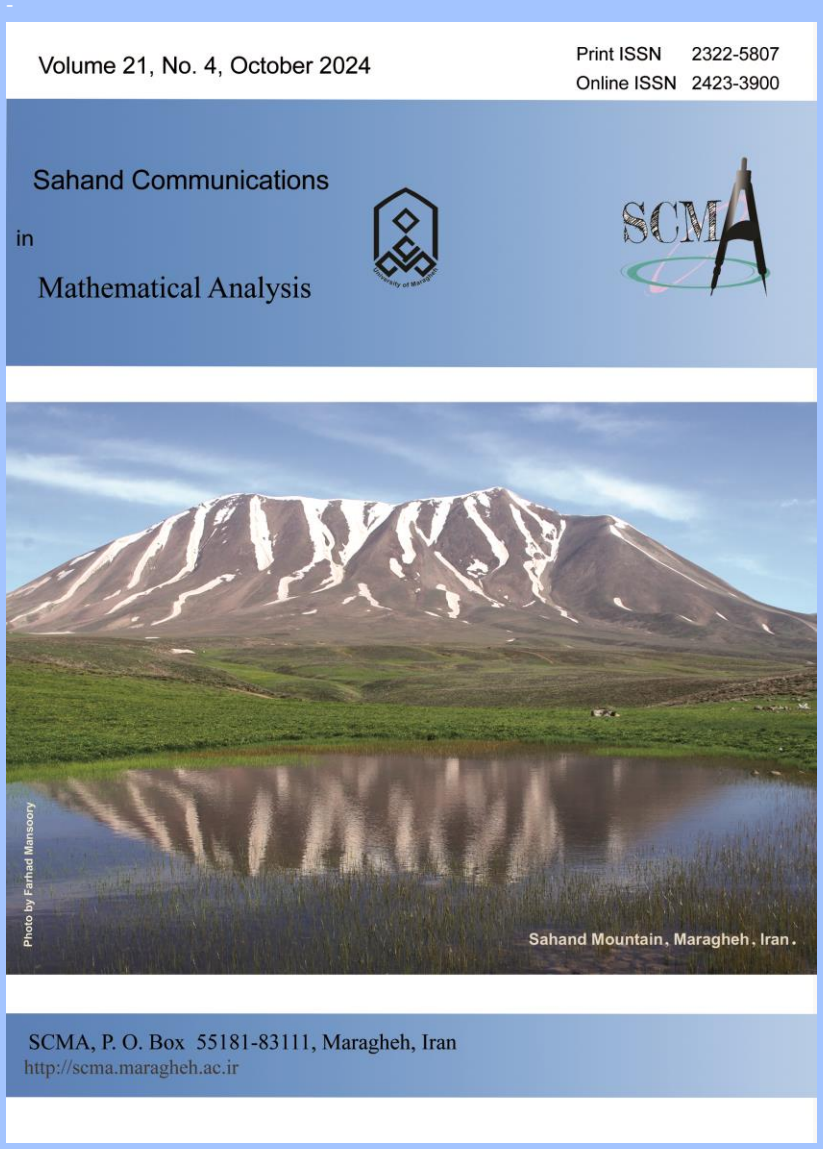
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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 4
Pages: 109-122

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2024.2011065.1457



Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 4 (2024), 109-122
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2024.2011065.1457

Brief Study On A New Family of Analytic Functions

Jamiu Olusegun Hamzat[1], Gangadharan Murugusundaramoorthy[2] and Matthew Olanrewaju Oluwayemi[3,4,5*]

Abstract. The authors introduced a new class of analytic functions in this study by means of convolution principle and obtain its relations with some well-known subclasses of analytic univalent functions in geometric functions theory in the open unit disk $\mathbb{D}$. Furthermore, as a special case, we show that convex functions of order $1/2$ are also members of the family $\mathcal{B}(\mu, \beta, \lambda, \theta)$, for $0 \leq \mu \leq 1$, $0 \leq \beta < 1$, $\lambda \geq 0$ and $|\theta| < \frac{\pi}{2}$. Thereafter, we obtain the bounds of the first three Taylorr- Maclaurin coefficients for functions $f(z)$ belonging to the class $\mathcal{B}(\mu, \alpha, \beta, \lambda, \theta; \breve{p})$ associated with shell-like curve.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be class of functions which are analytic and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

and are univalent in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ with normalization $f'(0) - 1 = 0 = f(0)$. Some of the subclasses of $\mathcal{A}$, are starlike, convex , spiralike, Pseudo-starlike, Pseudo-spiralike and bounded turning of order β $(0 \leq \beta < 1)$. The functions satisfy the following conditions:

(i) $\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \right\}$

(ii) $\mathcal{K}(\beta) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta \right\}$

(iii) $\mathcal{S}_p(\beta, \theta) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left\{ e^{i\theta} \frac{zf'(z)}{f(z)} \right\} > \beta,\ |\theta| < \frac{\pi}{2} \right\}$

