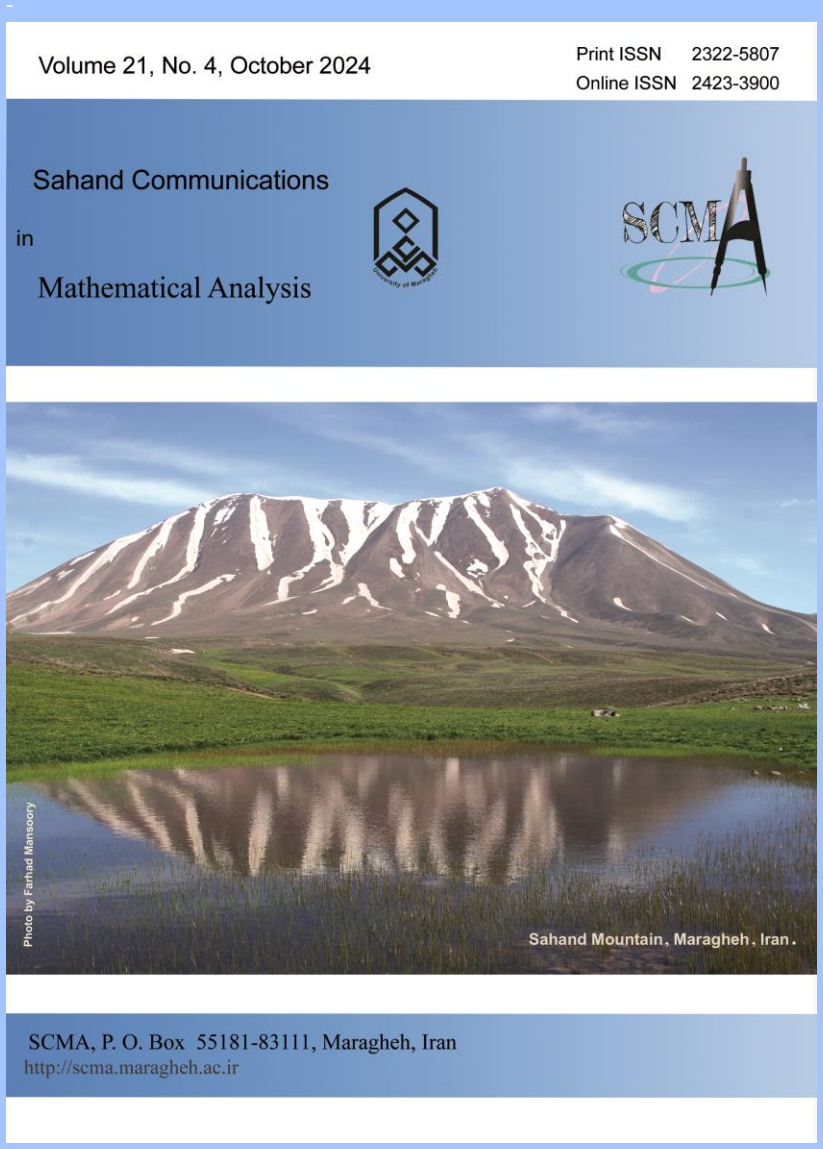
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Pseudo-Duals of Frames and Modular Riesz Bases in Hilbert C^*-Modules

Morteza Mirzaee Azandaryani

Abstract. In the present article, duals, approximate duals and pseudo-duals (generated by bounded and not necessarily adjointable operators) of a frame in a Hilbert C^*-module are characterized and some of their properties are obtained. Especially, the ones constructed by multiplication operators are discussed and their stability under the action of morphisms is focused and some equivalent conditions for the stability are derived. Finally, we get some results on pseudo-duals of modular Riesz bases, mainly their preservation under the action of morphisms and their behavior in the presence of semi-normalized symbols are studied.

1. Introduction

Standard frames in Hilbert C^*-modules were introduced and studied in [8].

Definition 1.1. Let $\mathcal{A}$ be a unital C^*-algebra and assume that $\mathcal{X}$ is a countably generated Hilbert $\mathcal{A}$-module. A finite or countable set $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$ in $\mathcal{X}$ is a standard frame for $\mathcal{X}$, if the series $\sum_{i\in\mathbb{J}} \langle g, x_i\rangle \langle x_i, g\rangle$ is convergent with respect to the norm, for each $g \in \mathcal{X}$, and there are positive numbers $C_{\mathcal{E}}$ and $D_{\mathcal{E}}$, such that for each $g \in \mathcal{X}$, the following inequalities hold

$$C_{\mathcal{E}} \langle g, g\rangle \leq \sum_{i\in\mathbb{J}} \langle g, x_i\rangle \langle x_i, g\rangle \leq D_{\mathcal{E}} \langle g, g\rangle .$$

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A sequence that satisfies the second inequality is called a standard Bessel sequence.

In the rest of this note, for two Hilbert $\mathcal{A}$-modules $\mathcal{X}$ and $\mathcal{Y}$, we denote the set of all bounded operators from $\mathcal{X}$ into $\mathcal{Y}$ by $\mathcal{B}(\mathcal{X},\mathcal{Y})$ and $\mathcal{B}(\mathcal{X},\mathcal{X})$ is denoted by $\mathcal{B}(\mathcal{X})$. Also, $\mathfrak{L}(\mathcal{X},\mathcal{Y})$ is the set of all adjointable operators from $\mathcal{X}$ into $\mathcal{Y}$ and $\mathfrak{L}(\mathcal{X},\mathcal{X})$ is denoted by $\mathfrak{L}(\mathcal{X})$. The space $\ell^2(\mathbb{J},\mathcal{A})$ which is defined by

$$\ell^2(\mathbb{J},\mathcal{A})=\left\{\{c_i\}_{i\in\mathbb{J}} \text{ in } \mathcal{A}:\sum_{i\in\mathbb{J}}c_ic_i^* \text{ converges with respect to the norm}\right\}$$

is a Hilbert $\mathcal{A}$-module with inner product $\langle\{c_i\}_{i\in\mathbb{J}},\{d_i\}_{i\in\mathbb{J}}\rangle=\sum_{i\in\mathbb{J}}c_id_i^*$. Many interesting properties of Hilbert C^*-modules can be found in [14].

Let $\mathcal{E}=\{x_i\}_{i\in\mathbb{J}}$ be a standard Bessel sequence with an upper bound $D_{\mathcal{E}}$. The operator $T_{\mathcal{E}}\colon \ell^2(\mathbb{J},\mathcal{A})\to\mathcal{X}$ which is defined by

$$T_{\mathcal{E}}(\{c_i\}_{i\in\mathbb{J}})=\sum_{i\in\mathbb{J}}c_ix_i$$

is called the synthesis operator of $\mathcal{E}$. It is adjointable with $T_{\mathcal{E}}^*(g)=\{\langle g,x_i\rangle\}_{i\in\mathbb{J}}$ and $\|T_{\mathcal{E}}\|\leq\sqrt{D_{\mathcal{E}}}$. $T_{\mathcal{E}}^*$ is the analysis operator of $\mathcal{E}$. Now the operator $S_{\mathcal{E}}\colon\mathcal{X}\to\mathcal{X}$ is defined by

$$\begin{aligned}S_{\mathcal{E}}(g)&=T_{\mathcal{E}}T_{\mathcal{E}}^*(g)\\&=\sum_{i\in\mathbb{J}}\langle g,x_i\rangle\,x_i.\end{aligned}$$

If $\mathcal{E}$ is a standard frame, then $S_{\mathcal{E}}$ is invertible. The operator $S_{\mathcal{E}}$ is called the frame operator of $\mathcal{E}$.

