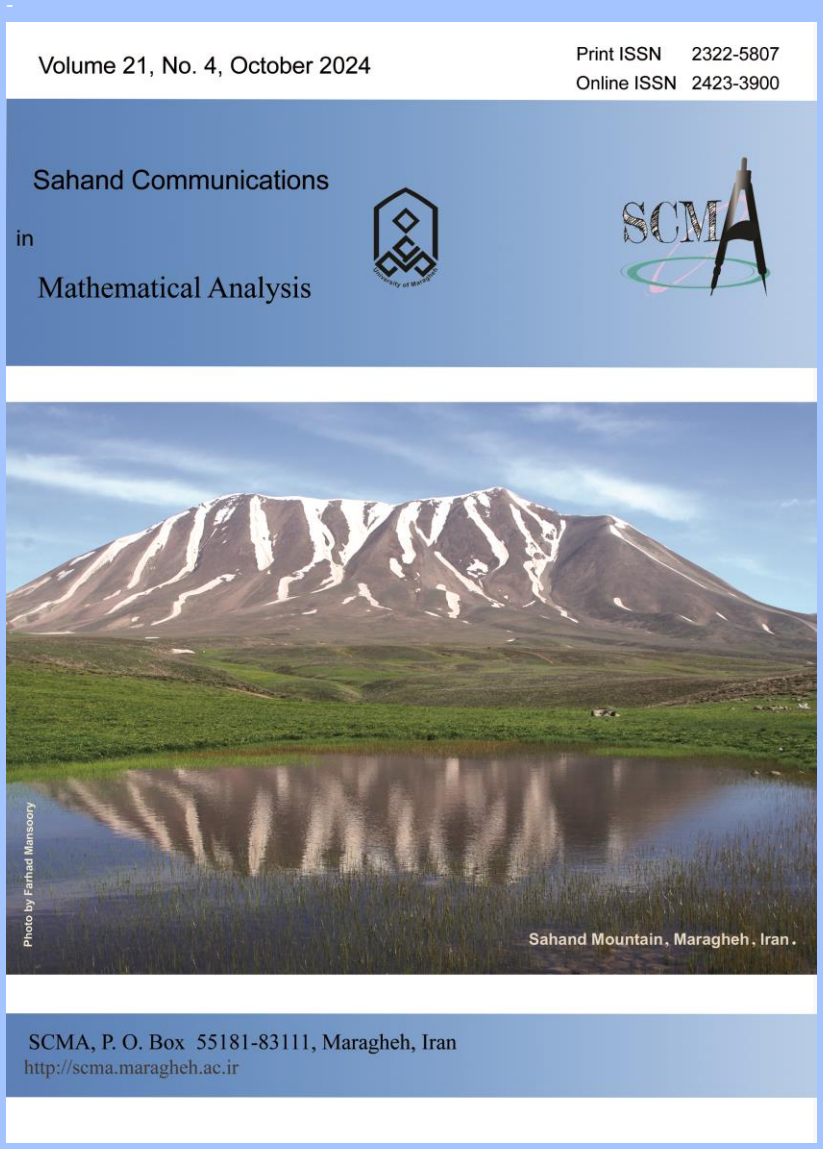
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Existence Results for an Implicit Coupled System Involving ξ-Caputo and p-Laplacian Operators

Walid Benhadda[1,*], Abderrazak Kassidi[2], Ali El Mfadel[3,4] and M'hamed Elomari[5]

Abstract. This paper aims to establish the existence and uniqueness of a solution to a coupled system of ξ-Caputo fractional differential equations involving the p-Laplacian operator in an arbitrary Banach space. The results of the existence derived using the tools of topological degree theory for condensing maps and the Banach contraction principle. An illustrative example is given at the end to demonstrate the practical application of the theoretical results.

1. Introduction

Nonlinear operators play an important role in differential equations. One of the most popular types of these operators applied to fractional differential equations is the p-Laplacian operator, denoted by ϕ_p, such that $\phi_p(s) = \|s\|^{p-2} s, (p > 1)$, see [6] for more information.

In recent times, fractional differential equations have emerged as crucial in applications in the real world due to their ability to accurately represent a wide range of physical phenomena spanning diverse fields such as chemistry, physics, biology, engineering, viscoelasticity, electrical engineering, signal processing, electrochemistry and controllability (for detailed references, see [7, 12–14]). Recently, a new class of fractional derivatives known as ξ-Caputo fractional derivatives, introduced by Almeida [2], has emerged. These derivatives represent a Caputo-type fractional derivative concerning a function. The concept of ξ-fractional or fractional calculus concerning a function, has gained prominence in the literature. This approach provides flexibility in modeling different

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phenomena, depending on the choice of the function ξ. The ξ-Caputo fractional derivative includes well-known fractional derivatives such as the Caputo fractional derivative for $\xi(t) = t$, the Caputo-Hadamard fractional derivative for $\xi(t) = \ln(t)$ and the Caputo-Erdelyi-Kober fractional derivative for $\xi(t) = t^{\sigma}$. This flexibility in modeling has sparked significant interest due to its practical applications. Numerous researchers have conducted extensive studies investigating systems incorporating the ξ-Caputo fractional derivative and p-Laplacian operators (see [3–5, 8]).

In 2019, Samina et al. [16], investigated the existence and uniqueness of solutions to the following coupled system of fractional differential equations(FDE) by employing Schauder's and Banach's fixed point theorems

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\alpha} x(t) = g_1\left(t, y(t), {}^{c}\mathcal{D}_{0^+}^{\alpha} x(t)\right), & t \in \mathcal{J}, \\ {}^{c}\mathcal{D}_{0^+}^{\beta} y(t) = g_2\left(t, x(t), {}^{c}\mathcal{D}_{0^+}^{\beta} y(t)\right), & t \in \mathcal{J}, \\ x(0) = -x(T), \\ y(0) = -y(T), \\ x'(0) = -x'(T), \\ y'(0) = -y'(T). \end{cases}$$

Where $\mathcal{J} = [0, T]$, $T \in (0, +\infty)$, ${}^{c}\mathcal{D}_{0^+}^{\alpha}$ and ${}^{c}\mathcal{D}_{0^+}^{\beta}$ are Caputo fractional derivatives of order α and β respectively, with $\alpha, \beta \in (1, 2]$. Additionally, g_1 and $g_2 : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are nonlinear continuous functions.

In 2023, El Mfadel et al. [9] used Mönch's fixed-point theorem to investigate the existence of solutions for the following anti-periodic coupled system of FDEs with a p-Laplacian operator within an arbitrary Banach space $\mathcal{X}$

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\beta_1,\xi}\phi_p\left({}^{c}\mathcal{D}_{0^+}^{\alpha_1,\xi} x(t)\right) = g_1(t, x(t), y(t)), & t \in \mathcal{J}, \\ {}^{c}\mathcal{D}_{0^+}^{\beta_2,\xi}\phi_p\left({}^{c}\mathcal{D}_{0^+}^{\alpha_2,\xi} y(t)\right) = g_2(t, x(t), y(t)), & t \in \mathcal{J}, \\ x(0) = -x(T), \\ y(0) = -y(T). \end{cases}$$

Where $\mathcal{J} = [0, T]$, $T > 0$, ${}^{c}\mathcal{D}_{0^+}^{\beta_i,\xi}$ and ${}^{c}\mathcal{D}_{0^+}^{\alpha_i,\xi}$ are ξ-Caputo fractional derivatives of orders $\beta_i, \alpha_i \in (0, 1)$ respectively, satisfying $\alpha_i + \beta_i \in (1, 2)$. Additionally, g_1 and $g_2 : \mathcal{J} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ are Carathéodory functions.