2010 *Mathematics Subject Classification.* 34B24, 34B27.
Key words and phrases. Univalent, Analytic, Starlike, Convex, Spiralike, Bounded Turning.
Received: 08 September 2023, Accepted: 06 January 2024.
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(iv) $\mathcal{S}^*(\beta,\lambda)=\left\{f\in\mathcal{A}:\mathrm{Re}\left\{\frac{(zf'(z))^\lambda}{f(z)}\right\}>\beta\right\}$

(v) $\mathcal{S}_p(\beta,\lambda,\theta)=\left\{f\in\mathcal{A}:\mathrm{Re}\left\{e^{i\theta}\frac{(zf'(z))^\lambda}{f(z)}\right\}>\beta\right\}$

(vi) $\mathcal{R}_\beta=\{f\in\mathcal{A}:\mathrm{Re}\{f'(z)\}>\beta\}$

respectively. It is also noted that $f\in\mathcal{K}\Leftrightarrow zf'(z)\in\mathcal{S}^*$. It is well-known that the class of Caratheodory function P is of the form

$$q(z)=1+\sum_{k=1}^{\infty}c_kz^k,\quad z\in D$$

such that $\Re q(z)>0$ and $q(0)=1$. We note that

$$q(z)=\frac{1+w(z)}{1-w(z)},\quad q(z)\in P.$$

In the recent time, efforts have been made to investigate the classes $SL(\check{q})$ and $KSL(\check{q})$, of starlike shell-like and convex shell-like functions, which are characterized by

$$\frac{zf'(z)}{f(z)}\prec\check{q}(z)=\frac{1+\alpha^2z^2}{1-\alpha z-\alpha^2z^2},$$

and

$$1+\frac{zf''(z)}{f'(z)}\prec\check{q}(z)=\frac{1+\alpha^2z^2}{1-\alpha z-\alpha^2z^2},$$

where $\alpha=\frac{1-\sqrt{5}}{2}\approx-0.618$ (see [1, 2, 10, 15, 16] among others). However, the function $\check{q}(z)$ is non-univalent in the disk $|z|=\frac{3-\sqrt{5}}{2}\approx0.38$. The image of the circle $|z|=1$ under $\check{q}(z)$ is given by

$$\left(10x-\sqrt{5}\right)y^2=\left(\sqrt{5}-2x\right)\left(\sqrt{5x}-1\right)^2.$$

With the above exposition, it is therefore pertinent to note the following properties of the curve $\check{q}(re^{i\psi})$ holds:

- (i) it has a vertical asymptote, for $r=1$;
- (ii) it is a closed curve without any loops, for $0<r\leq r_0$; and
- (iii) it is a loop, for $r_0<r<1$.

Of course, it is trivial to see that α satisfies the algebraic equation

$$\alpha^2=\alpha+1. \tag{1.2}$$

Then, (1.2) can be generalised as α^m which is a linear function as a linear combination of 1 and α. The recurring relationships so obtained produce the famous Fibonacci numbers u_m such that $\alpha^m=u_m\alpha+u_{m-1}$.

In [16] it has been shown that

$$\check{q}(z)=1+\sum_{k=2}^{\infty}(u_{m-1}+u_{m+1})$$

where

$$u_k = \frac{(1-\alpha)^k - \alpha}{\sqrt{5}}, \quad k = 1, 2, 3, \ldots.$$

Obviously, this has shown the relationship between the function $\check{q}(z)$ and the sequence of Fibonacci numbers u_k, such that

$$u_0 = 0, \;\; u_1 = 1, \qquad u_{k+2} = u_k + u_{k+1}, \quad k = 0, 1, 2, \ldots.$$

Therefore we can write that

$$\begin{aligned} \check{q}(z) &= 1 + \sum_{k=1}^{\infty} \check{q}_k z^k \\ &= 1 + (u_0 + u_2)\alpha z + (u_1 + u_3)\alpha^2 z^2 \\ &\quad + \sum_{k=3}^{\infty} (u_{k-3} + u_{k-2} + u_{k-1} + u_k)\alpha^k z^k \\ &= 1 + \alpha z + 3\alpha^2 z^2 + 4\alpha^3 z^3 + 7\alpha^4 z^4 + 11\alpha^5 z^5 + \cdots. \end{aligned} \tag{1.3}$$

For further details, see also [18] .

In 2013, the authors in [3] defined $h(z)$ as

$$\begin{aligned} \phi(z) &= 2\left(\frac{1}{1+e^{-z}}\right) \\ &= \frac{2}{1-e^{-z}}. \end{aligned} \tag{1.4}$$

The authors further proved that the modified sigmoid function $\phi(z)$ belong to the family of Caratheodory functions P, with series representation:

$$\phi(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \cdots. \tag{1.5}$$

See also [4, 8–10, 12, 14, 17, 19] for details.