A Bessel sequence $\mathcal{E}$ (in a Hilbert space or a Hilbert C^*-module) is a frame if and only if there is some Bessel sequence $\mathcal{P}$ such that $T_{\mathcal{P}}$ is a left-inverse of $T_{\mathcal{E}}^*$. In this case, the sequence $\mathcal{P}$ is said to be a dual for $\mathcal{E}$. It is worth noting that duals of a frame in a Hilbert space have been characterized in [6] and they are very important in applications. Approximate duals and pseudo-duals can also be helpful, particularly when it is difficult to find a dual. Approximate duality for frames (in Hilbert spaces) has been investigated and studied in [7] and its generalization to Hilbert C^*-modules has been presented in [15]. We also mention that some properties of pseudo-duals, especially facts about their excesses, for frames in Hilbert spaces (discrete ones) were obtained in [3].

In [11], the authors, using a bounded operator Q (inserted between the synthesis and analysis operators), defined a new kind of duals for a fusion frame (see also [5], for the definition of a multiplier). Then, Q-duals and

Q-approximate duals for frames, g-frames and fusion frames (in Hilbert spaces) were introduced in [16, 18], see also [21, 22]. Moreover, Q-pseudo-duals for continuous frames were introduced and studied in [20].

Recently in [19], Q-duals and Q-approximate duals were considered for frames in Hilbert C^*-modules. In this case, Q is assumed to be bounded and not necessarily adjointable.

Since there are many bounded operators on Hilbert C^*-modules that are not adjointable, in this note, in the definition and characterizations of pseudo-duals, just the boundedness of the operator Q is assumed and the pseudo-duals for standard frames and modular Riesz bases constructed by these operators are considered. This restriction makes the approach and the way to get the results different from the Hilbert space case, because the adjointability condition is frequently used for obtaining useful properties of pseudo-duals in Hilbert spaces.

2. Some Characterizations for Pseudo-Duals

Here, all duals, approximate duals and pseudo-duals associated with a given frame $\mathcal{E}$ are characterized. First, their definitions are presented.

Definition 2.1. Suppose that $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$ and $\mathcal{P} = \{y_i\}_{i\in\mathbb{J}}$ are two standard Bessel sequences for $\mathcal{X}$ and Q belongs to $\mathcal{B}\left(\ell^2\left(\mathbb{J},\mathcal{A}\right)\right)$. Then

(i) The sequence $\mathcal{P}$ is called a Q-pseudo-dual of $\mathcal{E}$ if the operator $S_{\mathcal{P},Q,\mathcal{E}}\colon \mathcal{X}\to\mathcal{X}$, $S_{\mathcal{P},Q,\mathcal{E}} = T_{\mathcal{P}}QT_{\mathcal{E}}^*$ is invertible.

(ii) The sequence $\mathcal{P}$ is said to be a Q-approximate dual of $\mathcal{E}$ if $\|S_{\mathcal{P},Q,\mathcal{E}} - I_{\mathcal{X}}\| < 1$.

(iii) We say that $\mathcal{P}$ is a Q-dual of $\mathcal{E}$ if $S_{\mathcal{P},Q,\mathcal{E}} = I_{\mathcal{X}}$.

In the case $Q = I_{\ell^2(\mathbb{J},\mathcal{A})}$, the sequence $\mathcal{P}$ in (i) (respectively (ii), (iii)) is called a pseudo-dual (respectively approximate dual, dual) of $\mathcal{E}$.

Obviously, every Q-approximate dual or a Q-dual of $\mathcal{E}$ is a Q-pseudo-dual for $\mathcal{E}$.

Theorem 2.2. *Let $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$ be a Bessel sequence in $\mathcal{X}$ and assume that Q belongs to $\mathcal{B}\left(\ell^2\left(\mathbb{J},\mathcal{A}\right)\right)$. Then*

(i) *If $\mathcal{P} = \{y_i\}_{i\in\mathbb{J}}$ is a Q-pseudo-dual of $\mathcal{E}$, then $\mathcal{P}$ and $\mathcal{E}$ are standard frames.*

(ii) *A frame $\mathcal{P}$ is a Q-pseudo-dual of $\mathcal{E}$ if and only if there exist some operator R in $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A}),\mathcal{X}\right)$ and an invertible operator T in $\mathcal{B}(\mathcal{X})$ such that*

$$T_{\mathcal{P}}Q = T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right).$$

(iii) *If $\mathcal{P} = \{y_i\}_{i\in\mathbb{J}}$ is a Q-pseudo-dual of $\mathcal{E}$, then there are some operator R in $\mathcal{B}(\ell^2(\mathbb{J},\mathcal{A}),\mathcal{X})$, an invertible operator T in $B(\mathcal{X})$*

and $\{L_i\}_{i\in\mathbb{J}} \subseteq \ell^2(\mathbb{J},\mathcal{A})$ *such that*

$$y_i = T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^* S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right) L_i.$$

(iv) *A frame* $\mathcal{P}$ *is a* Q*-approximate dual of* $\mathcal{E}$ *if and only if there exist some* R *in* $\mathcal{B}(\ell^2(\mathbb{J},\mathcal{A}),\mathcal{X})$ *and* T *in* $\mathcal{B}(\mathcal{X})$ *with* $\|T - I_{\mathcal{X}}\| < 1$ *such that*

$$T_{\mathcal{P}}Q = T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^* S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right).$$

(v) *If* $\mathcal{P} = \{y_i\}_{i\in\mathbb{J}}$ *is a* Q*-approximate dual of* $\mathcal{E}$*, then there exist an operator* R *in* $\mathcal{B}(\ell^2(\mathbb{J},\mathcal{A}),\mathcal{X})$*,* T *in* $B(\mathcal{X})$ *with* $\|T - I_{\mathcal{X}}\| < 1$ *and* $\{L_i\}_{i\in\mathbb{J}} \subseteq \ell^2(\mathbb{J},\mathcal{A})$ *such that*

$$y_i = T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^* S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right) L_i.$$

(vi) *A frame* $\mathcal{P}$ *is a* Q*-dual of* $\mathcal{E}$ *if and only if there exists some* R *in* $\mathcal{B}(\ell^2(\mathbb{J},\mathcal{A}),\mathcal{X})$ *such that*

$$T_{\mathcal{P}}Q = S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^* S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right).$$

(vii) *If* $\mathcal{P} = \{y_i\}_{i\in\mathbb{J}}$ *is a* Q*-dual of* $\mathcal{E}$*, then there exist some* R *in* $\mathcal{B}(\ell^2(\mathbb{J},\mathcal{A}),\mathcal{X})$ *and* $\{L_i\}_{i\in\mathbb{J}} \subseteq \ell^2(\mathbb{J},\mathcal{A})$ *with*

$$y_i = \left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^* S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right) L_i.$$

Proof. (i) Since $S_{\mathcal{P},Q,\mathcal{E}}$ is invertible, a similar calculation as Theorem 1 in [19] yields that

$$\frac{\|f\|^2}{\left\|S_{\mathcal{P},Q,\mathcal{E}}^{-1}\right\|^2 D_{\mathcal{P}}\|Q\|^2} \leq \left\|\sum_{i\in\mathbb{J}} |\langle f, x_i\rangle|^2\right\|,$$

and

$$\frac{\|f\|^2}{\left\|QT_{\mathcal{E}}^* S_{\mathcal{P},Q,\mathcal{E}}^{-1}\right\|^2} \leq \left\|\sum_{i\in\mathbb{J}} |\langle f, y_i\rangle|^2\right\|,$$

for each $f \in \mathcal{X}$. Now, [1, Theorem 2.6] implies that $\mathcal{E}$ and $\mathcal{P}$ are standard frames.

(ii) Let $\mathcal{P}$ be a Q-pseudo-dual of $\mathcal{E}$. Put $R := S_{\mathcal{P},Q,\mathcal{E}}^{-1}T_{\mathcal{P}}Q$ and $T := S_{\mathcal{P},Q,\mathcal{E}}$. Then T is invertible and we have

$$\begin{aligned} &S_{\mathcal{P},Q,\mathcal{E}}\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + S_{\mathcal{P},Q,\mathcal{E}}^{-1}T_{\mathcal{P}}Q\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^* S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right) \\ &\quad = S_{\mathcal{P},Q,\mathcal{E}}S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + T_{\mathcal{P}}Q - T_{\mathcal{P}}QT_{\mathcal{E}}^* S_{\mathcal{E}}^{-1}T_{\mathcal{E}} \\ &\quad = T_{\mathcal{P}}Q. \end{aligned}$$

Consequently, $T_{\mathcal{P}}Q = T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^* S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right)$.

