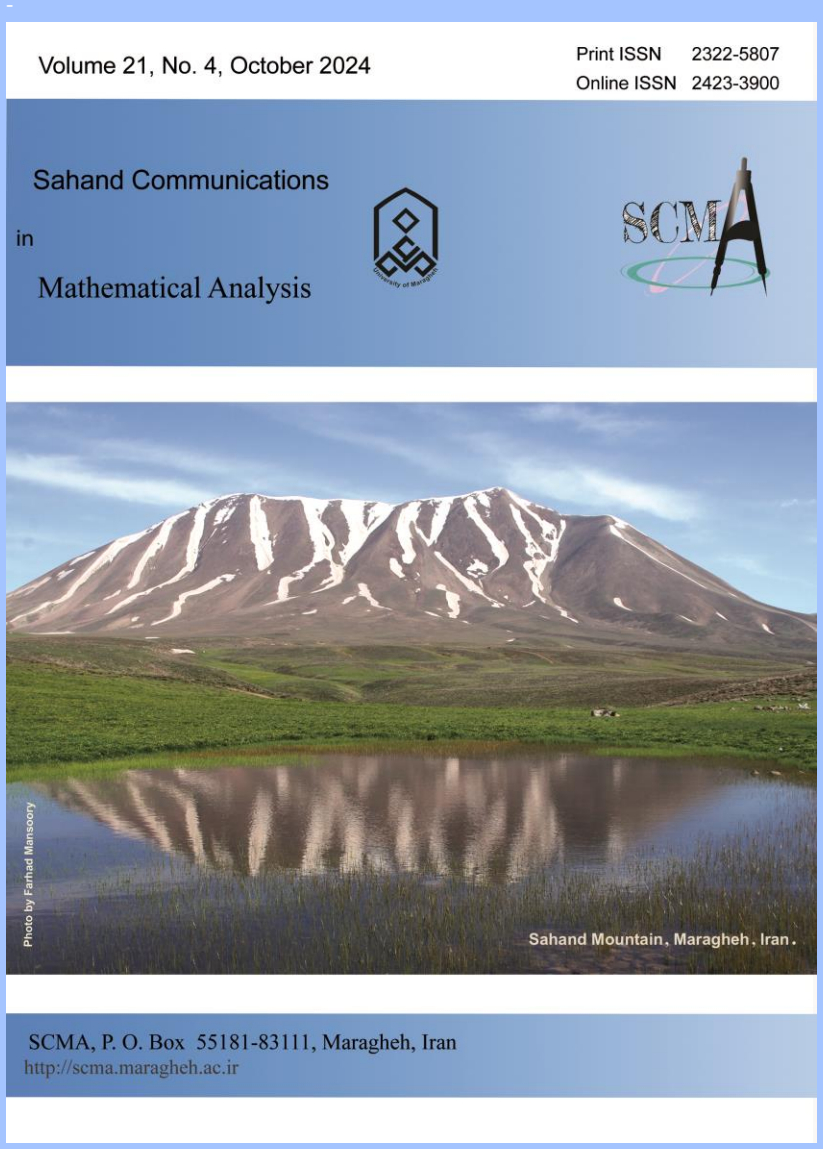
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A New Approach for Solving Nonlinear 3D-Volterra Fuzzy Integral Equations

Hamid Mottaghi Golshan[1*], Kianoush Fathi Vajargah[2] and Reza Ezzati[3]

Abstract. In this paper, we provide a quadrature-based iterative approach to solve three-dimensional nonlinear fuzzy Volterra integral equations of the second kind. The error estimation as well as convergence analysis of the proposed method are also provided. Finally, numerical experiments validate the theoretical findings. The suggested method's advantages include precision, accuracy and ease of use.

1. Introduction

As we know, Zadeh [36] was the first to develop and examine the idea of fuzzy numbers. This space, combined with a Hausdorff metric, forms a metric space. The Henstock and Riemann integrals for fuzzy-valued functions was described and researched in [34, 35], and it has been used to analyze many significant issues in partial differential equations, mathematics, physics, engineering and other fields that result from various physical models [1].

Several authors have also examined the possible existence of solutions to fuzzy integral equations. Also, many authors have investigated the existence of the solution to fuzzy integral equations by using Banach's contraction principle and other methods [2, 7, 8, 18, 19, 21, 24, 25, 28, 32, 33, 37]. In fuzzy numerical approaches, various methods have been used to investigate Volterra integral equations: Bernstein polynomials [20, 26], fuzzy differential transforms [31], open fuzzy cubature rule [16], Fuzzy Galerkin method [11], successive approximations and

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iterative procedures [14, 15, 17, 29], fuzzy bivariate triangular functions [23], Lagrange interpolation [6], trapezoidal cubature rule [13], fuzzy Bezier splines [12], fuzzy quasi-interpolation [3], and etc.