We now consider the following definitions for the purpose of this study:

Definition 1.1. Given $f \in \mathcal{A}$, then $f(z)$ belongs to class $\mathcal{B}(\mu, \beta, \lambda, \theta, \alpha; \check{q})$, $0 \leq \mu \leq 1$, $\alpha = \frac{1-\sqrt{5}}{2}$, $\lambda \geq 1$ and $|\theta| < \frac{\pi}{2}$ if the following geometric condition is satisfied:

$$\begin{aligned} \frac{(f'(z))^{\lambda}\left(\frac{e^{i\theta}z}{f(z)}\right)^{\mu} - \beta}{e^{i\mu\theta} - \beta} &\prec \check{q}(z) \\ &= \frac{1 + \alpha^2 z^2}{1 - \alpha z - \alpha^2 z^2}. \end{aligned} \tag{1.6}$$

Definition 1.2. Let $\mathcal{B}(\mu,\beta,\lambda,\theta)$, $0 \leq \mu \leq 1$, $0 \leq \beta < 1$, $\alpha = \frac{1-\sqrt{5}}{2}$, $\lambda \geq 1$ and $|\theta| < \frac{\pi}{2}$ denote the family of analytic functions which consist of the functions $f \in \mathcal{A}$ with the provision that

$$\left| \left(f'(z)\right)^{\lambda} \left(\frac{e^{i\theta}z}{f(z)}\right)^{\mu} - e^{i\mu\theta} \right| < e^{i\mu\theta} - \beta, \quad (z \in D). \tag{1.7}$$

That is,

$$\Re\left\{ \left(f'(z)\right)^{\lambda} \left(\frac{e^{i\theta}z}{f(z)}\right)^{\mu} \right\} > \beta.$$

This family is a generalized class of analytic functions with inclusion of several well-known classes of univalent functions as well as new ones. For instance, $\mathcal{B}(1,\beta,\lambda,0) = \mathcal{S}^*(\beta,\lambda)$; $\mathcal{B}(1,\beta,1,0) = \mathcal{S}^*(\beta)$; $\mathcal{B}(1,\beta,\lambda,\theta) = \mathcal{S}_p(\beta,\lambda,\theta)$; $\mathcal{B}(1,\beta,1,\theta) = \mathcal{S}_p(\beta,\theta)$; and $\mathcal{B}(0,\beta,\lambda,0) = \mathcal{R}_{\beta,\lambda}$; $\mathcal{B}(0,\beta,1,0) = \mathcal{R}_{\beta}$. See also [5].

Motivated by earlier works [6, 7, 13] in this paper, we investigate a sufficient condition for functions to be in the class $\mathcal{B}(\mu,\beta,\lambda,\theta)$. Also we established that convex functions of order $\frac{1}{2}$ and $\frac{3}{4}$ are also members of the aforementioned family $\mathcal{B}(\mu,\beta,\lambda,\theta)$. In addition, the bounds of functions $f(z)$ in class $\mathcal{B}(\mu,\beta,\lambda,\theta,\alpha;\check{p})$ connected with shell-like curve are obtained.

2. Main Results

In order to derive our main results, we have to recall here the following lemmas due to Jack [11] .

Lemma 2.1. *Suppose that $q(z)$ is analytic in $\mathbb{D}$ with the condition $q(0) = 1$. Also let $\Re\left(1 + \frac{zq'(z)}{q(z)}\right) > \frac{3\beta-1}{2\beta}$. Then $\Re\{q(z)\} > \beta$ for $z \in \mathbb{D}$ and $\frac{1}{2} \leq \beta < 1$ (see also [7])*

Theorem 2.2. *Suppose that $f \in \mathcal{A}$. If*

$$\Re\left(1 + \frac{\lambda z f''(z)}{f'(z)} + \mu\left(1 - \frac{zf'(z)}{f(z)}\right)\right) > \frac{3\beta-1}{2\beta}. \tag{2.1}$$

Then $f \in \mathcal{B}(\mu,\beta,\lambda,\theta)$, where $\frac{1}{2} \leq \beta < 1$.

Proof. Define

$$q(z) = \left(f'(z)\right)^{\lambda} \left(\frac{e^{i\theta}z}{f(z)}\right)^{\mu}. \tag{2.2}$$

Then, $q(z)$ is regular in $\mathbb{D}$ and $q(0) = 1$. Differentiate 2.2 so that

$$\frac{zq'(z)}{q(z)} = \lambda z \frac{f''(z)}{f'(z)} + \mu\left[1 - \frac{zf'(z)}{f(z)}\right] \tag{2.3}$$

and

$$1+\frac{zq'(z)}{q(z)}=1+\lambda z\frac{f''(z)}{f'(z)}+\mu\left[1-\frac{zf'(z)}{f(z)}\right]. \tag{2.4}$$

Appealing to Lemma 2.1 we conclude that $\Re\left(q(z)\right)>\beta$. That is,

$$\Re\left(\left(f'(z)\right)^{\lambda}\left(\frac{e^{i\theta}z}{f(z)}\right)^{\mu}\right)>\beta. \tag{2.5}$$

Therefore, $f\in\mathcal{B}(\mu,\beta,\lambda,\theta)$ and this concludes the proof. □

The following are the consequences of Theorem 2.2.