Conversely, let $T_{\mathcal{P}}Q = T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right)$. Then we have

$$\begin{aligned} T_{\mathcal{P}}QT_{\mathcal{E}}^* &= T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right)T_{\mathcal{E}}^* \\ &= TS_{\mathcal{E}}^{-1}T_{\mathcal{E}}T_{\mathcal{E}}^* + TRT_{\mathcal{E}}^* - TRT_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}T_{\mathcal{E}}^* \\ &= T, \end{aligned}$$

so $S_{\mathcal{P},Q,\mathcal{E}}$ is invertible.

(iii) Let $\mathcal{P}$ be a Q-pseudo-dual of $\mathcal{E}$. Then by (ii),

$$T_{\mathcal{P}}Q = T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right).$$

Since $T_{\mathcal{P}}QT_{\mathcal{E}}^*$ is invertible, $T_{\mathcal{P}}Q$ is surjective. Thus, there is some $\{L_i\}_{i\in\mathbb{J}} \subseteq \ell^2(\mathbb{J},\mathcal{A})$ with

$$\begin{aligned} T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right)L_i &= T_{\mathcal{P}}QL_i \\ &= y_i. \end{aligned}$$

(iv) Let $\mathcal{P}$ be a Q-approximate dual of $\mathcal{E}$. Put $R := S_{\mathcal{P},Q,\mathcal{E}}^{-1}T_{\mathcal{P}}Q$ and $T := S_{\mathcal{P},Q,\mathcal{E}}$. Then $\|T - I_{\mathcal{X}}\| < 1$ and we have

$$\begin{aligned} &S_{\mathcal{P},Q,\mathcal{E}}\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + S_{\mathcal{P},Q,\mathcal{E}}^{-1}T_{\mathcal{P}}Q\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right) \\ &\quad = S_{\mathcal{P},Q,\mathcal{E}}S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + T_{\mathcal{P}}Q - T_{\mathcal{P}}QT_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}} \\ &\quad = T_{\mathcal{P}}Q. \end{aligned}$$

Consequently, $T_{\mathcal{P}}Q = T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right)$. Conversely, let $T_{\mathcal{P}}Q = T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right)$. Then we have

$$\begin{aligned} T_{\mathcal{P}}QT_{\mathcal{E}}^* &= T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right)T_{\mathcal{E}}^* \\ &= TS_{\mathcal{E}}^{-1}T_{\mathcal{E}}T_{\mathcal{E}}^* + TRT_{\mathcal{E}}^* - TRT_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}T_{\mathcal{E}}^* \\ &= T, \end{aligned}$$

so $\|S_{\mathcal{P},Q,\mathcal{E}} - I_{\mathcal{X}}\| < 1$.

(v) Let $\mathcal{P}$ be a Q-approximate dual of $\mathcal{E}$. Then by (iv),

$$T_{\mathcal{P}}Q = T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right).$$

Since $T_{\mathcal{P}}QT_{\mathcal{E}}^*$ is invertible, $T_{\mathcal{P}}Q$ is surjective. Thus, there is some $\{L_i\}_{i\in\mathbb{J}} \subseteq \ell^2(\mathbb{J},\mathcal{A})$ with

$$\begin{aligned} T\left(S_{\mathcal{E}}^{-1}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T_{\mathcal{E}}^*S_{\mathcal{E}}^{-1}T_{\mathcal{E}}\right)\right)L_i &= T_{\mathcal{P}}QL_i \\ &= y_i. \end{aligned}$$

For (vi) and (vii), since $T := S_{\mathcal{P},Q,\mathcal{E}} = I_{\mathcal{X}}$, the results follow from parts (iv) and (v), respectively. □

Corollary 2.3. *Let $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$ be a Bessel sequence in $\mathcal{X}$. Then, the following statements are equivalent:*

(i) *$\mathcal{E}$ admits a Q-dual, for some bounded operator Q on $\ell^2(\mathbb{J},\mathcal{A})$.*
(ii) *$\mathcal{E}$ admits a Q-approximate dual, for some bounded operator Q on $\ell^2(\mathbb{J},\mathcal{A})$.*
(iii) *$\mathcal{E}$ admits a Q-pseudo-dual, for some bounded operator Q on $\ell^2(\mathbb{J},\mathcal{A})$.*
(iv) *$\mathcal{E}$ is a standard frame.*
(v) *$\mathcal{E}$ is its own Q-dual, for some bounded operator Q on $\ell^2(\mathbb{J},\mathcal{A})$.*
(vi) *$\mathcal{E}$ is its own Q-approximate dual, for some bounded operator Q on $\ell^2(\mathbb{J},\mathcal{A})$.*
(vii) *$\mathcal{E}$ is its own Q-pseudo-dual, for some bounded operator Q on $\ell^2(\mathbb{J},\mathcal{A})$.*

Proof. The implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (v) $\Rightarrow$ (vi), (vi) $\Rightarrow$ (vii) and (v) $\Rightarrow$ (i) are trivial. The implications (iii) $\Rightarrow$ (iv) and (vii) $\Rightarrow$ (iv) are consequences of part (i) in Theorem 2.2. The implication (iv) $\Rightarrow$ (v) is obtained by considering $Q := T^*_{\tilde{\mathcal{E}}}T_{\tilde{\mathcal{E}}}$, where $\tilde{\mathcal{E}}$ is the standard Bessel sequence $\left\{S^{-1}_{\mathcal{E}}x_i\right\}_{i\in\mathbb{J}}$. □

If $e_i := \{\mathbf{1}_{\mathcal{A}}\delta_{i,j}\}_{j\in\mathbb{J}}$, for each $i \in \mathbb{J}$, then $e_i \in \ell^2(\mathbb{J},\mathcal{A})$ ($\{e_i\}_{i\in\mathbb{J}}$ is called the standard orthonormal basis for $\ell^2(\mathbb{J},\mathcal{A})$) and $T_{\mathcal{P}}e_i = y_i$, for each $i \in \mathbb{J}$. Therefore, by Theorem 2.2, the next characterizations for pseudo-duals, approximate duals and duals of a frame in a Hilbert C^*-module are obtained.

Corollary 2.4. *Let $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$ be a frame for $\mathcal{X}$. Then*

(i) *The pseudo-duals of $\mathcal{E}$ are precisely the sequences*

$$\left\{T\left(S^{-1}_{\mathcal{E}}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T^*_{\mathcal{E}}S^{-1}_{\mathcal{E}}T_{\mathcal{E}}\right)\right)e_i\right\}_{i\in\mathbb{J}},$$

where R belongs to $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A}),\mathcal{X}\right)$, T is an invertible operator in $\mathcal{B}(\mathcal{X})$ and $\{e_i\}_{i\in\mathbb{J}}$ is the standard orthonormal basis for $\ell^2(\mathbb{J},\mathcal{A})$.

(ii) *The approximate duals of $\mathcal{E}$ are precisely the sequences*

$$\left\{T\left(S^{-1}_{\mathcal{E}}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T^*_{\mathcal{E}}S^{-1}_{\mathcal{E}}T_{\mathcal{E}}\right)\right)e_i\right\}_{i\in\mathbb{J}},$$

where R belongs to $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A}),\mathcal{X}\right)$, T is an operator in $\mathcal{B}(\mathcal{X})$ with $\|T - I_{\mathcal{X}}\| < 1$ and $\{e_i\}_{i\in\mathbb{J}}$ is the standard orthonormal basis for $\ell^2(\mathbb{J},\mathcal{A})$.