Inspired by the works mentioned above, we investigate the existence and uniqueness of a solution for the following problem

(1.1)
$$\begin{cases} {}^c\mathcal{D}_{0^+}^{\beta_1,\xi}\phi_p\big({}^c\mathcal{D}_{0^+}^{\alpha_1,\xi}x(t)\big) = g_1\left(t, x(t), y(t), {}^c\mathcal{D}_{0^+}^{\beta_1,\xi}\phi_p\left({}^c\mathcal{D}_{0^+}^{\alpha_1,\xi}x(t)\right)\right), & t \in \mathcal{J}, \\ {}^c\mathcal{D}_{0^+}^{\beta_2,\xi}\phi_p\left({}^c\mathcal{D}_{0^+}^{\alpha_2,\xi}y(t)\right) = g_2\left(t, x(t), y(t), {}^c\mathcal{D}_{0^+}^{\beta_2,\xi}\phi_p({}^c\mathcal{D}_{0^+}^{\alpha_2\xi}y(t))\right), & t \in \mathcal{J}, \\ x\left(0\right) = \sigma_1 x\left(T\right), \\ y(0) = \sigma_1 y(T), \\ x'\left(0\right) = \sigma_2 x'\left(T\right), \\ y'(0) = \sigma_2 y'(T). \end{cases}$$

Where $\sigma_1, \sigma_2 \in \mathbb{R}$ such that $\sigma_1 \neq 1$ and $\sigma_2 \neq \frac{\xi'(0)}{\xi'(T)}$, ${}^c\mathcal{D}_{0^+}^{\beta_i,\xi}$ and ${}^c\mathcal{D}_{0^+}^{\alpha_i,\xi}$ are ξ-Caputo fractional derivatives of orders $\beta_i \in (0,1)$ and $\alpha_i \in (1,2)$, respectively, for $i = 1, 2$. Additionally, $g_1, g_2 : \mathcal{J} \times \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ are continuous functions.

This paper is organized as follows: In Section 2, we present some basic tools from ξ-Caputo fractional calculus. Section 3 is devoted to proving our existence theorem using topological degree methods. In the last section, an example is given to validate the results of the existence theorem.

2. Preliminaries

This section presents some notations, basic definitions and properties of tempered ξ-fractional calculus.

We denote $\mathcal{C}(\mathcal{J}, \mathcal{X})$ the Banach space of all continuous functions $x : \mathcal{J} \to \mathcal{X}$ is equipped with the supremum norm

$$\|x\| = \sup_{t \in \mathcal{J}} \|x(t)\|.$$

Additionally, $L^p(\mathcal{J}, \mathcal{X})$ for $1 \leq p < \infty$ denotes the Banach space of all Bochner integrable functions $x : \mathcal{J} \to \mathcal{X}$ with the norm

$$\|x\|_{L^p} = \left(\int_0^t \|x\left(t\right)\|^p dt\right)^{\frac{1}{p}}.$$

We introduce the set $\mathcal{E} = \{(x, y) : (x, y) \in \mathcal{C}(\mathcal{J}, \mathcal{X}) \times \mathcal{C}(\mathcal{J}, \mathcal{X})\}$ This set is a Banach space with the norm

$$\|(x, y)\|_{\mathcal{E}} = \|x\| + \|y\|.$$

Definition 2.1 ([2])**.** Let $g \in L^1(\mathcal{J}, \mathbb{R})$ and $\xi \in C^1(\mathcal{J}, \mathbb{R})$ with $\xi'(t) > 0$ for every $t \in \mathcal{J}$. The ξ-Riemann-Liouville fractional integral of the function g at order $\gamma > 0$ is given by

$$\mathcal{I}_{0^+}^{\gamma,\xi} g(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \xi'(s) \Upsilon_\xi^{\gamma-1}(t, s) g(s) ds,$$

where $\Upsilon_\xi^\gamma(t,s) = (\xi(t) - \xi(s))^\gamma$ and Γ is the gamma function

Definition 2.2 ([2])**.** Let $g \in C^n(\mathcal{J}, \mathbb{R})$ and $\xi \in C^1(\mathcal{J}, \mathbb{R})$ with $\xi'(t) > 0$ for every $t \in \mathcal{J}$. The ξ-Caputo fractional derivative of the function g at order $\gamma > 0$ is given by

$$^C\mathcal{D}_{0^+}^{\gamma,\xi} g(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \xi'(s) \Upsilon_\xi^{n-\gamma-1}(t,s) g_\xi^{[n]}(s) ds,$$

where $g_\xi^{[n]}(s) = \left(\frac{1}{\xi'(s)} \frac{d}{ds}\right)^n g(s)$ and $n = [\gamma] + 1$ with $[\gamma]$ is the integer part of γ.

Proposition 2.3 ([2])**.** *For $\gamma > 0$, if $g \in \mathcal{C}^n(\mathcal{J}, \mathbb{R})$, then we have*

1) $^c\mathcal{D}_{0^+}^{\gamma,\xi} \mathcal{I}_{0^+}^{\gamma,\xi} g(t) = g(t)$
2) $\mathcal{I}_{0^+}^{\gamma,\xi}\ ^c\mathcal{D}_{0^+}^{\gamma,\xi} g(t) = g(t) - \sum_{k=0}^{n-1} c_k \Upsilon_\xi^k(t,0)$, *where* $c_k = \frac{g_\xi^{[k]}(0)}{k!}$

Remark 2.4. Note that if g is an abstract function with values in $\mathcal{X}$, then the integrals and derivatives appearing in Definitions 2.1 and 2.2 are taken in Bochner's sense (see, for instance, [17]).

Definition 2.5 ([7])**.** Let $O \subset \mathcal{X}$ be bounded. The map $\lambda : O \to \mathbb{R}_+$ given by

$$\lambda(O) = \inf \left\{ \rho > 0 : O \subseteq \bigcup_{i=1}^n O_i \text{ and } \operatorname{diam}(O_i) \leq \rho \right\}.$$

is called Kuratowski measure of non-compactness.

Proposition 2.6 ([7])**.** *Let $O, O_1, O_2 \subset \mathcal{X}$ be bounded, Then the following properties are satisfied*

(i) $\lambda(O) = 0 \Leftrightarrow O$ *is relatively compact.*
(ii) $\lambda(\kappa O) = |\kappa| \lambda(O), \quad \kappa \in \mathbb{R}$.
(iii) $\lambda(O_1 + O_2) \leq \lambda(O_1) + \lambda(O_2)$.
(iv) $O_1 \subset O_2 \Rightarrow \lambda(O_1) \leq \lambda(O_2)$.
(v) $\lambda(O_1 \cup O_2) = \max\{\lambda(O_1), \lambda(O_2)\}$.
(vi) $\lambda(O) = \lambda(\overline{O}) = \lambda(conv\, O)$, *where $\overline{O}$ and $conv\, O$ represent the closure and the convex hull of O, respectively.*

Definition 2.7 ([7])**.** Let $\mathcal{Q} : \Omega \subset \mathcal{X} \to \mathcal{X}$ be a continuous bounded map. $\mathcal{Q}$ is called

(i) λ-Lipschitz if there exists $\kappa \geq 0$ such that

$$\lambda(\mathcal{Q}(O)) \leq \kappa \lambda(O), \quad \text{for all } O \subset \Omega \text{ bounded.}$$

Furthermore, $\mathcal{Q}$ is called a strict λ-contraction if $\kappa < 1$.

(ii) λ-condensing if

$$\lambda(\mathcal{Q}(O)) < \lambda(O), \quad \text{for all } O \subset \Omega \text{ bounded with } \lambda(O) > 0.$$

Definition 2.8 ([7]). Let $\mathcal{Q} : \Omega \subset \mathcal{X} \to \mathcal{X}$, Then $\mathcal{Q}$ is called Lipschitz if there exists $\kappa > 0$ such that

$$\|\mathcal{Q}x_1 - \mathcal{Q}x_2\| \leq \kappa \|x_1 - x_2\|, \quad \text{for all } x_1, x_2 \in \Omega.$$

Furthermore, $\mathcal{Q}$ is called a strict contraction if $\kappa < 1$.

