frac{|\zeta(u)|}{3} du.$$

If we take

$$\begin{aligned} &\lambda = \frac{3}{20}; \\ &f(\tau) = \tau^2 + 4; \\ &\kappa(\tau,u) = (\tau^3+1)(2u+4); \\ &g(u,\zeta(u)) = \frac{|\zeta(u)|}{3}. \end{aligned}$$

Then (5.2) will be in form of (5.1).

It is clear that function $f(\tau) = \tau^2 + 4, \forall \tau \in [0,1]$ is continuous.

For each $\tau, u \in [0,1]$,

$$\begin{aligned} \int_0^1 \kappa(\tau,u) &= \int_0^1 (\tau^3+1)(2u+4) du \\ &\leq 10. \end{aligned}$$

For $\zeta_1, \zeta_2 \in Y; u \in [0,1]$,

$$\begin{aligned} \big| g(u,\zeta_1(u)) - g(u,\zeta_2(u)) \big| &= \left| \frac{|\zeta_1(u)|}{3} - \frac{|\zeta_2(u)|}{3} \right| \\ &= \frac{1}{3} \Big| |\zeta_1(u)| - |\zeta_2(u)| \Big| \\ &\leq \frac{1}{3} \Big| \zeta_1(u) - \zeta_2(u) \Big| \end{aligned}$$

since all assumptions of Theorem 5.2 are satisfied with $K = 10$ and $\lambda KL \leq 1$. Therefore nonlinear FIE (5.2) has a solution.

6. Conclusion

In conclusion, this paper investigated the application of the SPK iterative algorithm in approximating fixed points of monotone generalized α-nonexpansive mappings within partially ordered hyperbolic spaces. Convergence results for the algorithm were rigorously established, showcasing its effectiveness. Furthermore, the practical utilization of the algorithm in solving nonlinear integral equations exemplified its versatility. These findings significantly contributed to enhancing our understanding of the SPK algorithm's convergence properties and its efficacy in addressing problems within various mathematical contexts involving monotone mappings. The impact of this research extends to both theoretical advancements and practical applications in the realm of iterative algorithms.

References

1. M. Abbas and T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems*, Math. Vesn., 66 (2014), pp. 223-234.
2. B. Abdullatif, B. Dehaish and M. A. Khamsi, *Browder and Göhde fixed point theorem for monotone nonexpansive mappings*, Fixed Point Theory Appl., 2016 (1) (2016), pp. 1-9.
3. R.P. Agarwal, D. O'Regan and D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal., 8 (1) (2007), pp. 61-79.
4. S. Aggrawal, I. Uddin and S. Mujahid, *Convergence theorems for SP-iteration scheme in a orederd Hyperbolic spaces*, Nonlinear Funct. Anal. Appl., 26 (5) (2021), pp. 961-969.
5. K. Aoyama and F. Kohsaka, *Fixed point theorem for a nonexpansive mappings in Banach spaces*, Nonlinear Analysis, 2011 (13) (2011), pp. 4387–4391.
6. B.A. Bin Dehaish and M.A. Khamsi, *Browder and Göhde fixed point theorem for monotone nonexpansive mappings*, Fixed Point Theory Appl., 2016 (1) (2016), pp. 1-9.
7. H. Busemann, *Spaces with non-positive curvature*, Acta Math., 80 (1948), pp. 259-310.
8. F.E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Natl. Acad. Sci. U.S.A., 54 (4) (1965), pp. 1041-1044.

9. J. García-Falset, E. Llorens-Fuster and T. Suzuki, *Fixed point theory for a class of generalized nonexpansive mappings*, J. Appl. Math. Anal. Appl., 375 (1) (2011), pp. 185–195.
10. K. Goebel and W.A. Kirk, *Iteration processes for nonexpansive mappings*, Contemp. Math., 21 (1983), pp. 115-123.
11. D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr., 30 (1965), pp. 251-258.
12. M. Gromov, M. Katz, P. Pansu and S. Semmes, *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser Boston, Inc., Boston, 1999.
13. S. Ishikawa, *Fixed points by a new iteration method*, Proc. Am. Math. Soc., 44 (1) (1974), pp 147-150.
14. N. Kadioglu and I. Yildrim, *Approximating fixed points of nonexpansive mappings by faster iteration process*, J. Adv. Math. Stud., 8 (2) (2015), pp. 257-264.
15. E. Karapınar, T. Abdeljawad, and F. Jarad, *Applying new fixed point theorems on fractional and ordinary differential equations*, Adv. Differ. Equ., 2019, 421 (2019).
16. A.R. Khan, H. Fukhar-ud-din and M. Khan, *An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces*, Fixed Point Theory Appl., 54 (12) (2012), pp. 1-12.
17. M.A. Khamsi, *On metric spaces with uniform normal structure*, Proc. Amer. Math. Soc., 106 (3) (1989), pp. 723–726.
18. W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, 72 (9) (1965), pp. 1004-1006.
19. W.A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type*, Isr. J. Math., 17 (1974), pp. 339–346.
20. W.A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Analysis, 68 (2008), pp. 3689-3969.
21. J.K. Kim, S. Dashputre, Padmavati and K. Sakure, *Generalized α-nonexpansive mappings in hyperbolic spaces*, Nonlinear Funct. Anal. Appl., 27 (3) (2022), pp. 449-469.
22. U. Kohlenbach, *Some logical metatheorems with applications in functional analysis*, Trans. Amer. Math. Soc., 357 (1) (2005), pp. 89-128.
23. L. Leuştean, *A quadratic rate of asymptotic regularity for* CAT(0) *spaces*, J. Math. Anal. Appl., 325 (1) (2007), pp. 386-399.
24. L. Leuştean, *Nonexpansive iterations in uniformly convex W-hyperbolic spaces*, Nonlinear analysis and optimization I. Nonlinear analysis, Contemp. Math., 513 (2010), pp. 193-209.