(iii) *The duals of $\mathcal{E}$ are precisely the sequences*

$$\left\{\left(S^{-1}_{\mathcal{E}}T_{\mathcal{E}} + R\left(I_{\ell^2(\mathbb{J},\mathcal{A})} - T^*_{\mathcal{E}}S^{-1}_{\mathcal{E}}T_{\mathcal{E}}\right)\right)e_i\right\}_{i\in\mathbb{J}},$$

where R belongs to $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A}),\mathcal{X}\right)$ and $\{e_i\}_{i\in\mathbb{J}}$ is the standard orthonormal basis for $\ell^2(\mathbb{J},\mathcal{A})$.

Since the operator $Q := T^*_{\tilde{\mathcal{E}}} T_{\tilde{\mathcal{E}}}$, appeared in the proof of Corollary 2.3, is positive, we get the following result.

Corollary 2.5. *Let $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$ be a Bessel sequence in $\mathcal{X}$. Then the following statements are equivalent:*

(i) *$\mathcal{E}$ is a standard frame.*
(ii) *$\mathcal{E}$ is its own Q-dual, for some positive operator Q in $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A})\right)$.*
(iii) *$\mathcal{E}$ is its own Q-approximate dual, for some positive operator Q in $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A})\right)$.*
(iv) *$\mathcal{E}$ is its own Q-pseudo-dual, for some positive operator Q in $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A})\right)$.*
(v) *$\mathcal{E}$ is its own Q-dual, for some self-adjoint operator Q in $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A})\right)$.*
(vi) *$\mathcal{E}$ is its own Q-approximate dual, for some self-adjoint operator Q in $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A})\right)$.*
(vii) *$\mathcal{E}$ is its own Q-pseudo-dual, for some self-adjoint operator Q in $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A})\right)$.*

As we see, the conditions in Corollaries 2.3 and 2.5 are equivalent. These equivalences occur because the focus is on the existence of a bounded or self-adjoint or positive operator. Now, we get the following result for a fixed operator Q in $\mathfrak{L}\left(\ell^2(\mathbb{J},\mathcal{A})\right)$.

Corollary 2.6. *Let $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$ and $\mathcal{P} = \{y_i\}_{i\in\mathbb{J}}$ be Bessel sequences in $\mathcal{X}$ and let $Q \in \mathfrak{L}\left(\ell^2(\mathbb{J},\mathcal{A})\right)$. Then, the following statements are equivalent:*

(i) *$\mathcal{P}$ is a Q-pseudo-dual (respectively Q-approximate dual) of the Bessel sequence $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$.*
(ii) *There exists some invertible operator T in $\mathfrak{L}(\mathcal{X})$ (respectively some operator T in $\mathfrak{L}(\mathcal{X})$ with $\left\|T^{-1} - I_{\mathcal{X}}\right\| < 1$) such that $\{T(y_i)\}_{i\in\mathbb{J}}$ is a Q-dual of the Bessel sequence $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$.*
(iii) *The Bessel sequence $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$ is a Q^*-pseudo-dual (respectively Q^*-approximate dual) of $\mathcal{P}$.*

Proof. (i) $\Rightarrow$ (ii). Since $\mathcal{P}$ is a Q-pseudo-dual (respectively Q-approximate dual) of $\mathcal{E}$, the operator $S_{\mathcal{P},Q,\mathcal{E}}$ is invertible (respectively $\|S_{\mathcal{P},Q,\mathcal{E}} - I_{\mathcal{X}}\| < 1$). Now, for $T := S^{-1}_{\mathcal{P},Q,\mathcal{E}}$, the sequence $T\mathcal{P} := \{Ty_i\}_{i\in\mathbb{J}}$ is a standard Bessel sequence and $T_{T\mathcal{P}} = TT_{\mathcal{P}}$, so $S_{T\mathcal{P},Q,\mathcal{E}} = TT_{\mathcal{P}}QT^*_{\mathcal{E}} = I_{\mathcal{X}}$, meaning that $\{T(y_i)\}_{i\in\mathbb{J}}$ is a Q-dual of $\mathcal{E}$.

(ii) $\Rightarrow$ (i). The fact that $\{T(y_i)\}_{i\in\mathbb{J}}$ is a Q-dual of $\mathcal{E}$ is equivalent to say that $TT_{\mathcal{P}}QT^*_{\mathcal{E}} = I_{\mathcal{X}}$, consequently the operator $T_{\mathcal{P}}QT^*_{\mathcal{E}}$ is invertible with $T_{\mathcal{P}}QT^*_{\mathcal{E}} = T^{-1}$ and so the condition (i) is verified.

The equivalence between (i) and (iii) is obtained using the equality $(T_{\mathcal{P}}QT_{\mathcal{E}}^*)^* = T_{\mathcal{E}}Q^*T_{\mathcal{P}}^*$. □

Morphisms of Hilbert C^*-modules are important operators in operator theory with nice properties. It recently has been shown in [17] that most of the properties of a frame are preserved under the action of the morphisms. Now, we state their definition and refer to [4] for more details about these operators.

Definition 2.7. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathfrak{B}$, respectively. Let $\psi : \mathcal{A} \to \mathfrak{B}$ be a morphism of C^*-algebras. A map $\Psi : \mathcal{X} \to \mathcal{Y}$ is said to be a *ψ-morphism of Hilbert C^*-modules* if

$$\langle \Psi(f), \Psi(g) \rangle = \psi(\langle f, g \rangle),$$

for all $f, g \in \mathcal{X}$.

It is easy to verify that every ψ-morphism is a linear operator and $\Psi(af) = \psi(a)\Psi(f)$, for each $a \in \mathcal{A}$ and $f \in \mathcal{X}$. Also, since ψ is a morphism of C^*-algebras, we have $\|\psi\| \leq 1$, so the relation

$$\begin{aligned} \|\Psi(f)\| &= \|\langle \Psi(f), \Psi(f) \rangle\|^{\frac{1}{2}} \\ &= \|\psi(\langle f, f \rangle)\|^{\frac{1}{2}} \\ &\leq \|f\| \end{aligned}$$

yields that Ψ is bounded with $\|\Psi\| \leq 1$.

The following proposition and corollary characterize standard frames using the properties of surjective ψ-morphisms.

Proposition 2.8. *Let $\mathcal{E} = \{x_i\}_{i \in \mathbb{J}}$ be a Bessel sequence in $\mathcal{X}$. Then, the following statements are equivalent:*

(i) *$\mathcal{E}$ is a standard frame.*
(ii) *For each C^*-algebra $\mathfrak{B}$, every Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, any morphism $\psi : \mathcal{A} \to \mathfrak{B}$ and every surjective ψ-morphism $\Psi : \mathcal{X} \to \mathcal{Y}$, the sequence $\{\Psi(x_i)\}_{i \in \mathbb{J}}$ has an L-dual, for some $L \in \mathcal{B}\left(\ell^2(\mathbb{J}, \mathfrak{B})\right)$.*
(iii) *For each C^*-algebra $\mathfrak{B}$, every Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, any morphism $\psi : \mathcal{A} \to \mathfrak{B}$ and every surjective ψ-morphism $\Psi : \mathcal{X} \to \mathcal{Y}$, the sequence $\{\Psi(x_i)\}_{i \in \mathbb{J}}$ has an L-approximate dual, for some $L \in \mathcal{B}\left(\ell^2(\mathbb{J}, \mathfrak{B})\right)$.*
(iv) *For each C^*-algebra $\mathfrak{B}$, every Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, any morphism $\psi : \mathcal{A} \to \mathfrak{B}$ and every surjective ψ-morphism $\Psi : \mathcal{X} \to \mathcal{Y}$, the sequence $\{\Psi(x_i)\}_{i \in \mathbb{J}}$ has an L-pseudo-dual, for some $L \in \mathcal{B}\left(\ell^2(\mathbb{J}, \mathfrak{B})\right)$.*
(v) *For each C^*-algebra $\mathfrak{B}$, every Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, any isomorphism $\psi : \mathcal{A} \to \mathfrak{B}$ and every surjective ψ-morphism Ψ :*

$\mathcal{X} \to \mathcal{Y}$, *the sequence* $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ *has an* L*-pseudo-dual, for some* $L \in \mathcal{B}\left(\ell^2(\mathbb{J},\mathfrak{B})\right)$.