Lemma 2.9 ([7]). *If $\mathcal{Q}_1$, $\mathcal{Q}_2 : \Omega \subset \mathcal{X} \to \mathcal{X}$ are λ-Lipschitz maps having constants κ_1, κ_2 respectively. Then the map $\mathcal{Q}_1 + \mathcal{Q}_2 : \Omega \to \mathcal{X}$ is λ-Lipschitz having constant $\kappa_1 + \kappa_2$.*

Lemma 2.10 ([7]). *Let $\mathcal{Q} : \Omega \subset \mathcal{X} \to \mathcal{X}$*

(i) *If $\mathcal{Q}$ is Lipschitz with constant κ, then $\mathcal{Q}$ is λ-Lipschitz having the constant κ.*
(ii) *If $\mathcal{Q}$ is compact, then $\mathcal{Q}$ is λ-Lipschitz having constant $\kappa = 0$.*

Theorem 2.11 ([11]). *Let $\mathcal{Q} : \mathcal{X} \to \mathcal{X}$ be λ-condensing, consider the set*

$$\mathcal{S}_\varepsilon = \{x \in \mathcal{X} : \text{ there exists } \varepsilon \in [0,1] \text{ such that } x = \varepsilon\mathcal{Q}(x)\}.$$

If $\mathcal{S}_\varepsilon$ is a bounded in $\mathcal{X}$, then there exists $\mathcal{R} > 0$ such that $\mathcal{S}_\varepsilon \subset \mathcal{B}_\mathcal{R}(0)$ and we have

$$deg(I - \varepsilon\mathcal{Q}, \mathcal{B}_\mathcal{R}(0), 0) = 1, \quad \text{for all } \varepsilon \in [0,1].$$

Consequently, $\mathcal{Q}$ has at least a fixed point and the fixed points set of $\mathcal{Q}$ lies in $\mathcal{B}_\mathcal{R}(0)$.

Lemma 2.12 ([5]). *Let ϕ_p be a p-Laplacian operator. Then, we have*

(i) *If $1 < p < 2$, $s_1 s_2 > 0$ and $\|s_1\|, \|s_2\| \geq m > 0$, we have*

$$\|\phi_p(s_1) - \phi_p(s_2)\| \leq (p-1)m^{p-2} \|s_1 - s_2\|.$$

(ii) *If $p > 2$, $s_1 s_2 > 0$ and $\|s_1\|, \|s_2\| \leq M$, we have*

$$\|\phi_p(s_1) - \phi_p(s_2)\| \leq (p-1)M^{p-2} \|s_1 - s_2\|.$$

(iii) *ϕ_p is invertible with $\phi_p^{-1}(s) = \phi_q(s)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

3. Main Results

In this section, our main objective is to establish the existence and uniqueness of solutions to the problem (1.1). To accomplish this, we will proceed with the following assumptions

($\mathcal{A}_1$) There exist $\ell_i > 0$, $i = 1, 2, 3$ with $\ell_3 < 1$ such that

$$\|g_1(t, x_1, y_1, z_1) - g_1(t, x_2, y_2, z_2)\| \leq \ell_1\|x_1 - x_2\| + \ell_2\|y_1 - y_2\| + \ell_3\|z_1 - z_2\|,$$

for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathcal{E}$.

($\mathcal{A}_2$) There exist $\ell_i' > 0,\ i = 1,2,3$ with $\ell_3' < 1$ such that

$$\|g_2(t,x_1,y_1,z_1) - g_2(t,x_2,y_2,z_2)\| \leq \ell_1'\|x_1 - x_2\| + \ell_2'\|y_1 - y_2\| + \ell_3'\|z_1 - z_2\|,$$

for all $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in \mathcal{E}$.

($\mathcal{A}_3$) There exist $k_i > 0,\ i = 1,2,3,4$ with $k_4 < 1$ such that

$$\|g_1(t,x_1,y_1,z_1)\| \leq k_1\|x_1\| + k_2\|y_1\| + k_3\|z_1\| + k_4,$$

for all $x_1, y_1, z_1 \in \mathcal{C}(\mathcal{J},\mathcal{X})$.

($\mathcal{A}_4$) There exist $k_i' > 0,\ i = 1,2,3,4$ with $k_4' < 1$ such that

$$\|g_2(t,x_1,y_1,z_1)\| \leq k_1'\|x_1\| + k_2'\|y_1\| + k_3'\|z_1\| + k_4',$$

for all $x_1, y_1, z_1 \in \mathcal{C}(\mathcal{J},\mathcal{X})$.

For brevity, we will use the following notations

$$\mathfrak{g}^i_{x_1,x_2}(t) = g_i\left(t, x_1(t), x_2(t), {}^c\mathcal{D}^{\beta_i,\xi}_{0^+}\phi_p\,{}^c\mathcal{D}^{\alpha_i,\xi}_{0^+}x_i(t)\right),\ \mathcal{M}_i^q = (q-1)M_i^{q-2},$$

$$\varpi_{\alpha_i,\beta_i} = \mathcal{M}_i^q|\eta|\,\omega_{\alpha_i,\beta_i}\left(\left(\left|\frac{\sigma_1}{1-\sigma_1}\right| + 1\right)(\alpha_i+\beta_i) + \left|\frac{\sigma_1}{\eta(1-\sigma_1)}\right|\right),$$

where $\eta = \dfrac{\sigma_2\xi'(T)}{(\xi'(0) - \sigma_2\xi'(T))}$ and $\omega_{\alpha_i,\beta_i} = \dfrac{\Upsilon_\xi^{\alpha_i+\beta_i}(T,0)}{\Gamma(\alpha_i+\beta_i+1)},\ i = 1,2.$

$$\mathbf{L}_1 = \max\left\{\frac{\ell_1\varpi_{\alpha_1,\beta_1}}{1-\ell_3} + \frac{\ell_1'\varpi_{\alpha_2,\beta_2}}{1-\ell_3'}, \frac{\ell_2\varpi_{\alpha_1,\beta_1}}{1-\ell_3} + \frac{\ell_2'\varpi_{\alpha_2,\beta_2}}{1-\ell_3'}\right\},$$

$$\mathbf{L}_2 = \max\left\{\frac{\ell_1\omega_{\alpha_1,\beta_1}}{1-\ell_3} + \frac{\ell_1'\omega_{\alpha_2,\beta_2}}{1-\ell_3'}, \frac{\ell_2\omega_{\alpha_1,\beta_1}}{1-\ell_3} + \frac{\ell_2'\omega_{\alpha_2,\beta_2}}{1-\ell_3'}\right\},$$

$$\mathbf{K}_1 = \max\left\{\frac{k_1\varpi_{\alpha_1,\beta_1}}{1-k_3} + \frac{k_1'\varpi_{\alpha_2,\beta_2}}{1-k_3'}, \frac{k_2\varpi_{\alpha_1,\beta_1}}{1-k_3} + \frac{k_2'\varpi_{\alpha_2,\beta_2}}{1-k_3'}\right\},$$

$$\mathbf{K}_2 = \max\left\{\frac{k_1\omega_{\alpha_1,\beta_1}}{1-k_3} + \frac{k_1'\omega_{\alpha_2,\beta_2}}{1-k_3'}, \frac{k_2\omega_{\alpha_1,\beta_1}}{1-k_3} + \frac{k_2'\omega_{\alpha_2,\beta_2}}{1-k_3'}\right\},$$

$$\mathbf{D}_1 = \frac{k_4\varpi_{\alpha_1,\beta_1}}{1-k_3} + \frac{k_4'\varpi_{\alpha_2,\beta_2}}{1-k_3'},$$

$$\mathbf{D}_2 = \frac{k_4\omega_{\alpha_1,\beta_1}}{1-k_3} + \frac{k_4'\omega_{\alpha_2,\beta_2}}{1-k_3'}.$$

Definition 3.1. A pair $(x,y) \in \mathcal{E}$ is considered a solution to the coupled system (1.1) if it satisfies all the equations in (1.1).