In this work, we describe and develope an iterative approach for solving three-dimensional nonlinear fuzzy Volterra integral equations of the second kind (3DNFVIE) as follows:

$$T(F)(s) := f(s) \oplus \left[\lambda \otimes (FH) \int_{I(s)} K(s,t,F(t))dt\right], \quad s,t \in I. \tag{1.1}$$

It is according to the 3-D trapezoidal quadrature rule and has a simple proof, where "$(FH)\int$" denotes the fuzzy Henstock integral, and $I := [a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$ is a 3-D cube, $I(s) := [a_1,s_1]\times[a_2,s_2]\times[a_3,s_3] \subseteq I, s = (s_1,s_2,s_3) \in I$, $a_i \leq s_i \leq b_i, i = 1,2,3$ and F is an unknown fuzzy-valued function, f and K are the given appropriate fuzzy functions.

Section 2 provides an overview of fuzzy set theory definitions and results. Section 3 investigates definitions and results regarding 3-D Henstock integrals. Section 4 investigates the requirements for the existence of a unique 3DNFVIE solution. Finally, Section 5 presents two nonlinear numerical experiments and explains the proposed method to obtain a numerical solution for 3DNFVIE through an iterative procedure. The error estimation of the presented approach is provided in terms of the modulus of continuity and the method's convergence is demonstrated.

2. Preliminaries

In this section, we review some essential terminology, findings and illustrations related to the theory of fuzzy numbers and the modulus of continuity. For acquaintance, the standard reference is [1, 4, 9].

Definition 2.1 ([9, 36])**.** A fuzzy number is defined as a function $v : \mathbb{R} \to [0,1]$ such that

(i) v is fuzzy convex set (i.e. $v(\alpha t+(1-\alpha)s) \geq \min\{v(t),v(s)\}, \forall t,s \in \mathbb{R}, \alpha \in [0,1]$),
(ii) v is normal, i.e., $\exists t_0 \in \mathbb{R}$ such that $v(t_0) = 1$,
(iii) v is upper semicontinuous,
(iv) $\overline{\{t \in \mathbb{R} : v(t) > 0\}}$ is a compact set, where $\overline{X}$ represents the closure of X.

Definition 2.2 ([35])**.** For any $0 < r \leq 1$, in parametric form, an arbitrary fuzzy number is represented by an ordered pair of functions $\big(\underline{v}(r), \overline{v}(r)\big)$, which meet the following requirements:

(i) $\underline{v}(r) \leq \overline{v}(r)$ on $[0,1]$,
(ii) $\overline{v}$ is non-increasing, left continuous and bounded function,
(iii) $\underline{v}$ is non-decreasing, left continuous and bounded function.

The definitions of scaler multiplication and fuzzy number addition in $\mathbb{R}_{\mathcal{F}}$ are as follows:

(i) $(v \oplus u)(r) = (\underline{v}(r) + \underline{u}(r), \overline{v}(r) + \overline{u}(r))$,

(ii) $(\alpha \otimes v)(r) = \begin{cases} (\alpha\underline{v}(r), \alpha\overline{v}(r)) & \alpha \geq 0 \\ (\alpha\overline{v}(r), \alpha\underline{v}(r)) & \alpha < 0 \end{cases}$

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the set of all fuzzy numbers. Note that the triple $(\mathbb{R}_{\mathcal{F}}, \oplus, \otimes)$ is a vector space over $\mathbb{R}$. Every $r \in \mathbb{R}$ can be regarded as a fuzzy point $\tilde{r} = \chi_r$ and thus $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$ and $\tilde{0}$ and $\tilde{1}$ are neutral elements for these operations, respectively.