(vi) *There exist some* C^**-algebra* $\mathfrak{B}$*, a Hilbert* $\mathfrak{B}$*-module* $\mathcal{Y}$*, an isomorphism* $\psi : \mathcal{A} \to \mathfrak{B}$ *and some surjective* ψ*-morphism* $\Psi : \mathcal{X} \to \mathcal{Y}$ *such that the sequence* $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ *has an* L*-dual, for some* $L \in \mathcal{B}\left(\ell^2(\mathbb{J},\mathfrak{B})\right)$.

(vii) *There exist some* C^**-algebra* $\mathfrak{B}$*, a Hilbert* $\mathfrak{B}$*-module* $\mathcal{Y}$*, an isomorphism* $\psi : \mathcal{A} \to \mathfrak{B}$ *and some surjective* ψ*-morphism* $\Psi : \mathcal{X} \to \mathcal{Y}$ *such that the sequence* $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ *has an* L*-approximate dual, for some* $L \in \mathcal{B}\left(\ell^2(\mathbb{J},\mathfrak{B})\right)$.

(viii) *There exist some* C^**-algebra* $\mathfrak{B}$*, a Hilbert* $\mathfrak{B}$*-module* $\mathcal{Y}$*, an isomorphism* $\psi : \mathcal{A} \to \mathfrak{B}$ *and some surjective* ψ*-morphism* $\Psi : \mathcal{X} \to \mathcal{Y}$ *such that the sequence* $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ *has an* L*-pseudo-dual, for some* $L \in \mathcal{B}\left(\ell^2(\mathbb{J},\mathfrak{B})\right)$.

Proof. (i) $\Rightarrow$ (ii). It follows from Proposition 2.6 in [17] that $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ is a standard frame, so it admits one dual (its canonical dual). Hence, the result can be obtained using $L := I_{\ell^2(\mathbb{J},\mathfrak{B})}$.

The implications (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) are obvious.

(v) $\Rightarrow$ (i). Using (v) for $\mathfrak{B} := \mathcal{A}$, $\mathcal{Y} := \mathcal{X}$, $\psi := I_{\mathcal{A}}$ and $\Psi := I_{\mathcal{X}}$, we obtain that $\{\Psi(x_i)\}_{i\in\mathbb{J}} = \{x_i\}_{i\in\mathbb{J}}$ has an L-pseudo-dual, for some $L \in \mathcal{B}\left(\ell^2(\mathbb{J},\mathfrak{B})\right)$. Therefore, Theorem 2.2 yields that $\mathcal{E}$ is a standard frame.

(i) $\Rightarrow$ (vi). Since $\mathcal{E}$ is a standard frame, it possesses a dual. Now, it is enough to consider $\mathfrak{B} := \mathcal{A}$, $\mathcal{Y} := \mathcal{X}$, $\psi := I_{\mathcal{A}}$, $\Psi := I_{\mathcal{X}}$ and $L := I_{\ell^2(\mathbb{J},\mathcal{A})}$.

The implications (vi) $\Rightarrow$ (vii) and (vii) $\Rightarrow$ (viii) are obvious.

(viii) $\Rightarrow$ (i). Since $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ has an L-pseudo-dual, for some $L \in \mathcal{B}\left(\ell^2(\mathbb{J},\mathfrak{B})\right)$, by Theorem 2.2, $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ is a standard frame. Also, it follows from the obtained results in [4] for ψ-morphisms that if $\Psi : \mathcal{X} \to \mathcal{Y}$ is a ψ-morphism, where $\psi : \mathcal{A} \to \mathfrak{B}$ is an isomorphism, then Ψ is an isometry and since Ψ is surjective, it is invertible and Ψ^{-1} is a ψ^{-1}-morphism. Hence, by Proposition 2.6 in [17], $\{x_i\}_{i\in\mathbb{J}} = \{\Psi^{-1}\Psi(x_i)\}_{i\in\mathbb{J}}$ is a standard frame. □

Corollary 2.9. *Let* $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$ *be a Bessel sequence in* $\mathcal{X}$*. Then the following statements are equivalent:*

(i) $\mathcal{E}$ *is a standard frame.*

(ii) *For each* C^**-algebra* $\mathfrak{B}$*, every Hilbert* $\mathfrak{B}$*-module* $\mathcal{Y}$*, any isomorphism* $\psi : \mathcal{A} \to \mathfrak{B}$ *and every surjective* ψ*-morphism* $\Psi : \mathcal{X} \to \mathcal{Y}$*, the sequence* $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ *is a standard frame for* $\mathcal{Y}$*.*

(iii) *There are some* C^**-algebra* $\mathfrak{B}$*, a Hilbert* $\mathfrak{B}$*-module* $\mathcal{Y}$*, an isomorphism* $\psi : \mathcal{A} \to \mathfrak{B}$ *and some surjective* ψ*-morphism* $\Psi : \mathcal{X} \to \mathcal{Y}$ *such that the sequence* $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ *is a standard frame for* $\mathcal{Y}$*.*

(iv) *For each C^*-algebra $\mathfrak{B}$, every Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, any morphism $\psi : \mathcal{A} \to \mathfrak{B}$ and every surjective ψ-morphism $\Psi : \mathcal{X} \to \mathcal{Y}$, the sequence $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ is a standard frame for $\mathcal{Y}$.*

Proof. Since a necessary and sufficient condition for the Bessel sequence $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ to be a standard frame is the existence of an L-pseudo-dual, the equivalences between (i),(ii) and (iii) can be obtained using Proposition 2.8.

The implication (iv) $\Rightarrow$ (ii) is obvious and the implication (i) $\Rightarrow$ (iv) follows from [17, Proposition 2.6]. □

Bessel multipliers are operators defined on a separable Hilbert space by inserting a multiplication operator between the synthesis and analysis operators of two Bessel sequences. The multiplication operator is defined using a bounded sequence of complex numbers, see [5]. A generalization of these operators on Hilbert C^*-modules is stated in [13]. In this case, the multiplication operator inserted between the synthesis and analysis operators is defined using a bounded sequence in the center of the C^*-algebra. As we see in [13], Bessel multipliers on Hilbert C^*-modules are adjointable.

In this section, the boundedness of the sequence in the multiplication operator is just assumed and its members do not necessarily belong to the center of the C^*-algebra. Using this assumption, new pseudo-duals, approximate duals and duals are constructed. Now, we define the following two multiplication operators on $\ell^2(\mathbb{J},\mathcal{A})$ using a sequence $m = \{m_i\}_{i\in\mathbb{J}}$ in $\mathcal{A}$.

$$\mathcal{R}_m \colon \ell^2(\mathbb{J},\mathcal{A}) \to \ell^2(\mathbb{J},\mathcal{A}), \qquad \mathcal{R}_m(\{a_i\}_{i\in\mathbb{J}}) = \{a_i m_i\}_{i\in\mathbb{J}},$$

and

$$\mathcal{L}_m \colon \ell^2(\mathbb{J},\mathcal{A}) \to \ell^2(\mathbb{J},\mathcal{A}), \qquad \mathcal{L}_m(\{a_i\}_{i\in\mathbb{J}}) = \{m_i a_i\}_{i\in\mathbb{J}}.$$

Proposition 2.10. *Let $m = \{m_i\}_{i\in\mathbb{J}}$ be a sequence in $\mathcal{A}$. Then the following statements are equivalent:*

(i) *$\mathcal{R}_m$ is bounded.*
(ii) *$m \in \ell^\infty(\mathbb{J},\mathcal{A})$, where*

$$\ell^\infty(\mathbb{J},\mathcal{A}) := \left\{\{c_i\}_{i\in\mathbb{J}} \subseteq \mathcal{A} : \|\{c_i\}_{i\in\mathbb{J}}\|_\infty = \sup\{\|c_i\| : i \in \mathbb{J}\} < \infty\right\}.$$

(iii) *$\mathcal{R}_{m^*}$ is bounded, where $m^* := \{m_i^*\}_{i\in\mathbb{J}}$.*
(iv) *$\mathcal{R}_m$ is adjointable.*
(v) *$\mathcal{R}_{m^*}$ is adjointable.*

In case the equivalent conditions are satisfied, then we have $\mathcal{R}_m^* = \mathcal{R}_{m^*}$ and $\|\mathcal{R}_m\| = \|m\|_\infty$.