Lemma 3.2. *Let $\beta \in (0,1)$, $\alpha \in (1,2)$ and $x, g \in \mathcal{C}(\mathcal{J},\mathcal{X})$. Then the function x is a solution of the following system*

$$\begin{cases} {}^c\mathcal{D}^{\beta,\xi}_{0^+}\phi_p\left({}^c\mathcal{D}^{\alpha,\xi}_{0^+}x(t)\right) = g(t), & t \in \mathcal{J}, \\ x(0) = \sigma_1 x(T), \\ x'(0) = \sigma_2 x'(T), \end{cases} \tag{3.1}$$

if and only if it fulfills the following fractional integral equation

$$
\begin{aligned}
x(t) = &\int_0^t \frac{\xi'(s)\Upsilon_\xi^{\alpha-1}(t,s)}{\Gamma(\alpha)}\phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g\left(s\right)\right)ds \\
&+ \frac{\sigma_1\Upsilon_\xi^1(T,0))}{1-\sigma_1}\eta\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha-2}(T,s)}{\Gamma(\alpha-1)}\phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g\left(s\right)\right)ds \\
&+ \frac{\sigma_1}{1-\sigma_1}\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha-1}(T,s)}{\Gamma(\alpha)}\phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g\left(s\right)\right)ds \\
&+ \Upsilon_\xi^1(t,0)\eta\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha-2}(T,s)}{\Gamma(\alpha-1)}\phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g\left(s\right)\right)ds,
\end{aligned} \tag{3.2}
$$

Proof. Assuming that $x \in \mathcal{C}(\mathcal{J},\mathcal{X})$ is a solution to the system (3.1), applying $\mathcal{I}_{0^+}^{\beta,\xi}$ to both sides of equation (3.1), we get:

$$\phi_p\left({}^c\mathcal{D}_{0^+}^{\alpha,\xi}x(t)\right) = c_0 + \mathcal{I}_{0^+}^{\beta,\xi}g(t),$$

it follows that

$$\phi_p\left({}^c\mathcal{D}_{0^+}^{\alpha,\xi}x(t)\right) = c_0 + \frac{1}{\Gamma(\beta)}\int_0^t \xi'(s)\Upsilon_\xi^{\beta-1}(t,s)g(s)ds, \quad \text{where } c_0 \in \mathbb{R}. \tag{3.3}$$

Since ${}^c\mathcal{D}_{0^+}^{\alpha,\xi}x(0) = 0$, we conclude that $c_0 = 0$. By applying the q-Laplacian operator ϕ_q on both sides of (3.3) and using the proposition 2.3 we get

$${}^c\mathcal{D}_{0^+}^{\alpha,\xi}x(t) = \phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g(t)\right). \tag{3.4}$$

Next, applying $\mathcal{I}_{0^+}^{\alpha,\xi}$ to both sides of (3.4) and using Proposition 2.3, it follows

$$x(t) = c_1 + c_2\Upsilon_\xi^1(t,0) + \mathcal{I}_{0^+}^{\alpha,\xi}\ \phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g(t)\right), \quad \text{where } c_1,\, c_2 \in \mathbb{R},$$

which implies that

$$x(t) = c_1 + c_2\Upsilon_\xi^1(t,0) + \frac{1}{\Gamma(\alpha)}\int_0^t \xi'(s)\Upsilon_\xi^{\alpha-1}(t,s)\phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g(s)\right)ds, \tag{3.5}$$

then, we have

$$x'(t) = c_2\xi'(t) + \xi'(t)\int_0^t \frac{\xi'(s)\Upsilon_\xi^{\alpha-2}(t,s)}{\Gamma(\alpha-1)}\phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g(s)\right)ds.$$

According to the conditions $x(0) = \sigma_1 x(T)$ and $x'(0) = \sigma_2 x'(T)$, we can conclude that

$$c_1 = \frac{\sigma_1\sigma_2\xi'(T)\Upsilon_\xi^1(T,0)}{(1-\sigma_1)(\xi'(0)-\sigma_2\xi'(T))}\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha-2}(t,s)}{\Gamma(\alpha-1)}\phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g\left(s\right)\right)ds$$

$$+\frac{\sigma_1}{1-\sigma_1}\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha-1}(T,s)}{\Gamma(\alpha)}\phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g\left(s\right)\right)ds.$$

$$c_2=\frac{\sigma_2\xi'(T)}{(\xi'(T)-\sigma_2\xi'(0))}\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha-2}(T,s)}{\Gamma(\alpha-1)}\phi_q\Big(\mathcal{I}_{0^+}^{\beta,\xi}g(s)\Big)ds.$$

By substituting c_1 and c_2 in (3.5), we get

$$\begin{aligned}x(t)=&\int_0^t \frac{\xi'(s)\Upsilon_\xi^{\alpha-1}(t,s)}{\Gamma(\alpha)}\phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g\left(s\right)\right)ds\\&+\frac{\sigma_1\Upsilon_\xi^1(T,0)}{1-\sigma_1}\eta\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha-2}(T,s)}{\Gamma(\alpha-1)}\phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g\left(s\right)\right)ds\\&+\frac{\sigma_1}{1-\sigma_1}\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha-1}(T,s)}{\Gamma(\alpha)}\phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g\left(s\right)\right)ds\\&+\Upsilon_\xi^1(t,0)\eta\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha-2}(t,s)}{\Gamma(\alpha-1)}\phi_q\left(\mathcal{I}_{0^+}^{\beta,\xi}g\left(s\right)\right)ds.\end{aligned}$$

Conversely, a direct computation shows that if x satisfies the integral equation (3.2), the system (3.1) holds, completing the proof. □

We introduce the operators

$\mathcal{G}_i:\mathcal{E}\mapsto\mathcal{C}(\mathcal{J},\mathcal{X}),\ i=1,2$

$$\begin{aligned}\mathcal{G}_i(x,y)(t)=&\frac{\sigma_1}{1-\sigma_1}\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha_i-1}(T,s)}{\Gamma(\alpha_i)}\phi_q\left(\mathcal{I}_{0^+}^{\beta_i,\xi}\mathfrak{g}_{x,y}^i\left(s\right)\right)ds\\&+\Upsilon_\xi^1(t,0)\eta\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha_i-2}(T,s)}{\Gamma(\alpha_i-1)}\phi_q\left(\mathcal{I}_{0^+}^{\beta_i,\xi}\mathfrak{g}_{x,y}^i\left(s\right)\right)ds\\&+\frac{\sigma_1\Upsilon_\xi^1(T,0)}{1-\sigma_1}\eta\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha_i-2}(T,s)}{\Gamma(\alpha_i-1)}\phi_q\left(\mathcal{I}_{0^+}^{\beta_i,\xi}\mathfrak{g}_{x,y}^i\left(s\right)\right)ds,\end{aligned}$$

$\mathcal{Q}_i:\mathcal{E}\mapsto\mathcal{C}(\mathcal{J},\mathcal{X}),\ i=1,2$

$$\mathcal{Q}_i(x,y)(t)=\int_0^t \frac{\xi'(s)\Upsilon_\xi^{\alpha_i-1}(t,s)}{\Gamma(\alpha_i)}\phi_q\left(\mathcal{I}_{0^+}^{\beta_i,\xi}\mathfrak{g}_{x,y}^i\left(s\right)\right)ds,$$

and

$\mathcal{F}_i:\mathcal{E}\mapsto\mathcal{C}(\mathcal{J},\mathcal{X}),\ i=1,2$

$\mathcal{F}_i(x,y)=\mathcal{G}_i(x,y)+\mathcal{Q}_i(x,y).$

We define the operator: $\mathcal{F}:\mathcal{E}\to\mathcal{E}$ given by

$$\mathcal{F}(x,y)=\left(\mathcal{F}_1\left(x,y\right),\mathcal{F}_2\left(x,y\right)\right).$$

It is clear that $\mathcal{F}$ is well defined. By Lemma 3.2, we can deduce that

$$\mathcal{F}(x,y)=(x,y).$$

Thus, demonstrating that the system (1.1) has a solution is equivalent to demonstrating that the operator $\mathcal{F}$ has a fixed point.
In the following section, we will show that the operator $\mathcal{F}$ satisfies all the conditions given in (2.11).