Corollary 2.3. *Let $f\in\mathcal{A}$. If*

$$\Re\left(1+\frac{\lambda zf''(z)}{f'(z)}\right)>\frac{1}{2}, \tag{2.6}$$

then

$$\Re\left(f'(z)\right)^{\lambda}>\frac{1}{2},\quad z\in D.$$

Hence $f\in\mathcal{B}\left(0,\frac{1}{2},\lambda,0\right)$.

Corollary 2.4. *Let $f\in\mathcal{A}$. If*

$$\Re\left(\frac{\lambda zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)>-\frac{3}{2}. \tag{2.7}$$

Then

$$\Re\left(e^{i\theta}\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\right)>\frac{1}{2},\quad z\in\mathbb{D}.$$

Then, $f\in\mathcal{B}(1,\frac{1}{2},\lambda,\theta)$ is pseudo-spiralike of order $\frac{1}{2}$.

Corollary 2.5. *Let $f\in\mathcal{A}$. If*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\frac{1}{2}, \tag{2.8}$$

then

$$\Re\left(f'(z)\right)>\frac{1}{2},\quad z\in\mathbb{D}.$$

Therefore $f\in\mathcal{B}\left(0,\frac{1}{2},1,\theta\right)=\mathcal{R}_{\frac{1}{2}}$ is a convex function of order $\frac{1}{2}$.

Corollary 2.6. *Let $f\in\mathcal{A}$. If*

$$\Re\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)>-\frac{3}{2}. \tag{2.9}$$

Then

$$\Re\left(e^{i\theta}\frac{zf'(z)}{f(z)}\right)>\frac{1}{2},\quad z\in\mathbb{D}.$$

Let $f\in\mathcal{B}(1,\frac{1}{2},1,\theta)$. Then, $f(z)$ is a spiralike function of order $\frac{1}{2}$.

Corollary 2.7. *Let $f \in \mathcal{A}$. If*

$$\Re\left(\frac{\lambda z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)}\right) > -\frac{3}{2}. \tag{2.10}$$

Then

$$\Re\left(\frac{z\,(f'(z))^{\lambda}}{f(z)}\right) > \frac{1}{2}, \quad z \in \mathbb{D}.$$

Therefore, if $f(z)$ is pseudo-starlike of order $\frac{1}{2}$, then $f \in \mathcal{B}(1, \frac{1}{2}, \lambda, 0)$.

Corollary 2.8. *Suppose $f \in \mathcal{A}$. If*

$$\Re\left(\frac{\lambda z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)}\right) > -\frac{3}{2}. \tag{2.11}$$

Then

$$\Re\left(\frac{z f'(z)}{f(z)}\right) > \frac{1}{2}, \quad z \in \mathbb{D}.$$

Then suppose $f \in \mathcal{B}(1, \frac{1}{2}, 1, 0)$, $f(z)$ is starlike of order $\frac{1}{2}$.

Corollary 2.9. *Let $f \in \mathcal{A}$. If*

$$\Re\left(\frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)}\right) > -\frac{7}{6}. \tag{2.12}$$

Then

$$\Re\left(e^{i\theta}\frac{z f'(z)}{f(z)}\right) > \frac{3}{4}, \quad z \in \mathbb{D}.$$

Therefore, if $f(z)$ is spiralike of order $\frac{3}{4}$, then $f \in \mathcal{B}(1, \frac{3}{4}, 1, \theta)$.

Corollary 2.10. *Let $f \in \mathcal{A}$. If*

$$\Re\left(\frac{\lambda z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)}\right) > -\frac{7}{6}. \tag{2.13}$$

Then

$$\Re\left(\frac{z\,(f'(z))^{\lambda}}{f(z)}\right) > \frac{3}{4}, \quad z \in \mathbb{D}.$$

That is, if $f(z)$ is pseudo-starlike of order $\frac{3}{4}$, then $f \in \mathcal{B}\left(1, \frac{3}{4}, \lambda, 0\right)$.

Corollary 2.11. *Let $f \in \mathcal{A}$. If*

$$\Re\left(\frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)}\right) > -\frac{7}{6}. \tag{2.14}$$

Then

$$\Re\left(\frac{z f'(z)}{f(z)}\right) > \frac{3}{4}, \quad z \in \mathbb{D}.$$

Therefore, if $f(z)$ is spiralike of order $\frac{3}{4}$, then $f \in \mathcal{B}\left(1, \frac{3}{4}, 1, 0\right)$.

Corollary 2.12. *Let $f \in \mathcal{A}$. If*

$$\Re\left(1+\frac{\lambda z f''(z)}{f'(z)}\right) > \frac{5}{6}. \tag{2.15}$$

Then

$$\Re\left(\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\right) > \frac{3}{4}, \quad z \in \mathbb{D}.$$

Therefore, $f \in \mathcal{B}(0, \frac{3}{4}, \lambda, 0) = \mathcal{R}_{\frac{3}{4},\lambda}$.