Proof. (i) $\Rightarrow$ (ii). Let $\{e_i\}_{i\in\mathbb{J}}$ be the standard orthonormal basis for $\ell^2(\mathbb{J},\mathcal{A})$. Then, $\|m_i\| = \|\mathcal{R}_m(e_i)\| \leq \|\mathcal{R}_m\|$, for each $i \in \mathbb{J}$, so $m \in \ell^\infty(\mathbb{J},\mathcal{A})$ with $\|m\|_\infty \leq \|\mathcal{R}_m\|$.

(ii) $\Rightarrow$ (iii). For every $\{a_i\}_{i\in\mathbb{J}} \in \ell^2(\mathbb{J},\mathcal{A})$ and finite set $\Omega \subseteq \mathbb{J}$, we have

$$\sum_{i\in\Omega} a_i m_i^* m_i a_i^* \leq \|m\|_\infty^2 \sum_{i\in\Omega} a_i a_i^*.$$

Therefore $\sum_{i\in\mathbb{J}} a_i m_i^* m_i a_i^*$ converges in norm, so $\mathcal{R}_{m^*}$ is well defined and bounded with $\|\mathcal{R}_{m^*}\| \leq \|m\|_\infty$.

(iii) $\Rightarrow$ (iv). Since $\mathcal{R}_{m^*}$ is bounded, using the implication (i) $\Rightarrow$ (ii), we have $m^* \in \ell^\infty(\mathbb{J},\mathcal{A})$ and the implication (ii) $\Rightarrow$ (iii) implies that $\mathcal{R}_m$ is bounded. Now, for each $\{a_i\}_{i\in\mathbb{J}}, \{b_i\}_{i\in\mathbb{J}} \in \ell^2(\mathbb{J},\mathcal{A})$, we get

$$\begin{aligned}\langle \{a_i\}_{i\in\mathbb{J}}, \mathcal{R}_{m^*}(\{b_i\}_{i\in\mathbb{J}})\rangle &= \langle \{a_i\}_{i\in\mathbb{J}}, \{b_i m_i^*\}_{i\in\mathbb{J}}\rangle \\ &= \sum_{i\in\mathbb{J}} a_i m_i b_i^* \\ &= \langle \mathcal{R}_m(\{a_i\}_{i\in\mathbb{J}}), \{b_i\}_{i\in\mathbb{J}}\rangle .\end{aligned}$$

The above relation shows that $\mathcal{R}_m$ is adjointable with $\mathcal{R}_m^* = \mathcal{R}_{m^*}$. Now, the implications (iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i) are easy to verify and also the relations $\mathcal{R}_m^* = \mathcal{R}_{m^*}$ and $\|\mathcal{R}_m\| = \|m\|_\infty$ are obvious. □

The above result does not hold for the left multiplication operator. For example, if a is an element in a C^*-algebra $\mathcal{A}$ which does not belong to the center of $\mathcal{A}$, then the operator $\mathcal{L}_a \colon \mathcal{A} \to \mathcal{A}$ defined by $\mathcal{L}_a(b) = ab$ is bounded but not adjointable.

Corollary 2.11. *Let $m = \{m_i\}_{i\in\mathbb{J}}$ be a sequence in $\mathcal{A}$. Then the following statements are equivalent:*

(i) *$\mathcal{L}_m$ is adjointable with $\mathcal{L}_m^* = \mathcal{L}_{m^*}$.*
(ii) *$m \in \ell^\infty(\mathbb{J},\mathcal{A})$ and m_i belongs to the center of $\mathcal{A}$, for each $i \in \mathbb{J}$.*

Proof. (i) $\Rightarrow$ (ii). Let $a \in \mathcal{A}$. If $\{e_i\}_{i\in\mathbb{J}}$ is the standard orthonormal basis of $\ell^2(\mathbb{J},\mathcal{A})$, then for each $i_0 \in \mathbb{J}$, we have

$$\langle \mathcal{L}_m(\{a\delta_{i_0,i}\}_{i\in\mathbb{J}}), e_{i_0}\rangle = \langle \{a\delta_{i_0,i}\}_{i\in\mathbb{J}}, \mathcal{L}_{m^*} e_{i_0}\rangle ,$$

consequently, $m_{i_0} a = a m_{i_0}$, so m_{i_0} belongs to the center of $\mathcal{A}$. Thus $\mathcal{L}_m = \mathcal{R}_m$ and by Proposition 2.10, we have $m \in \ell^\infty(\mathbb{J},\mathcal{A})$.

The implication (ii) $\Rightarrow$ (i) follows from Proposition 2.10 since the fact that m_i belongs to the center of $\mathcal{A}$, for each $i \in \mathbb{J}$, yields $\mathcal{L}_m = \mathcal{R}_m$. □

Example 2.12. Suppose that $\mathcal{A}$ is a non-commutative, unital C^*-algebra and it is regarded as a Hilbert $\mathcal{A}$-module. Assume that a is an invertible element in $\mathcal{A}$ which does not belong to the center of $\mathcal{A}$.

(i) The operator $Q : \mathcal{A} \to \mathcal{A}$ defined by $Q(x) = axa^{-1}$ is bounded and non-adjointable. Also, $\mathcal{P} := \{a\}$ is a Q-pseudo-dual of $\mathcal{E} := \{\mathbf{1}_{\mathcal{A}}\}$. Moreover, if $\|a - \mathbf{1}_{\mathcal{A}}\| < 1$, then $\mathcal{P}$ is a Q-approximate dual of $\mathcal{E}$.

(ii) Let $m := \{a\}$. Then, by Corollary 2.11, the operator $\mathcal{L}_m$ is bounded but it is not adjointable. Also, $\mathcal{E} := \{\mathbf{1}_{\mathcal{A}}\}$ is an $\mathcal{L}_m$-pseudo-dual of itself. Moreover, if $\|a - \mathbf{1}_{\mathcal{A}}\| < 1$, then $\mathcal{E}$ is an $\mathcal{L}_m$-approximate dual of itself.

We mention that the results obtained in this section generalize Theorems 2.2, 2.5, 2.7, Proposition 2.3, Corollary 2.6 in [18] and the obtained results for characterizations of Q-pseudo-duals, Q-duals and Q-approximate duals of continuous frames in [20] to Hilbert C^*-modules (without the adjointability condition for the operator Q).

3. Pseudo-Duality of Modular Riesz Bases

In Hilbert spaces, a frame which admits a unique dual is called a Riesz basis. There are several equivalent conditions for a frame to be a Riesz basis, for example, it is equivalent to say that the frame is ω-independent or it is exact.

Riesz bases in Hilbert C^*-modules were introduced in [8]. There are many similarities between Riesz bases in Hilbert spaces and Hilbert C^*-modules but Riesz bases in Hilbert C^*-modules can behave completely different from those in Hilbert spaces. For example, it was shown in [9] that a Riesz basis in a Hilbert C^*-module can possess more than one dual and a dual of a Riesz basis is not necessarily a Riesz basis. Moreover, some characterizations for duals of a Riesz basis in a Hilbert C^*-module were presented in [9].