Lemma 3.3. *The operator $\mathcal{G}$ is λ-Lipschitz having constant $\mathbf{L}_1$. Furthermore, $\mathcal{G}$ satisfies the following growth condition*

$$\|\mathcal{G}(x,y)\|_{\mathcal{E}}\leq \mathbf{K}_1\|(x,y)\|+\mathbf{D}_1. \tag{3.6}$$

Proof. Let (x,y), $(\bar{x},\bar{y})\in\mathcal{E}$. Then, we have

$$\begin{aligned}
&\|\mathcal{G}_1(x,y)-\mathcal{G}_1(\bar{x},\bar{y})\|\\
&\leq \frac{|\sigma_1|}{|1-\sigma_1|}\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha_1-1}(T,s)}{\Gamma(\alpha_1)}\left\|\phi_q\mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{x,y}(s)-\phi_q\mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{\overline{x},\overline{y}}(s)\right\|ds\\
&\quad+\frac{|\sigma_1|}{|1-\sigma_1|}\Upsilon^1_\xi(T,0)\,|\eta|\int_0^T\frac{\xi'(s)\Upsilon_\xi^{\alpha_1-2}(T,s)}{\Gamma(\alpha_1-1)}\\
&\quad\times\left\|\phi_q\mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{x,y}(s)-\phi_q\mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{\overline{x},\overline{y}}(s)\right\|ds\\
&\quad+\Upsilon^1_\xi(t,0)\,|\eta|\int_0^T\frac{\xi'(s)\Upsilon_\xi^{\alpha_1-2}(T,s)}{\Gamma(\alpha_1-1)}\\
&\quad\times\left\|\phi_q\mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{x,y}(s)-\phi_q\mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{\overline{x},\overline{y}}(s)\right\|ds.
\end{aligned}$$

By Lemma 2.12 and $(\mathcal{A}_1)$, we obtain

$$\left\|\phi_q\mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{x,y}(s)-\phi_q\mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{\overline{x},\overline{y}}(s)\right\|\leq\mathcal{M}_1^q\mathcal{I}_{0^+}^{\beta_1,\xi}\left(\left\|\mathfrak{g}^1_{x,y}(s)-\mathfrak{g}^1_{\overline{x},\overline{y}}(s)\right\|\right),$$

it follows

$$\begin{aligned}
\|\mathcal{G}_1(x,y)-\mathcal{G}_1(\bar{x},\bar{y})\|\leq\mathcal{M}_1^q&\left|\frac{\sigma_1}{1-\sigma_1}\right|\int_0^T\frac{\xi'(s)\Upsilon_\xi^{\alpha_1-1}(T,s)}{\Gamma(\alpha_1)}\\
&\times\mathcal{I}_{0^+}^{\beta_1,\xi}\left(\frac{\ell_1}{1-\ell_3}\|x-\overline{x}\|+\frac{\ell_2}{1-\ell_3}\|y-\overline{y}\|\right)ds\\
&+\mathcal{M}_1^q\frac{|\sigma_1|}{|1-\sigma_1|}\Upsilon^1_\xi(T,0)\,|\eta|\int_0^T\frac{\xi'(s)\Upsilon_\xi^{\alpha_1-2}(T,s)}{\Gamma(\alpha_1-1)}\\
&\times\mathcal{I}_{0^+}^{\beta_1,\xi}\left(\frac{\ell_1}{1-\ell_3}\|x-\overline{x}\|+\frac{\ell_2}{1-\ell_3}\|y-\overline{y}\|\right)ds\\
&+\mathcal{M}_1^q\Upsilon^1_\xi(t,0)\,|\eta|\int_0^T\frac{\xi'(s)\Upsilon_\xi^{\alpha_1-2}(T,s)}{\Gamma(\alpha_1-1)}
\end{aligned}$$

$$\times \mathcal{I}_{0^+}^{\beta_1,\xi}\left(\frac{\ell_1}{1-\ell_3}\|x-\overline{x}\|+\frac{\ell_2}{1-\ell_3}\|y-\overline{y}\|\right)ds.$$

Then

$$\|\mathcal{G}_1(x,y)-\mathcal{G}_1(\bar{x},\bar{y})\| \leq \left(\frac{\ell_1\varpi_{\alpha_1,\beta_1}}{1-\ell_3}\right)\|x-\overline{x}\|+\left(\frac{\ell_2\varpi_{\alpha_1,\beta_1}}{1-\ell_3}\right)\|y-\overline{y}\|.$$

Similarly, according to Lemma 2.12 and $(\mathcal{A}_2)$, we get

$$\|\mathcal{G}_2(x,y)-\mathcal{G}_2(\bar{x},\bar{y})\| \leq \left(\frac{\ell_1'\varpi_{\alpha_2,\beta_2}}{1-\ell_3'}\right)\|x-\overline{x}\|+\left(\frac{\ell_2'\varpi_{\alpha_2,\beta_2}}{1-\ell_3'}\right)\|y-\overline{y}\|.$$

Hence

$$\|\mathcal{G}(x,y)-\mathcal{G}(\overline{x},\overline{y})\| \leq \mathbf{L}_1\left(\|x-\bar{x}\|+\|y-\bar{y}\|\right).$$

Then $\mathcal{G}$ is Lipschitz having constant $\mathbf{L}_1$.
To show the growth condition (3.6), let $(x,y)\in\mathcal{E}$. Thus by Lemma 2.12 and $(\mathcal{A}_3)$, we obtain

$$\begin{aligned}
\|\mathcal{G}_1(x,y)\| \leq \mathcal{M}_1^q\left|\frac{\sigma_1}{1-\sigma_1}\right|\int_0^T \frac{\xi'(s)\Upsilon_\xi^{\alpha_1-1}(T,s)}{\Gamma(\alpha_1)}&\\
\times \mathcal{I}_{0^+}^{\beta_1,\xi}\left(\frac{k_1}{1-k_3}\|x\|+\frac{k_2}{1-k_3}\|y\|+\frac{k_4}{1-k_3}\right)ds&\\
+\mathcal{M}_1^q\frac{|\sigma_1|\,\Upsilon_\xi^1(T,0)}{|1-\sigma_1|\,|\eta|}\int_0^T\frac{\xi'(s)\Upsilon_\xi^{\alpha_1-2}(T,s)}{\Gamma(\alpha_1-1)}&\\
\times \mathcal{I}_{0^+}^{\beta_1,\xi}\left(\frac{k_1}{1-k_3}\|x\|+\frac{k_2}{1-k_3}\|y\|+\frac{k_4}{1-k_3}\right)ds&\\
+\mathcal{M}_1^q\Upsilon_\xi^1(t,0)\,|\eta|\int_0^T\frac{\xi'(s)\Upsilon_\xi^{\alpha_1-2}(T,s)}{\Gamma(\alpha_1-1)}&\\
\times \mathcal{I}_{0^+}^{\beta_1,\xi}\left(\frac{k_1}{1-k_3}\|x\|+\frac{k_2}{1-k_3}\|y\|+\frac{k_4}{1-k_3}\right)ds.&
\end{aligned}$$