Corollary 2.13. *Let $f \in \mathcal{A}$. If*

$$\Re\left(\frac{\lambda z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)}\right) > -\frac{7}{6}. \tag{2.16}$$

Then

$$\Re\left(\frac{e^{i\theta} z\left(f'(z)\right)^{\lambda}}{f(z)}\right) > \frac{3}{4}, \quad z \in D$$

is a pseudo-spiralike of order $\frac{3}{4}$ and $f \in \mathcal{B}\left(1, \frac{3}{4}, \lambda, \theta\right)$.

Corollary 2.14. *Let $f \in \mathcal{A}$. If*

$$\Re\left(1+\frac{\lambda z f''(z)}{f'(z)}\right) > \frac{5}{6}. \tag{2.17}$$

Then

$$\Re\left(f'(z)\right) > \frac{3}{4}, \quad z \in D$$

implies $f \in \mathcal{B}(0, \frac{3}{4}, 1, 0) = \mathcal{R}_{\frac{3}{4}}$, a convex of order $\frac{3}{4}$.

We now consider the bounds on the first three coefficients for functions $f(z) \in \mathcal{B}\left(\mu, \alpha, \lambda, \theta; \check{p}\right)$ associated with shell-like curve.

Theorem 2.15. *Let $f \in \mathcal{A}$ be in the class $\mathcal{B}(\mu, \alpha, \beta, \lambda, \theta; \check{p})$ of analytic functions associated with shell-like curve with the provision that $0 \leq \mu \leq 1$, $0 \leq \beta < 1$, $\alpha = \frac{1-\sqrt{5}}{2}$, $\lambda \geq 1$ and $|\theta| < \frac{\pi}{2}$. Then*

$$|a_2| \leq \frac{|\alpha|(e^{i\mu\theta} - \beta)}{4(2\lambda_1 - \mu)}, \tag{2.18}$$

$$|a_3| \leq \frac{|\alpha|(e^{i\mu\theta} - \beta)}{16(2\lambda_1 - \mu)^2(3\lambda_1 - \mu)} \left[|\alpha|G + (2\lambda_1 - \mu)^2\right], \tag{2.19}$$

(2.20)

$$|a_4| \leq \frac{|\alpha|(e^{i\mu\theta} - \beta)}{192(2\lambda_1 - \mu)^3(3\lambda_1 - \mu)(4\lambda_1 - \mu)} \left[3|\alpha|^2[I - J - L] + 3|\alpha|M + N\right],$$

and

(2.21)

$$|a_3-\gamma a_2^2| \leq \frac{|\alpha|(e^{i\mu\theta}-\beta)}{16(2\lambda_1-\mu)^2(3\lambda_1-\mu)} \times \left[|\alpha|\left[G-\gamma(e^{i\mu\theta}-\beta)(2\lambda_1-\mu)^2(3\lambda_1-\mu)\right]+(2\lambda_1-\mu)^2\right],$$

where

$$G = 3(2\lambda_1-\mu)^2-(e^{i\mu\theta}-\beta)\left(4\lambda_2-2\lambda_1\mu+\frac{\mu(\mu+1)}{2}\right),$$

$$I = 4(2\lambda_1-\mu)^3(3\lambda_1-\mu),$$

$$J = (e^{i\mu\theta}-\beta)\left(12\lambda_2-5\lambda_1\mu+\mu(\mu+1)\right) \times\left(3(2\lambda_1-\mu)^2-(e^{i\mu\theta}-\beta)\left(4\lambda_2-2\lambda_1\mu+\frac{\mu(\mu+1)}{2}\right)\right),$$

$$L = (e^{i\mu\theta}-\beta)^2(3\lambda_1-\mu)\left(8\lambda_3-4\lambda_2\mu+\lambda_1\mu(\mu+1)+\frac{\mu(\mu+1)(\mu+2)}{6}\right),$$

$$M = (2\lambda_1-\mu)^2\left[6(2\lambda_1-\mu)(3\lambda_1-\mu)-(e^{i\mu\theta}-\beta)\left(12\lambda_2-5\lambda_1+\frac{\mu(\mu+1)}{2}\right)\right],$$

$$N = (2\lambda_1-\mu)^3(3\lambda_1-\mu)$$

and $\lambda_1=\lambda, \lambda_2=\frac{\lambda(\lambda-1)}{2!}, \lambda_3=\frac{\lambda(\lambda-1)(\lambda-2)}{3!},\ldots.$

Proof. Let $f\in\mathcal{A}\in\mathcal{B}(\mu,\alpha,\beta,\lambda,\theta;\check{p})$. Then it follows from definition 1.1 that

$$\frac{(f'(z))^{\lambda}\left(\frac{e^{i\theta}z}{f(z)}\right)^{\mu}-\beta}{e^{i\mu\theta}-\beta}=\check{q}\left(\phi(z)\right). \tag{2.22}$$