In [2, 9, 10, 12], Riesz bases with only one dual (the canonical dual) are focused. This kind of Riesz bases is called a modular Riesz basis and it was proved in [12] that a necessary and sufficient condition for a frame $\{x_i\}_{i\in\mathbb{J}}$ in a Hilbert $\mathcal{A}$-module to be a modular Riesz basis is that it is equivalent to the standard basis of $\ell^2(\mathbb{J}, \mathcal{A})$ via an adjointable, invertible operator, i.e., a sequence $\{x_i\}_{i\in\mathbb{J}}$ is a *modular Riesz basis* for $\mathcal{X}$ if there exists an invertible operator $U \in \mathfrak{L}\left(\ell^2(\mathbb{J}, \mathcal{A}), \mathcal{X}\right)$ such that $U(e_i) = x_i$, for each $i \in \mathbb{J}$, where $\{e_i\}_{i\in\mathbb{J}}$ is the standard orthonormal basis of $\ell^2(\mathbb{J}, \mathcal{A})$, i.e., $e_i = \{\delta_{i,j}\mathbf{1}_{\mathcal{A}}\}_{j\in\mathbb{J}}$. Hence, we can say that a frame is a modular Riesz basis if and only if its synthesis operator is invertible.

In this section, pseudo-duals of modular Riesz bases are considered. We begin with the following lemma which gives some equivalent conditions for the invertibility of multiplication operators in terms of semi-normalized and invertible symbols. We recall that a sequence $\{m_i\}_{i\in\mathbb{J}}$ in $\ell^\infty(\mathbb{J}, \mathcal{A})$ such that m_i belongs to the center of $\mathcal{A}$, for each $i \in \mathbb{J}$ is called

a *symbol* and a symbol $\{m_i\}_{i\in\mathbb{J}}$ is called *semi-normalized* if there exists some positive number A with $A.\mathbf{1}_{\mathcal{A}} \leq |m_i|$, for each $i \in \mathbb{J}$. Clearly, if m is a symbol, then $\mathcal{L}_m$ and $\mathcal{R}_m$ are identical and we denote them by $\mathcal{M}_m$. Also, in this case, we say m-pseudo-dual instead of $\mathcal{L}_m$-pseudo-dual or $\mathcal{R}_m$-pseudo-dual.

Lemma 3.1. *Let $m = \{m_i\}_{i\in\mathbb{J}}$ be a symbol in $\mathcal{A}$. Then, the following statements are equivalent:*

(i) *The symbol m is semi-normalized.*
(ii) *The element m_i is invertible, for all $i \in \mathbb{J}$ and $m^{-1} := \{m_i^{-1}\}_{i\in\mathbb{J}}$ is a symbol.*
(iii) *The operator $\mathcal{M}_m$ is invertible on $\ell^2(\mathbb{J}, \mathcal{A})$.*

Proof. The implication (i) $\Rightarrow$ (ii) can be obtained similar to the first part of the proof stated for [13, Theorem 4.3(iii)] using the inequalities $A.\mathbf{1}_{\mathcal{A}} \leq |m_i| \leq \|m\|_\infty .\mathbf{1}_{\mathcal{A}}$, $\|m\|_\infty^{-1} \leq \left\||m_i|^{-1}\right\| \leq \frac{1}{A}$ and the fact that m_i is invertible with $|m_i^{-1}| = |m_i|^{-1}$, for each $i \in \mathbb{J}$ (because m_i's belong to the center of the C^*-algebra).

(ii) $\Rightarrow$ (iii). Since m^{-1} is a symbol, the operator $\mathcal{M}_{m^{-1}}$ is well defined and bounded on $\ell^2(\mathbb{J}, \mathcal{A})$ and clearly $\mathcal{M}_{m^{-1}}\mathcal{M}_m = I_{\ell^2(\mathbb{J},\mathcal{A})} = \mathcal{M}_m\mathcal{M}_{m^{-1}}$.

(iii) $\Rightarrow$ (i). Assume that S is a bounded operator on $\ell^2(\mathbb{J}, \mathcal{A})$ with $\mathcal{M}_m S = I_{\ell^2(\mathbb{J},\mathcal{A})}$. Hence, for $e_i := \{\delta_{i,j}\mathbf{1}_{\mathcal{A}}\}_{j\in\mathbb{J}}$, we get $\mathcal{M}_m(Se_i) = e_i$. Consequently, if $P_i : \ell^2(\mathbb{J}, \mathcal{A}) \to \mathcal{A}$ is the coordinate operator, then m_i is invertible with $m_i^{-1} = P_i(Se_i)$, so $\left\|m_i^{-1}\right\| \leq \|S\|$, for each $i \in \mathbb{J}$. Thus,

$$\left(m_i^{-1}\right)^* m_i^{-1} \leq \left\|m_i^{-1}\right\|^2 .\mathbf{1}_{\mathcal{A}} \leq \|S\|^2 .\mathbf{1}_{\mathcal{A}},$$

which yields that

$$\frac{1}{\|S\|^2}.\mathbf{1}_{\mathcal{A}} \leq m_i^* m_i = |m_i|^2.$$

Therefore $\frac{1}{\|S\|}.\mathbf{1}_{\mathcal{A}} \leq |m_i|$, for each $i \in \mathbb{J}$ and the result follows. □

Note that the fact that m^{-1} is a symbol (in the condition (ii) of Lemma 3.1) cannot be omitted. For example, if $m_n := \frac{1}{n}$, for each $n \in \mathbb{N}$, then $m := \{m_n\}_{n=1}^\infty$ is a symbol in $\mathbb{C}$ but $\mathcal{M}_m$ is not invertible on $\ell^2(\mathbb{N}, \mathbb{C})$.

Proposition 3.2. *Let $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$ be a modular Riesz basis. Then*

(i) *A Q-pseudo-dual of $\mathcal{E}$ is a modular Riesz basis if and only if Q is invertible.*
(ii) *An m-pseudo-dual of $\mathcal{E}$ is a modular Riesz basis if and only if m is a semi-normalized symbol.*
(iii) *Every pseudo-dual of $\mathcal{E}$ is a modular Riesz basis.*

Proof. (i) Let $\mathcal{P}$ be a Q-pseudo-dual of $\mathcal{E}$. Thus, $T_{\mathcal{P}}QT_{\mathcal{E}}^*$ is invertible. Since $\mathcal{E}$ is a modular Riesz basis, the operator $T_{\mathcal{E}}^*$ is invertible. Hence, if Q is invertible, then $T_{\mathcal{P}}$ is also invertible, so $\mathcal{P}$ is a modular Riesz basis.
Conversely, let $\mathcal{P}$ be a modular Riesz basis. Therefore, $T_{\mathcal{P}}$, $T_{\mathcal{E}}^*$ and $T_{\mathcal{P}}QT_{\mathcal{E}}^*$ are invertible which yields that Q is invertible.

(ii) By Lemma 3.1, $\mathcal{M}_m$ is invertible if and only if m is semi-normalized. Now, the result follows from part (i).

(iii) It is enough to consider $m = \{m_i\}_{i\in\mathbb{J}}$, where $m_i = \mathbf{1}_{\mathcal{A}}$, for each $i \in \mathbb{J}$ and use part (ii).

□

Theorem 3.3. *Let $\mathcal{X}$ be a Hilbert C^*-module over $\mathcal{A}$ and let $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$ be a standard frame in $\mathcal{X}$. Then, the following statements are equivalent:*

(i) *The sequence $\mathcal{E}$ is a modular Riesz basis.*

(ii) *For each Hilbert $\mathcal{A}$-module $\mathcal{Y}$ and every invertible operator T in $\mathfrak{L}(\mathcal{X},\mathcal{Y})$, the sequence $T\mathcal{E} := \{Tx_i\}_{i\in\mathbb{J}}$ is a modular Riesz basis.*

(iii) *There exist some Hilbert $\mathcal{A}$-module $\mathcal{Y}$ and some invertible operator T in $\mathfrak{L}(\mathcal{X},\mathcal{Y})$ such that $T\mathcal{E} := \{Tx_i\}_{i\in\mathbb{J}}$ is a modular Riesz basis.*

(iv) *For each invertible operator Q in $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A})\right)$, any Q-pseudo-dual (respectively Q-dual, Q-approximate dual) of $\mathcal{E}$ is a modular Riesz basis.*

(v) *For each semi-normalized symbol m, any m-pseudo-dual (respectively m-dual, m-approximate dual) of $\mathcal{E}$ is a modular Riesz basis.*

(vi) *Every pseudo-dual (respectively dual, approximate dual) of $\mathcal{E}$ is a modular Riesz basis.*

(vii) *There exists some pseudo-dual (respectively dual, approximate dual) of $\mathcal{E}$ which is a modular Riesz basis.*

(viii) *There exist some semi-normalized symbol m and some m-pseudo-dual (respectively m-dual, m-approximate dual) of $\mathcal{E}$ which is a modular Riesz basis.*

(ix) *There exist some invertible operator Q in $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A})\right)$ and some Q-pseudo-dual (respectively Q-dual, Q-approximate dual) of $\mathcal{E}$ which is a modular Riesz basis.*

Proof. (i) ⇒ (ii). Let T be an invertible operator in $\mathfrak{L}(\mathcal{X},\mathcal{Y})$. Then, it is easy to see that $T\mathcal{E}$ is a standard Bessel sequence with the synthesis operator $T_{T\mathcal{E}} = TT_{\mathcal{E}}$. Because T and $T_{\mathcal{E}}$ are invertible, we conclude that $T_{T\mathcal{E}}$ is invertible which is equivalent to say that $T\mathcal{E}$ is a modular Riesz basis.