Therefore

$$\|\mathcal{G}_1(x,y)\| \leq \left(\frac{k_1\varpi_{\alpha_1,\beta_1}}{1-k_3}\right)\|x\|+\left(\frac{k_2\varpi_{\alpha_1,\beta_1}}{1-k_3}\right)\|y\|+\left(\frac{k_4\varpi_{\alpha_1,\beta_1}}{1-k_3}\right).$$

By the same manner, using Lemma 2.12 and $(\mathcal{A}_4)$, we find that

$$\|\mathcal{G}_2(x,y)\| \leq \left(\frac{k_1'\varpi_{\alpha_2,\beta_2}}{1-k_3'}\right)\|x\|+\left(\frac{k_2'\varpi_{\alpha_2,\beta_2}}{1-k_3'}\right)\|y\|+\left(\frac{k_4'\varpi_{\alpha_1,\beta_1}}{1-k_3'}\right).$$

Hence

$$\|\mathcal{G}(x,y)\|_{\mathcal{E}} \leq \mathbf{K}_1\|(x,y)\|+\mathbf{D}_1. \qquad \square$$

Lemma 3.4. *The operator $\mathcal{Q}$ is continuous. Additionally, $\mathcal{Q}$ satisfies the following growth condition:*

$$\|\mathcal{Q}(x,y)\|_{\mathcal{E}} \leq \mathbf{K}_2\|(x,y)\|+\mathbf{D}_2. \tag{3.7}$$

Proof. Take $\mathcal{B}_{\mathcal{R}} = \{(w_1, w_2) \in \mathcal{E} : \|(w_1, w_2)\|_{\mathcal{E}} \leq \mathcal{R}\}$ and consider a sequence $(x_n, y_n) \in \mathcal{B}_{\mathcal{R}}$ that converges to $(x, y) \in \mathcal{B}_{\mathcal{R}}$.
By utilizing Lemma 2.12, we get

$$\begin{aligned}
&\|\mathcal{Q}_1(x_n, y_n) - \mathcal{Q}_1(x, y)\| \\
&\quad = \left\| \mathcal{I}_{0^+}^{\alpha_1,\xi} \phi_q \left(\mathcal{I}_{0^+}^{\beta_1,\xi} \mathfrak{g}^1_{x_n,y_n}(t) \right) - \mathcal{I}_{0^+}^{\alpha_1,\xi} \phi_q \left(\mathcal{I}_{0^+}^{\beta_1,\xi} \mathfrak{g}^1_{x,y}(t) \right) \right\| \\
&\quad \leq \mathcal{M}_1^q \mathcal{I}_{0^+}^{\alpha_1,\xi} \left(\mathcal{I}_{0^+}^{\beta_1,\xi} \left\| \mathfrak{g}^1_{x_n,y_n}(t) - \mathfrak{g}^1_{x,y}(t) \right\| \right).
\end{aligned}$$

By means of $(\mathcal{A}_3)$, we obtain

$$\begin{aligned}
&\frac{\xi'(s)\Upsilon_\xi^{\alpha_1-1}(t,s)\Upsilon_\xi^{\beta_1}(s,0)}{\Gamma(\alpha_1)\Gamma(\beta_1+1)} \left\| \mathfrak{g}^1_{x_n,y_n}(t) - \mathfrak{g}^1_{x,y}(t) \right\| \\
&\leq 2\frac{\xi'(s)\Upsilon_\xi^{\alpha_1-1}(t,s)\Upsilon_\xi^{\beta_1}(s,0)}{\Gamma(\beta_1+1)\Gamma(\alpha_1)} \left(\left(\frac{k_1+k_2}{1-k_3} \right) \mathcal{R} + \frac{k_4}{1-k_3} \right).
\end{aligned}$$

Since the function $s \mapsto \dfrac{\xi'(s)\Upsilon_\xi^{\alpha_1-1}(t,s)\Upsilon_\xi^{\beta_1}(s,0)}{\Gamma(\alpha_1)\Gamma(\beta_1+1)}$ is Lebesgue integrable over $[0,t]$. Then, by the Lebesgue Dominated Convergence Theorem, we deduce that

$$\left\| \mathcal{I}_{0^+}^{\alpha_1,\xi} \phi_q \left(\mathcal{I}_{0^+}^{\beta_1,\xi} \mathfrak{g}^1_{x_n,y_n}(t) \right) - \mathcal{I}_{0^+}^{\alpha_1,\xi} \phi_q \left(\mathcal{I}_{0^+}^{\beta_1,\xi} \mathfrak{g}^1_{x,y}(t) \right) \right\| \to 0, \quad \text{as } n \to \infty.$$

Then

$$\|\mathcal{Q}_1(x_n, y_n) - \mathcal{Q}_1(x, y)\| \to 0, \quad \text{as } n \to \infty.$$

In the same way, using $(\mathcal{A}_4)$, we obtain

$$\|\mathcal{Q}_2(x_n, y_n) - \mathcal{Q}_2(x, y)\| \to 0, \quad \text{as } n \to \infty,$$

Therefore

$$\|\mathcal{Q}(x_n, y_n) - \mathcal{Q}(x, y)\|_{\mathcal{E}} \to 0, \quad \text{as } n \to \infty.$$

Hence, $\mathcal{Q}$ is a continuous.

Now, let us demonstrate the growth condition (3.7), let $(x, y) \in \mathcal{E}$. Then, by using Lemma 2.12 and $(\mathcal{A}_3)$, we obtain

$$\begin{aligned}
\|\mathcal{Q}_1(x, y)\| &\leq \int_0^t \frac{\xi'(s)\Upsilon_\xi^{\alpha_i-1}(t,s)}{\Gamma(\alpha_1)} \phi_q \left(\mathcal{I}_{0^+}^{\beta_1,\xi} \left\| \mathfrak{g}^1_{x,y}(s) \right\| \right) ds \\
&\leq \mathcal{M}_1^q \int_0^t \frac{\xi'(s)\Upsilon_\xi^{\alpha_1-1}(t,s)}{\Gamma(\alpha_1)} \mathcal{I}_{0^+}^{\beta_1,\xi} \Big(\frac{k_1 \|x\|}{1-k_3} + \frac{k_2 \|y\|}{1-k_3} + \frac{k_4}{1-k_3} \Big) ds \\
&\leq \left(\frac{k_1 \omega_{\alpha_1,\beta_1}}{1-k_3} \right) \|x\| + \left(\frac{k_1 \omega_{\alpha_1,\beta_1}}{1-k_3} \right) \|y\| + \left(\frac{k_4 \omega_{\alpha_1,\beta_1}}{1-k_3} \right).
\end{aligned}$$

Similarly, by using Lemma 2.12 and $(\mathcal{A}_4)$, we obtain

$$\|\mathcal{Q}_2(x,y)\| \leq \left(\frac{k_1'\omega_{\alpha_2,\beta_2}}{1-k_3'}\right)\|x\| + \left(\frac{k_2'\omega_{\alpha_2,\beta_2}}{1-k_3'}\right)\|y\| + \left(\frac{k_4'\omega_{\alpha_2,\beta_2}}{1-k_3'}\right).$$

Therefore

$$\|\mathcal{Q}(x,y)\| \leq \mathbf{K}_2\,\|(x,y)\| + \mathbf{D}_2. \qquad \square$$

Lemma 3.5. *The operator $\mathcal{Q}$ is compact. Consequently $\mathcal{Q}$ is λ-Lipschitz with zero constant.*