25. T.C. Lim, *On the center of convex set*, Proc. Amer. Math. Soc., 81 (2) (1981), pp. 345-346.
26. W.R. Mann, *Mean value methods in iteration*, Proc. Am. Math. Soc., 4 (1953), pp. 506-610.
27. J. Martínez-Moreno, K. Calderón, P. Kumam and E. Rojas, *Approximating Fixed Points of Suzuki* (α, β)*-Nonexpansive Mappings in Ordered Hyperbolic Metric Spaces*, In: Cho, Y.J., Jleli, M., Mursaleen, M., Samet, B., Vetro, C. (eds) Advances in Metric Fixed Point Theory Appl., Springer, Singapore, 2021.
28. K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl., 279 (2) (2003), pp. 372-379.
29. M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., 251 (1) (2000), pp. 217-229.
30. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., 73 (4) (1967), pp. 591-597.
31. R. Pandey, R. Pant and A. Al-Rawashdeh, *Fixed point results for a class of monotone nonexpansive type mappings in hyperbolic spaces*, J. Funct. Spaces, 2018 Article ID 5850181, 11 pages.
32. R. Shukla, R. Pant and M. De la Sen, *Generalized* α*-nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., 2017 4 (2017).
33. H. Piri, B. Daraby, S. Rahrovi and M. Ghasemi, *Approximating fixed points of generalized* $\alpha-$*nonexpansive mappings in Banach spaces by new faster iteration process*, Numer. Algor., 81 (2019), pp. 1129-1148.
34. S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal., 15 (6) (1990), pp. 537-558.
35. S. Rezapour, T.M. Thabet, I. Kedim, M. Vivas-Cortez and M. Ghaderi, *A computational method for investigating a quantum integrodifferential inclusion with simulations and heatmaps*, AIMS Math., 8 (11) (2023), pp. 27241-27267.
36. D.R. Sahu, *Application of the s-iteration process to constrained minimization problems and feasibility problems*, Fixed Point Theory, 12 (2011), pp. 187-204.
37. H.F. Senter and W.G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Am. Math. Soc., 44 (1974), pp. 375-380.
38. R. Shukla, R. Pant and P. Kumam, *On the* α*-nonexpansive mappings in partially ordered hyperbolic metric spaces*, J. Math. Anal., 8 (1) (2017), pp. 1-15.

39. C. Suanoom, K. Sriwichai, C. Klin-Eam and W. Khuangsatung, *The generalized α-nonexpansive mappings and related convergence theorems in hyperbolic spaces*, J. Inform. Math. Sci., 11 (1) (2019), pp. 1-17.
40. T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Appl. Math. Anal. Appl., 340 (2) (2008), pp. 1088–1095. 2008.
41. W. Takahashi, *A convexity in metric space and nonexpansive mappings*, I, Kodai Math. Sem. Rep., 22 (2) (1970), pp. 142- 149.
42. B.S. Thakur, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, Appl. Math. Comput., 275 (2016), pp. 147–155.
43. Y. Yao, Y.J. Cho and Y.C. Liou, *Algorithms of common solutions of variational inclusions, mixed equilibrium problems and fixed point problems*, Eur. J. Oper. Res., 212 (2) (2011), pp. 242-250.

---

[1] Department of Mathematics, Govt. College, Arjunda, Balod, Chhattisgarh, India.
Email address: samir231973@gmail.com

[2] Department of Mathematics, Govt. Digvijay Auto. P.G. College, Rajnandgaon, Chhattisgarh, India.
Email address: kavitaaage@gmail.com