Definition 2.3 (see [4, 22])**.** Let $x = (\underline{x}(r), \overline{x}(r))$ and $y = \left(\underline{y}(r), \overline{y}(r)\right)$ be arbitrary fuzzy numbers. The distance between x and y is the Hausdorff metric on $\mathbb{R}_{\mathcal{F}}$ and defined as follows:

$$D(x, y) = \sup_{r \in [0,1]} \max \left\{ |\underline{x}(r) - \underline{y}(r)|, |\overline{x}(r) - \overline{y}(r)| \right\}.$$

The following are widely used to obtain the main result:

Theorem 2.4 (see [4, 10])**.** *(i) $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space and for all $x, y, z, e \in \mathbb{R}_{\mathcal{F}}, \alpha, \alpha_1, \alpha_2 \in \mathbb{R}$*

(ii) $D(x \oplus y, z \oplus y) = D(x, z)$,

(iii) $D(k \otimes x, \alpha \otimes z) = |\alpha| D(x, z)$,

(iv) $D(x \oplus z, y \oplus e) \leq D(x, y) + D(z, e)$,

(v) $D(\alpha_1 \otimes x, \alpha_2 \otimes x) = |\alpha_1 - \alpha_2| D(x, \tilde{0})$.

With the metric $\widehat{D}(x_1, x_2) = \sup \{D(x_1(t), x_2(t)), t \in I\}, x_1, x_2 \in C_{\mathcal{F}}(I)$, let $\left(C_{\mathcal{F}}(I), \widehat{D}\right)$ be the space of all continuous fuzzy valued mappings x from I into $\mathbb{R}_{\mathcal{F}}$. It is well known that $\left(C_{\mathcal{F}}(I), \widehat{D}\right)$ is a complete metric space (see [4, 14, 16]). If $f : I \to \mathbb{R}_{\mathcal{F}}$ be a continues function then $\omega_\delta(f) = \sup \{D(f(y_1), f(y_2)) : y_1, y_2 \in I, d(y_1, y_2) \leq \delta\}$ is known as the modulus of continuity of f on I and we need to define

$$\omega_\delta^1(s)(f) := \sup \{D(f(s, y_1), f(s, y_2)) : y_1, y_2 \in I, d(y_1, y_2) \leq \delta\}, \ s \in I,$$

for a bounded function $f : I \times I \to \mathbb{R}_{\mathcal{F}}$. In the case of one dimension, the following characteristics will be demonstrated in a comparable manner [5, 10]:

(I) $\omega_{(\cdot)}(x)$ is an non-decreasing mapping in $\cdot$,

(II) $\omega_{(\cdot)}(x)$ is continuous at 0 iff $x \in C_{\mathcal{F}}(I)$,

(III) $D(x(t), x(s)) \leq \omega_{d(t,s)}(x)$,

for all $x \in \mathbb{R}_{\mathcal{F}}, t, s \in \mathbb{R}$.

3. 3-D Henstock Integral

The authors initially introduced the concept of the Henstock integral for a fuzzy-number-valued function in [30, 35]. We extend this approach to include the three-dimensional Henstock integral.

Definition 3.1. Let $f : I \to \mathbb{R}_{\mathcal{F}}, n_1, n_2, n_3 \in \mathbb{N}$ consider three partitions of the intervals $[a_1, b_1]$, $[a_2, b_2]$ and $[a_3, b_3]$, respectively

$$\begin{aligned} &\Delta_x^{n_1} : a_1 = s_0 < s_1 < s_2 < \cdots < s_{n_1-1} < s_{n_1} = b_1, \\ &\Delta_y^{n_2} : a_2 = t_0 < t_1 < t_2 < \cdots < t_{n_2-1} < t_{n_2} = b_2, \\ &\Delta_z^{n_3} : a_3 = u_0 < u_1 < u_2 < \cdots < u_{n_3-1} < u_{n_3} = b_3, \end{aligned} \tag{3.1}$$

and the intermediates points $\xi_i \in [x_{i-1}, x_i]$, $\eta_i \in [y_{j-1}, y_j]$ and $\zeta_k \in [z_{k-1}, z_k], i = 1, \ldots, n, j = 1, \ldots, n_1, k = 1, \ldots, n$. These partitions are denoted by $\Delta_x^{n_1} = ([x_{i-1}, x_i], \xi_i)$, $\Delta_y^{n_2} = ([y_{j-1}, y_j], \eta_j)$, and $\Delta_z^{n_3} = ([z_{k-1}, z_k], \rho_k)$. Consider the following three functions $\delta : [a_1, b_1] \to \mathbb{R}^+$, $\sigma : [a_2, b_2] \to \mathbb{R}^+$ and $\rho : [a_3, b_3] \to \mathbb{R}^+$, then above partitions are said to be δ-fine, σ-fine and ρ-fine respectively if $[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$, $[y_{j-1}, y_j] \subseteq (\eta_j - \sigma(\eta_j), \eta_j + \sigma(\eta_j))$ and $[z_{k-1}, z_k] \subseteq (\zeta_k - \rho(\zeta_k), \zeta_k + \rho(\zeta_k))$. The function f is said to be three-dimensional Henstock integrable to $S \in \mathbb{R}_{\mathcal{F}}$ if for every $\varepsilon > 0$ there are functions $\delta : [a_1, b_1] \to \mathbb{R}^+$, $\sigma : [a_2, b_2] \to \mathbb{R}^+$ and $\rho : [a_3, b_3] \to \mathbb{R}^+$ such that for any δ-fine, σ-fine and ρ-fine partitions we have