Proof. Consider $(x,y) \in \mathcal{B}_{\mathcal{R}}$ and $t_1, t_2 \in \mathcal{J}$ with $t_1 < t_2$, then we obtain

$$\begin{aligned}
&\|\mathcal{Q}_1(x,y)(t_2) - \mathcal{Q}_1(x,y)(t_1)\| \\
&= \left\| \int_{t_1}^{t_2} \frac{\xi'(s)\Upsilon_\xi^{\alpha_1-1}(t_2,s)}{\Gamma(\alpha_1)} \phi_q \mathcal{I}_{0^+}^{\beta_1,\xi} \mathfrak{g}_{x,y}^1(s)ds \right\| \\
&\quad + \left\| \int_0^{t_1} \frac{\xi'(s)\left[\Upsilon_\xi^{\alpha_1-1}(t_2,s) - \Upsilon_\xi^{\alpha_1-1}(t_1,s)\right]}{\Gamma(\beta_1)} \phi_q \mathcal{I}_{0^+}^{\beta_1,\xi} \mathfrak{g}_{x,y}^1(s)ds \right\|.
\end{aligned}$$

By Lemma 2.12 and $(\mathcal{A}_3)$, we get

$$\begin{aligned}
&\|\mathcal{Q}_1(x,y)(t_2) - \mathcal{Q}_1(x,y)(t_1)\| \\
&\leq \mathcal{M}_1^q \left[\int_{t_1}^{t_2} \frac{\xi'(s)\Upsilon_\xi^{\alpha_1-1}(t_2,s)}{\Gamma(\alpha_1)} \mathcal{I}_{0^+}^{\beta_1,\xi} \left(\frac{k_1+k_2}{1-k_3}\mathcal{R} + \frac{k_4}{1-k_3}\right) ds \right. \\
&\quad + \int_0^{t_1} \frac{\xi'(t)\left(\Upsilon_\xi^{\alpha_1-1}(t_2,s) - \Upsilon_\xi^{\alpha_1-1}(t_1,s)\right)}{\Gamma(\alpha_1)} \\
&\quad \left. \times \mathcal{I}_{0^+}^{\beta_1,\xi} \left(\frac{k_1+k_2}{1-k_3}\mathcal{R} + \frac{k_4}{1-k_3}\right) ds \right] \\
&\leq \mathcal{M}_1^q \left(\frac{k_1+k_2}{1-k_3}\mathcal{R} + \frac{k_4}{1-k_3}\right) \\
&\quad \times \frac{\left(\Upsilon_\xi^{\alpha_1+\beta_1}(t_1,0) + 2\Upsilon_\xi^{\alpha_1+\beta_1}(t_2,t_1) - \Upsilon_\xi^{\alpha_1+\beta_1}(t_2,0)\right)}{\Gamma(\alpha_1+\beta_1+1)}.
\end{aligned}$$

Since the function ξ is continuous, we can conclude that

$$\|\mathcal{Q}_1(x,y)(t_2) - \mathcal{Q}_1(x,y)(t_1)\| \to 0, \quad \text{as } t_1 \to t_2.$$

In a similar way, when we use Lemma 2.12 and $(\mathcal{A}_4)$, we obtain

$$\|\mathcal{Q}_2(x,y)(t_2) - \mathcal{Q}_2(x,y)(t_1)\| \to 0, \quad \text{as } t_1 \to t_2.$$

Therefore

$$\|\mathcal{Q}(x,y)(t_2) - \mathcal{Q}(x,y)(t_1)\| \to 0, \quad \text{as } t_1 \to t_2.$$

Then, $\mathcal{Q}(\mathcal{B}_{\mathcal{R}})$ is equicontinuous. According to the Arzelà-Ascoli Theorem [10], we can conclude that $\mathcal{Q}(\mathcal{B}_{\mathcal{R}})$ is relatively compact. Therefore, the operator $\mathcal{Q}$ is compact. Consequently, by Lemma 2.10, we can deduce that $\mathcal{Q}$ is λ-Lipschitz having a constant zero. □

Theorem 3.6. *If the assumptions* $(\mathcal{A}_1)-(\mathcal{A}_4)$ *and* $\mathbf{L}_1 < 1$ *are satisfied, then the problem* (1.1) *has at least one solution* $(x, y) \in \mathcal{E}$*. Additionally, the solution set of* (1.1) *is bounded in* $\mathcal{E}$*.*

Proof. The operators $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{F}$ are all continuous and bounded. Furthermore, $\mathcal{G}$ is λ-Lipschitz having constant $\mathbf{L}_1$ and $\mathcal{Q}$ is λ-Lipschitz having zero constant. By using Lemmas 2.9 and 2.10, we can deduce that the operator $\mathcal{F}$ is λ-condensing.
Now, let us consider the set

$$\mathcal{S}_\varepsilon = \{(x, y) \in \mathcal{E} : \text{ there exist } 0 \leq \varepsilon \leq 1 \text{ with } (x, y) = \varepsilon\mathcal{F}(x, y)\}.$$

Let $(x, y) \in \mathcal{S}_\varepsilon$. Then, we have

$$\begin{aligned}\|(x, y)\|_{\mathcal{E}} &= \varepsilon \|\mathcal{F}(x, y)\|_{\mathcal{E}} \\ &\leq \|\mathcal{G}(x, y)\|_{\mathcal{E}} + \|\mathcal{Q}(x, y)\|_{\mathcal{E}},\end{aligned}$$

Considering Lemmas 3.3 and 3.4, we obtain

$$\|(x, y)\|_{\mathcal{E}} \leq (\mathbf{K}_1 + \mathbf{K}_2)\|(x, y)\| + (\mathbf{D}_1 + \mathbf{D}_2).$$

This implies that $\mathcal{S}_\varepsilon$ is bounded in $\mathcal{E}$. Since all conditions of Theorem 2.11 are met, we can infer that the operator $\mathcal{F}$ has at least one fixed point, which is a solution to the problem (1.1). □

Theorem 3.7. *Assume that* $(\mathcal{A}_1)-(\mathcal{A}_2)$ *hold and* $\mathbf{L}_1 + \mathbf{L}_2 < 1$*, then the problem* (1.1) *has a unique solution* $(x, y) \in \mathcal{E}$*.*

Proof. Let (x, y), $(\overline{x}, \overline{y}) \in \mathcal{E}$, we have

$$\begin{aligned}&\|\mathcal{Q}(x, y) - \mathcal{Q}(\overline{x}, \overline{y})\| \\ &\quad = \|\mathcal{Q}_1(x, y) - \mathcal{Q}_1(\overline{x}, \overline{y})\| + \|\mathcal{Q}_2(x, y) - \mathcal{Q}_2(\overline{x}, \overline{y})\| \\ &\quad \leq \int_0^t \frac{\xi'(s)\Upsilon_\xi^{\alpha_1-1}(t, s)}{\Gamma(\alpha_1)} \left\|\phi_q \mathcal{I}_{0^+}^{\beta_1,\xi} \mathfrak{g}^1_{x,y}(s) - \phi_q \mathcal{I}_{0^+}^{\beta_1,\xi} \mathfrak{g}^1_{\overline{x},\overline{y}}(s)\right\| ds \\ &\qquad + \int_0^t \frac{\xi'(s)\Upsilon_\xi^{\alpha_2-1}(t, s)}{\Gamma(\alpha_2)} \left\|\phi_q \mathcal{I}_{0^+}^{\beta_2,\xi} \mathfrak{g}^2_{x,y}(s) - \phi_q \mathcal{I}_{0^+}^{\beta_2,\xi} \mathfrak{g}^2_{\overline{x},\overline{y}}(s)\right\| ds.\end{aligned}$$