Since $\phi(z)$ is defined as in (1.5) as a function in class P, with $\phi(z)\prec\check{q}(z)$, then we have $\phi(z)$ with $\phi(0)=1$ and $|\phi(z)|<1$ such that $\phi(z)=\check{q}\left(\phi(z)\right)$. Here we define

$$p(z)=\frac{1+\phi(z)}{1-\phi(z)} =1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{1}{64}z^6+\cdots. \tag{2.23}$$

A simple computation of (2.23) yields

$$\phi(z)=\frac{p(z)-1}{p(z)+1} =\frac{1}{4}z-\frac{1}{16}z^2-\frac{1}{192}z^3+\cdots$$

and

(2.24)

$$\check{q}\left(\phi(z)\right) = 1 + \frac{1}{4}\alpha z + \frac{\alpha}{16}\left(3\alpha - 1\right) z^2 + \frac{\alpha}{192}\left(12\alpha^2 - 18\alpha - 1\right) z^3 + \cdots .$$

In view of (2.22) and (2.24), we have that

(2.25)

$$\begin{aligned}
&1 + (2\lambda_1 - \mu)\, a_2 z + \left((3\lambda_1 - \mu)\, a_3 + \left(4\lambda_2 - 2\lambda_1\mu + \frac{\mu(\mu+1)}{2}\right) a_2^2\right) z^2 \\
&\quad + ((4\lambda_1 - \mu)a_4 + (12\lambda_2 - 5\lambda_1\mu + \mu(\mu+1))a_2 a_3 \\
&\quad + \left(8\lambda_3 - 4\lambda_2\mu + \lambda_1\mu(\mu+1) + \frac{\mu(\mu+1)(\mu+2)}{6}\right) a_2^3\Bigg) z^3 + \cdots \\
&= e^{-i\mu\theta}(e^{i\mu\theta} - \beta)\left(1 + \frac{1}{4}\alpha z + \frac{\alpha}{16}(3\alpha - 1)z^2 + \frac{\alpha}{96}(6\alpha^2 - 9\alpha - 2)z^3 + \cdots\right) \\
&\quad + e^{-i\mu\theta}\beta.
\end{aligned}$$

Equating the coefficients of the like powers of z, z^2 and z^3, we obtain

$$e^{i\mu\theta}(2\lambda_1 - \mu)a_2 = \frac{1}{4}\alpha(e^{i\mu\theta} - \beta), \tag{2.26}$$

$$\begin{aligned}
e^{i\mu\theta}&\left[(3\lambda_1 - \mu)a_3 + \left(4\lambda_2 - 2\lambda_1\mu + \frac{\mu(\mu+1)}{2}\right) a_2^2\right] \\
&= \frac{\alpha}{16}(3\alpha - 1)(e^{i\mu\theta} - \beta),
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
e^{i\mu\theta}&\Bigg[(4\lambda_1 - \mu)a_4 + (12\lambda_2 - 5\lambda_1\mu + \mu(\mu+1))\, a_2 a_3 \\
&\quad + \left(8\lambda_3 - 4\lambda_2\mu + \lambda_1\mu(\mu+1) + \frac{\mu(\mu+1)(\mu+2)}{6}\right) a_2^3\Bigg] \\
&= \frac{\alpha}{192}(12\alpha^2 - 18\alpha - 1)(e^{i\mu\theta} - \beta),
\end{aligned} \tag{2.28}$$

and

(2.29)

$$\begin{aligned}
&a_3 - \gamma a_2^2 \\
&= \frac{\alpha(e^{i\mu\theta} - \beta)\left[\begin{array}{c} \alpha\left(3e^{i\mu\theta}(2\lambda_1 - \mu)^2 - (e^{i\mu\theta} - \beta)\left(4\lambda_2 - 2\lambda_1\mu + \frac{\mu(\mu+1)}{2}\right)\right) \\ -e^{i\mu\theta}(2\lambda_1 - \mu)^2 \end{array}\right]}{16e^{2i\mu\theta}(2\lambda_1 - \mu)^2(3\lambda_1 - \mu)} \\
&\quad - \gamma\frac{\alpha^2(e^{i\mu\theta} - \beta)^2}{16e^{2i\mu\theta}(2\lambda_1 - \mu)^2}.
\end{aligned}$$

□

Applying the triangle inequality in (2.26)-(2.29), we obtain the desired results as contained in (2.18)-(2.21).

Let $\theta = 0$ in Theorem 2.15. We obtain the following corollary.