$$D\left(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3}(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}) \otimes f(\xi_i, \eta_j, \zeta_k), S\right) < \epsilon,$$

where $\sum$ stands for the fuzzy summation. Then S is called the three-dimensional Henstock integral of f and given as

$$S = (FH)\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x, y, z)dx\, dy\, dz,$$

or more briefly, $S = (FH)\int_I f(t)dt$. Let $FH(I)$ be the set of Henstock integrable functions $f : I \to \mathbb{R}_{\mathcal{F}}$. In the Lebesgue notion, we also utilize the notation $(L)\int_I f(t)dt$ for integrals. The Riemann integral is the result of this idea if the aforementioned δ, σ and ρ are constant functions.

Theorem 3.2. *If $f, g \in FH(I)$ and let $D(f(s), g(s)) \in L^1(I)$, then*

$$\begin{aligned} &D\left((FH)\int_I f(s,t,u)ds\, dt\, du, (FH)\int_I g(s,t,u)ds\, dt\, du\right) \\ &\leq (L)\int_I D(f(s,t,u), g(s,t,u))ds\, dt\, du. \end{aligned}$$

Proof. Gal [22] provided that for all integrable functions $h, r : [a, b] \to \mathbb{R}_{\mathcal{F}}$

$$D\left((FH)\int_a^b h(s)ds, (FH)\int_a^b r(s)ds\right) \leq (L)\int_a^b D(h(s), r(s))ds.$$

Clearly, we have

$$\begin{aligned}
&D\left((FH)\int_I f(s,t,u)ds\,dt\,du, (FH)\int_I g(s,t,u)ds\,dt\,du\right)\\
&\leq (L)\int_{a_1}^{b_1} D\left((FH)\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(s,t,u)dt\ du, (FH)\int_{a_2}^{b_2}\int_{a_3}^{b_3} g(s,t,u)dt\ du\right)ds\\
&\quad\vdots\\
&\leq (L)\int_I D(f(s,t,u), g(s,t,u))ds\,dt\,du.
\end{aligned}$$

This concludes the proof. □

Theorem 3.3. *(1) Suppose $x \in FH(I)$. Then, for any partition* (3.1)*, we have*

$$\begin{aligned}
&D\Bigg(D(FH)\int_I x(t)dt, \sum_{j_x=0}^{n_1-1}\sum_{j_y=0}^{n_2-1}\sum_{j_z=0}^{n_3-1}\\
&\qquad (x_{j_x+1}-x_{j_x})(y_{j_x+1}-y_{j_x})(z_{j_x+1}-z_{j_x})\otimes x(\xi_{j_x},\eta_{j_y},\zeta_{j_z})\Bigg)\\
&\quad\leq |I|\omega_\delta(x).
\end{aligned}$$

(2) Furthermore, for every s in I and for every partition (3.1)*, if $x : I \times I \to \mathbb{R}_{\mathcal{F}}$ is a continuous map and $x(s,\cdot) \in FH(I)$, then we get*

$$\begin{aligned}
&D\Bigg((FH)\int_I x(s,t,u)dt, \sum_{j_x=0}^{n_1-1}\sum_{j_y=0}^{n_2-1}\sum_{j_z=0}^{n_3-1}(x_{j_x+1}-x_{j_x})(y_{j_x+1}-y_{j_x})(z_{j_x+1}-z_{j_x})\\
&\otimes x(s,\xi_{j_x},\eta_{j_y},\zeta_{j_z})\Bigg)\\
&\leq |I|\omega_\delta^1(s)(x),
\end{aligned}$$

where,

$$\begin{aligned}
\delta = \max\big\{&d((x_{j_x}, y_{j_y}, z_{j_3}), (x_{j_1+1}, y_{j_2+1}, z_{j_3+1})),\\
&0 \leq j_x \leq n_1-1, 0 \leq j_y \leq n_2-1, 0 \leq j_z \leq n_3-1\big\},
\end{aligned}$$

and $|I| := Vol(I)$.