Combining Lemma 2.12 with $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we get

$$\begin{aligned}&\|\mathcal{Q}(x, y) - \mathcal{Q}(\overline{x}, \overline{y})\| \\ &\quad \leq \mathcal{M}_1^q \int_0^t \frac{\xi'(s)\Upsilon_\xi^{\alpha_1-1}(t, s)}{\Gamma(\alpha_1)} \mathcal{I}_{0^+}^{\beta_1,\xi} \left(\frac{\ell_1}{1-\ell_3}\|x - \overline{x}\| + \frac{\ell_2}{1-\ell_3}\|y - \overline{y}\|\right) ds\end{aligned}$$

$$+\mathcal{M}_2^q\int_0^t\frac{\xi'(s)\Upsilon_\xi^{\alpha_2-1}(t,s)}{\Gamma(\alpha_2)}\mathcal{I}_{0^+}^{\beta_2,\xi}\left(\frac{\ell_1'}{1-\ell_3'}\left\|x-\overline{x}\right\|+\frac{\ell_2'}{1-\ell_3'}\left\|y-\overline{y}\right\|\right)ds.$$

Therefore

$$\left\|\mathcal{Q}(x,y)-\mathcal{Q}(\overline{x},\overline{y})\right\|\leq\mathbf{L}_2\left(\left\|x-\overline{x}\right\|+\left\|y-\overline{y}\right\|\right).$$

On the other hand, by Lemma 3.3, we have

$$\left\|\mathcal{G}(x,y)-\mathcal{G}(\overline{x},\overline{y})\right\|\leq\mathbf{L}_1\left(\left\|x-\overline{x}\right\|+\left\|y-\overline{y}\right\|\right).$$

Hence

$$\left\|\mathcal{F}(x,y)-\mathcal{F}(\bar{x},\bar{y})\right\|\leq\left(\mathbf{L}_1+\mathbf{L}_2\right)\left\|(x,y)-(\overline{x},\overline{y})\right\|.$$

Due to the Banach Fixed Point Theorem, $\mathcal{F}$ has a unique fixed point. Hence, the system (1.1) has one solution. □

4. An Illustrative Example

In this section we consider the following fractional coupled system (4.1)

$$\begin{cases}{}^c\mathcal{D}_{0^+}^{\frac{1}{2},t^2}\phi_2{}^c\mathcal{D}_{0^+}^{\frac{3}{2},t^2}x(t)=\frac{e^{-t}|x|}{19+e^t}+\frac{\sin(|y|)+\left|{}^c\mathcal{D}_{0^+}^{\frac{1}{2},t^2}\phi_2{}^c\mathcal{D}_{0^+}^{\frac{3}{2},t^2}\right|}{10+t^2}, & t\in[0,1],\\ {}^c\mathcal{D}_{0^+}^{\frac{1}{2},t^2}\phi_2{}^c\mathcal{D}_{0^+}^{\frac{3}{2},t^2}y(t)=\frac{|x|}{13+e^t}+\frac{|y|}{7+t^2}+\frac{3\left|{}^c\mathcal{D}_{0^+}^{\frac{1}{2},t^2}\phi_2{}^c\mathcal{D}_{0^+}^{\frac{3}{2},t^2}y\right|}{22-t^2}, & t\in[0,1],\\ x(0)=-x(1),\\ y(0)=-y(1),\\ x'(0)=-x'(1),\\ y'(0)=-y'(1).\end{cases}$$

It is evident that the system (4.1) is a special case of the problem (1.1) when $\alpha_1=\alpha_2=\frac{1}{2},\ \beta_1=\beta_2=\frac{3}{2},\ \mathcal{J}=[0,1],\ \sigma_1=\sigma_2=-1,\ \xi(t)=t^2$, $p=q=2$,

$$g_1\left(t,x,y,{}^c\mathcal{D}_{0^+}^{\frac{1}{2},t^2}\phi_2{}^c\mathcal{D}_{0^+}^{\frac{3}{2},t^2}x\right)=\frac{e^{-t}|x|}{19+e^t}+\frac{\sin(|y|)+\left|{}^c\mathcal{D}_{0^+}^{\frac{1}{2},t^2}\phi_2{}^c\mathcal{D}_{0^+}^{\frac{3}{2},t^2}\right|}{10+t^2},$$

and

$$g_2\left(t,x,y,{}^c\mathcal{D}_{0^+}^{\frac{1}{2},t^2}\phi_2{}^c\mathcal{D}_{0^+}^{\frac{3}{2},t^2}y\right)=\frac{|x|}{13+e^t}+\frac{|y|}{7+t^2}+\frac{3\left|{}^c\mathcal{D}_{0^+}^{\frac{1}{2},t^2}\phi_2{}^c\mathcal{D}_{0^+}^{\frac{3}{2},t^2}y\right|}{22-t^2}.$$

In the following discussion, we will demonstrate that the assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ are satisfied.

Let $x_1,y_1,z_1,x_2,y_2,z_2\in\mathcal{C}(\mathcal{J},\mathbb{R})$, we have

$$\left|g_1\left(t,x_1,y_1,z_1\right)-g_1\left(t,x_2,y_2,z_2\right)\right|$$

$$\leq \left|\frac{e^{-t}(|x_1|-|x_2|)}{19+e^t}\right| + \left|\frac{\sin(|y_1|)-\sin(|y_2|)}{10+t^2}\right| + \left|\frac{|z_1|-|z_2|}{10+t^2}\right|$$
$$\leq \frac{1}{20}|x_1-x_2| + \frac{1}{10}|y_1-y_2| + \frac{1}{10}|z_1-z_2|.$$

Therefore, the assumption $(\mathcal{A}_1)$ is satisfied with $\ell_1 = \frac{1}{20}$, $\ell_2 = \frac{1}{10}$ and $\ell_3 = \frac{1}{10}$.

In the same way, we obtain

$$|g_2(t,x_1,y_1,z_1) - g_2(t,x_2,y_2z_2)| \leq \frac{1}{14}|x_1-x_2| + \frac{1}{7}|y_1-y_2| + \frac{1}{7}|z_1-z_2|.$$

Therefore the condition $(\mathcal{A}_2)$ holds with $\ell_1' = \frac{1}{14}, \ell_2' = \frac{1}{7}$ and $\ell_3' = \frac{1}{7}$. Furthermore, it can be demonstrated that

$$|g_1(t,x_1,y_1,z_1)| = \left|\frac{e^{-t}}{19+e^t}|x_1| + \frac{1}{10+t^2}\sin(|y_1|) + \frac{1}{10+t^2}|z_1|\right|$$
$$\leq \frac{1}{20}|x_1| + \frac{1}{10}|y_1| + \frac{1}{10}|z_1|.$$

Hence the condition $(\mathcal{A}_3)$ holds with $k_1 = \frac{1}{20}$, $k_2 = \frac{1}{10}$, $k_3 = \frac{1}{10}$ and $k_4 = 0$.

In similar way, we get

$$|g_2(t,x_1,y_1,z_1)| \leq \frac{1}{14}|x_1| + \frac{1}{7}|y_1| + \frac{1}{10}|z_1|.$$

It follows that $(\mathcal{A}_4)$ holds with $k_1' = \frac{1}{14}, k_2' = \frac{1}{7}, k_3' = \frac{1}{7}$ and $k_4' = 0$.

Moreover, a straightforward calculation demonstrates that

$$\mathbf{L}_1 \simeq 0.2363 < 1 \text{ and } \mathbf{L}_1 + \mathbf{L}_2 \simeq 0.4808 < 1$$

By applying Theorem 3.7, we conclude that a unique solution exists for the system (4.1).

5. Conclusion

By employing the topological degree theory for condensing maps and the Banach contraction mapping principle, we have established sufficient conditions for ensuring the existence and uniqueness of solutions for the system (1.1). Finally, we provide a test example to illustrate our findings.

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