Corollary 2.16. *If $f \in \mathcal{A} \in \mathcal{B}(\mu, \alpha, \beta, \lambda, 0; \check{p})$ of analytic functions associated with shell-like curve with the provision that $0 \leq \mu \leq 1$, $0 \leq \beta < 1$, $\alpha = \frac{1-\sqrt{5}}{2}$ and $\lambda \geq 1$. Then*

$$|a_2| \leq \frac{|\alpha|(1-\beta)}{4(2\lambda_1 - \mu)},$$

$$|a_3| \leq \frac{|\alpha|(1-\beta)}{16(2\lambda_1 - \mu)^2(3\lambda_1 - \mu)} \left[|\alpha|G + (2\lambda_1 - \mu)^2\right],$$

$$|a_4| \leq \frac{|\alpha|(1-\beta)}{192(2\lambda_1 - \mu)^3(3\lambda_1 - \mu)(4\lambda_1 - \mu)} \left[3|\alpha|^2[I - J - L] + 3|\alpha|M + N\right]$$

and

$$|a_3 - \gamma a_2^2| \leq \frac{|\alpha|(1-\beta)}{16(2\lambda_1 - \mu)^2(3\lambda_1 - \mu)} \times \left[|\alpha|\left[G - \gamma(1-\beta)(2\lambda_1 - \mu)^2(3\lambda_1 - \mu)\right] + (2\lambda_1 - \mu)^2\right],$$

where

$$G = 3(2\lambda_1 - \mu)^2 - (1-\beta)\left(4\lambda_2 - 2\lambda_1\mu + \frac{\mu(\mu+1)}{2}\right),$$

$$I = 4(2\lambda_1 - \mu)^3(3\lambda_1 - \mu),$$

$$J = (1-\beta)\left(12\lambda_2 - 5\lambda_1\mu + \mu(\mu+1)\right) \times \left(3(2\lambda_1 - \mu)^2 - (1-\beta)\left(4\lambda_2 - 2\lambda_1\mu + \frac{\mu(\mu+1)}{2}\right)\right),$$

$$L = (1-\beta)^2(3\lambda_1 - \mu)\left(8\lambda_3 - 4\lambda_2\mu + \lambda_1\mu(\mu+1) + \frac{\mu(\mu+1)(\mu+2)}{6}\right),$$

$$M = (2\lambda_1 - \mu)^2\left[6(2\lambda_1 - \mu)(3\lambda_1 - \mu) - (1-\beta)\left(12\lambda_2 - 5\lambda_1\mu + \frac{\mu(\mu+1)}{2}\right)\right],$$

$$N = (2\lambda_1 - \mu)^3(3\lambda_1 - \mu).$$

Setting $\theta = \beta = \mu = 0$ in Theorem 2.15. Then we have the following corollary.

Corollary 2.17. *If $f \in \mathcal{A}$ belongs to the class $\mathcal{B}(0, \alpha, 0, \lambda, 0; \check{p})$ of analytic functions associated with shell-like curve with the provision that $\alpha = \frac{1-\sqrt{5}}{2}$ and $\lambda \geq 1$. Then*

$$|a_2| \leq \frac{|\alpha|}{8\lambda_1},$$

$$|a_3| \leq \frac{|\alpha|}{48\lambda_1^3}\left[|\alpha|\left(3\lambda_1^2 - \lambda_2\right) + \lambda_1^2\right],$$

$$|a_4| \leq \frac{|\alpha|}{4(192)\lambda_1^5}\Big[3|\alpha|^2\left[4\lambda_1^4 - 6\lambda_1^2\lambda_2 + 2\lambda_2^2 - \lambda_1\lambda_3\right] + 3|\alpha|\left[6\lambda_4^1 - 2\lambda_2^1\lambda_2\right] + \lambda_1^4\Big]$$

and

$$|a_3 - \gamma a_2^2| \leq \frac{|\alpha|}{48\lambda_1^3}\left[|\alpha|\left[3\lambda_1^2(1-\gamma\lambda_1) - \lambda_2\right] + \lambda_1^2\right].$$

Corollary 2.18. *Suppose $f \in \mathcal{A}$ belongs to the class $\mathcal{B}(0,\alpha,0,1,0;\check{p})$ of analytic functions associated with shell-like curve with the provision that $\alpha = \frac{1-\sqrt{5}}{2}$. Then*

$$|a_2| \leq \frac{|\alpha|}{8},$$
$$|a_3| \leq \frac{|\alpha|}{48}\left[3|\alpha| + 1\right],$$
$$|a_4| \leq \frac{|\alpha|}{4(192)}\left[12|\alpha|^2 + 18|\alpha| + 1\right]$$

and

$$|a_3 - \gamma a_2^2| \leq \frac{|\alpha|}{48}\left[3|\alpha|(1-\gamma) + 1\right].$$