The implications (ii) ⇒ (iii) and (iv) ⇒ (v) are obvious.

(iii) $\Rightarrow$ (i). Assume that T is an invertible operator in $\mathfrak{L}(\mathcal{X},\mathcal{Y})$ such that $T\mathcal{E} := \{Tx_i\}_{i\in\mathbb{J}}$ is a modular Riesz basis. Hence, $T_{T\mathcal{E}}$ is invertible and the equality $T^{-1}T_{T\mathcal{E}} = T_{\mathcal{E}}$ yields that $T_{\mathcal{E}}$ is invertible, consequently $\mathcal{E}$ is a modular Riesz basis.

The implication (i) $\Rightarrow$ (iv) follows from Proposition 3.2.

The implications (v) $\Rightarrow$ (vi) and (vii) $\Rightarrow$ (viii) can be obtained by considering $m_i = \mathbf{1}_{\mathcal{A}}$, for each $i \in \mathbb{J}$. For (vi) $\Rightarrow$ (vii) and (viii) $\Rightarrow$ (ix), it is enough to consider $\widetilde{\mathcal{E}} := \{S_{\mathcal{E}}^{-1}x_i\}_{i\in\mathbb{J}}$ as a pseudo-dual (respectively dual, approximate dual) of $\mathcal{E}$ and $Q := \mathcal{M}_m$, respectively.

(ix) $\Rightarrow$ (i). Let Q be an invertible operator in $\mathcal{B}\left(\ell^2(\mathbb{J},\mathcal{A})\right)$ and assume that $\mathcal{P} = \{y_i\}_{i\in\mathbb{J}}$ is a Q-pseudo-dual of $\mathcal{E}$ which is a modular Riesz basis (the sequence $\mathcal{P}$ can be a Q-dual or a Q-approximate dual of $\mathcal{E}$). Hence, $T_{\mathcal{P}}$, Q and $T_{\mathcal{P}}QT_{\mathcal{E}}^*$ are invertible, consequently $T_{\mathcal{E}}^*$ is invertible which is equivalent to say that $\mathcal{E}$ is a modular Riesz basis. □

Note that Theorem 3.3 generalizes Propositions 3.7 and 3.9 in [20] to Hilbert C^*-modules.

Corollary 3.4. *Let $\mathcal{E} = \{x_i\}_{i\in\mathbb{J}}$ be a frame in $\mathcal{X}$. Then, the following statements are equivalent:*

(i) *$\mathcal{E}$ is a modular Riesz basis.*

(ii) *For each C^*-algebra $\mathfrak{B}$, every Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, any isomorphism $\psi : \mathcal{A} \to \mathfrak{B}$ and every surjective ψ-morphism $\Psi : \mathcal{X} \to \mathcal{Y}$, the sequence $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ is a modular Riesz basis.*

(iii) *There exist some C^*-algebra $\mathfrak{B}$, a Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, an isomorphism $\psi : \mathcal{A} \to \mathfrak{B}$ and some surjective ψ-morphism $\Psi : \mathcal{X} \to \mathcal{Y}$ such that the sequence $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ is a modular Riesz basis.*

(iv) *For each C^*-algebra $\mathfrak{B}$, every Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, any isomorphism $\psi : \mathcal{A} \to \mathfrak{B}$, every surjective ψ-morphism $\Psi : \mathcal{X} \to \mathcal{Y}$ and for every invertible operator Q in $\mathcal{B}\left(\ell^2(\mathbb{J},\mathfrak{B})\right)$, any Q-pseudo-dual (respectively Q-dual, Q-approximate dual) of $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ is a modular Riesz basis.*

(v) *For each C^*-algebra $\mathfrak{B}$, every Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, any isomorphism $\psi : \mathcal{A} \to \mathfrak{B}$, every surjective ψ-morphism $\Psi : \mathcal{X} \to \mathcal{Y}$ and for every semi-normalized symbol $m = \{m_i\}_{i\in\mathbb{J}}$ in $\mathcal{A}$, any $\psi(m) := \{\psi(m_i)\}_{i\in\mathbb{J}}$-pseudo-dual (respectively $\psi(m)$-dual, $\psi(m)$-approximate dual) of $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ is a modular Riesz basis.*

(vi) *For each C^*-algebra $\mathfrak{B}$, every Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, any isomorphism $\psi : \mathcal{A} \to \mathfrak{B}$ and every surjective ψ-morphism $\Psi : \mathcal{X} \to \mathcal{Y}$, any pseudo-dual (respectively dual, approximate dual) of $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ is a modular Riesz basis.*

(vii) *For some C^*-algebra $\mathfrak{B}$, a Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, an isomorphism $\psi : \mathcal{A} \to \mathfrak{B}$ and a surjective ψ-morphism $\Psi : \mathcal{X} \to \mathcal{Y}$, there exists some pseudo-dual (respectively dual, approximate dual) for $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ which is a modular Riesz basis.*

(viii) *For some C^*-algebra $\mathfrak{B}$, a Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, an isomorphism $\psi : \mathcal{A} \to \mathfrak{B}$, a surjective ψ-morphism $\Psi : \mathcal{X} \to \mathcal{Y}$ and a semi-normalized symbol $m = \{m_i\}_{i\in\mathbb{J}}$ in $\mathcal{A}$, there exists some $\psi(m) := \{\psi(m_i)\}_{i\in\mathbb{J}}$-pseudo-dual (respectively $\psi(m)$-dual, $\psi(m)$-approximate dual) of $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ which is a modular Riesz basis.*

(ix) *For some C^*-algebra $\mathfrak{B}$, a Hilbert $\mathfrak{B}$-module $\mathcal{Y}$, an isomorphism $\psi : \mathcal{A} \to \mathfrak{B}$, a surjective ψ-morphism $\Psi : \mathcal{X} \to \mathcal{Y}$ and an invertible operator Q in $\mathcal{B}\left(\ell^2(\mathbb{J}, \mathfrak{B})\right)$, there exists some Q-pseudo-dual (respectively Q-dual, Q-approximate dual) of $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ which is a modular Riesz basis.*

Proof. As we see in the proof of Proposition 2.8, $\Psi : \mathcal{X} \to \mathcal{Y}$ is invertible and Ψ^{-1} is a ψ^{-1}-morphism. Also, by Theorem 3.1 in [17], if $\{x_i\}_{i\in\mathbb{J}}$ is a modular Riesz basis, then $\{\Psi(x_i)\}_{i\in\mathbb{J}}$ is a modular Riesz basis. Now, the result follows from Theorem 3.3 and using the fact that for an isomorphism $\psi : \mathcal{A} \to \mathfrak{B}$, the sequence $m = \{m_i\}_{i\in\mathbb{J}}$ in $\mathcal{A}$ is a semi-normalized symbol if and only if $\psi(m) := \{\psi(m_i)\}_{i\in\mathbb{J}}$ is a semi-normalized symbol in $\mathfrak{B}$. □

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