Corollary 2.19. *If $f \in \mathcal{A}$ belongs to the class $\mathcal{B}(0,\alpha,0,1,0;\check{p})$ of analytic functions associated with shell-like curve with the provision that $\alpha = \frac{1-\sqrt{5}}{2} = -0.618$. Then*

$$|a_2| \leq 0.0773,$$
$$|a_3| \leq 0.0367,$$
$$|a_4| \leq 0.0134$$

and

$$|a_3 - \gamma a_2^2| \leq 0.0367 - 0.0239\gamma.$$

Corollary 2.20. *Let $f \in \mathcal{A}$ belong to the class $\mathcal{B}(1,\alpha,0,\lambda,0;\check{p})$ of analytic functions associated with shell-like curve with the provision that $\alpha = \frac{1-\sqrt{5}}{2}$ and $\lambda \geq 1$. Then*

$$|a_2| \leq \frac{|\alpha|}{4(2\lambda_1 - 1)},$$
$$|a_3| \leq \frac{|\alpha|}{16(2\lambda_1-1)^2(3\lambda_1-1)}\left\{|\alpha|G + (2\lambda_1-1)^2\right\},$$
$$|a_4| \leq \frac{|\alpha|}{192(2\lambda_1-1)^3(3\lambda_1-1)(4\lambda_1-1)}\left\{3|\alpha|^2[I-J-L] + 3|\alpha|M + N\right\}$$

and

$$|a_3-\gamma a_2^2| \leq \frac{|\alpha|}{16(2\lambda_1-1)^2(3\lambda_1-1)}\left\{|\alpha|\left[G-\gamma(3\lambda_1-1)\right]+(2\lambda_1-1)^2\right\},$$

where

$$\begin{aligned}
G &= 3(2\lambda_1-1)^2-(4\lambda_2-2\lambda_1+1),\\
I &= 4(2\lambda_1-1)^3(3\lambda_1-1),\\
J &= (12\lambda_2-5\lambda_1+2)\left(12\lambda_1^2-10\lambda_1-4\lambda_2+2\right),\\
L &= (3\lambda_1-1)\left(8\lambda_3-4\lambda_2+2\lambda_1+1\right),\\
M &= (2\lambda_1-1)^2\left[36\lambda_1^2-25\lambda_1-12\lambda_2+5\right],\\
N &= (2\lambda_1-1)^3(3\lambda_1-1).
\end{aligned}$$

Setting $\theta=\beta=0,\ \mu=\lambda=1$ in Theorem 2.15. Then we have the following corollary.

Corollary 2.21. *Let $f\in\mathcal{A}$ belong to the class $\mathcal{B}(1,\alpha,0,1,0;\check{p})$ of analytic functions associated with shell-like curve with the provision that $\alpha=\frac{1-\sqrt{5}}{2}$. Then*

$$\begin{aligned}
|a_2| &\leq \frac{|\alpha|}{4},\\
|a_3| &\leq \frac{|\alpha|}{32}\left[4|\alpha|+1\right],\\
|a_4| &\leq \frac{|\alpha|}{6(192)}\left[42|\alpha|^2+48|\alpha|+2\right]
\end{aligned}$$

and

$$|a_3-\gamma a_2^2| \leq \frac{|\alpha|\left[2|\alpha|(2-\gamma)+1\right]}{32}.$$

Corollary 2.22. *Let $f\in\mathcal{B}(1,\alpha,0,1,0;\check{p})$ of analytic functions associated with shell-like curve with the provision that $\alpha=\frac{1-\sqrt{5}}{2}=-0.618$. Then*

$$\begin{aligned}
|a_2| &\leq 0.1545,\\
|a_3| &\leq 0.0671,\\
|a_4| &\leq 0.0256
\end{aligned}$$

and

$$|a_3-\gamma a_2^2| \leq 0.0671-0.0239\gamma.$$

3. Discussion

The authors introduced and investigated the following new classes of univalent functions $\mathcal{B}(\mu, \beta, \lambda, \theta, \alpha; \breve{q})$ and $\mathcal{B}(\mu, \beta, \lambda, \theta)$ in the study. Several interesting results including corrolarries were obtained. The family of class of analytic functions considered in the work is a generalization of many known class of functions in the literature. In particular, we observed that $\mathcal{B}(1, \beta, \lambda, 0) = \mathcal{S}^*(\beta, \lambda)$; $\mathcal{B}(1, \beta, 1, 0) = \mathcal{S}^*(\beta)$, $\mathcal{B}(1, \beta, \lambda, \theta) = \mathcal{S}_p(\beta, \lambda, \theta)$, $\mathcal{B}(1, \beta, 1, \theta) = \mathcal{S}_p(\beta, \theta)$, $\mathcal{B}(0, \beta, \lambda, 0) = \mathcal{R}_{\beta,\lambda}$ and $\mathcal{B}(0, \beta, 1, 0) = \mathcal{R}_\beta$. We investigate the sufficient condition for functions to be in the class $\mathcal{B}(\mu, \beta, \lambda, \theta)$, established that convex functions of order $\frac{1}{2}$ and $\frac{3}{4}$ are also members of the aforementioned family $\mathcal{B}(\mu, \beta, \lambda, \theta)$ and obtained the bounds of functions $f(z)$ in class $\mathcal{B}(\mu, \beta, \lambda, \theta, \alpha; \breve{p})$ connected with shell-like curve.

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