Proof. The proofs of (1) and (2) are similar, we only show (1). Using equalities

$$\begin{aligned} S &= (FH)\int_I x(t)dt \\ &= \sum_{j_x=0}^{n_1-1}\sum_{j_y=0}^{n_2-1}\sum_{j_z=0}^{n_3-1}\int_{x_{j_x}}^{x_{j_x+1}}\int_{y_{j_x}}^{y_{j_x+1}}\int_{z_{j_x}}^{z_{j_x+1}} x(t_1,t_2,t_3)dt_1dt_2dt_3 \end{aligned}$$

$$\begin{aligned} W &= \sum_{j_x=0}^{n_1-1}\sum_{j_y=0}^{n_2-1}\sum_{j_z=0}^{n_3-1}(x_{j_x+1}-x_{j_x})(y_{j_x+1}-y_{j_x})(z_{j_x+1}-z_{j_x})\otimes x(\xi_{j_x},\eta_{j_y},\zeta_{j_z}) \\ &= \sum_{j_x=0}^{n_1-1}\sum_{j_y=0}^{n_2-1}\sum_{j_z=0}^{n_3-1}(FH)\int_{x_{j_x}}^{x_{j_x+1}}\int_{y_{j_x}}^{y_{j_x+1}}\int_{z_{j_x}}^{z_{j_x+1}} x(\xi_{j_x},\eta_{j_y},\zeta_{j_z})dt_1dt_2dt_3, \end{aligned}$$

and [35, Remark 3.2] and properties (I)-(III) we get

$$\begin{aligned} &D(S,W) \\ &\leq \sum_{j_x=0}^{n_1-1}\sum_{j_y=0}^{n_2-1}\sum_{j_z=0}^{n_3-1} D\Bigg((FH)\int_{x_{j_x}}^{x_{j_x+1}}\int_{y_{j_x}}^{y_{j_x+1}}\int_{z_{j_x}}^{z_{j_x+1}} x(t_1,t_2,t_3) \\ &\quad dt_1dt_2dt_3, (FH)\int_{x_{j_x}}^{x_{j_x+1}}\int_{y_{j_x}}^{y_{j_x+1}}\int_{z_{j_x}}^{z_{j_x+1}} x(\xi_{j_x},\eta_{j_y},\zeta_{j_z})dt_1dt_2dt_3\Bigg) \\ &\leq \sum_{j_x=0}^{n_1-1}\sum_{j_y=0}^{n_2-1}\sum_{j_z=0}^{n_3-1}(L)\int_{x_{j_x}}^{x_{j_x+1}}\int_{y_{j_x}}^{y_{j_x+1}}\int_{z_{j_x}}^{z_{j_x+1}} D(x(t_1,t_2,t_3) \\ &\quad ,x(\xi_{j_x},\eta_{j_y},\zeta_{j_z}))dt_1dt_2dt_3 \\ &\leq \sum_{j_x=0}^{n_1-1}\sum_{j_y=0}^{n_2-1}\sum_{j_z=0}^{n_3-1}(x_{j_x+1}-x_{j_x})(y_{j_x+1}-y_{j_x})(z_{j_x+1}-z_{j_x}) \\ &\quad \omega_\delta\left(x,[x_{j_x},x_{j_1+1}]\times\cdots\times[t_{1,j_z},t_{1,j_z+1}]\right) \\ &\leq |I|\omega_\delta(x,I). \end{aligned}$$

It brings the proof to end. □

4. The Numerical Solution Method

In this section, we offer an iterative technique to approximate the solution of Eq. (1.1). Suppose $n_1, n_2, n_3 \in \mathbb{N}$ and select a partition as (3.1) on I. Also, suppose

$$s_x = a_1 + xh_1, \qquad t_y = a_2 + yh_2, \qquad u_z = a_3 + zh_3, \tag{4.1}$$

where $h_i = \frac{b_i - a_i}{n_i}, i = 1, 2, 3$ and $I_0 = \{(s_x, t_y, u_z) : x = 0, \ldots, n_1, y = 0, \ldots, n_2, z = 0, \ldots, n_3\}$. Let $x \in C_{\mathcal{F}}(I)$ and $p = (s_x, t_y, u_z) \in I_0$. Then define $x^{\circ}(p) = f(p)$ for all $p = (a_1, \cdot, \cdot), (\cdot, a_2, \cdot), (\cdot, \cdot, a_3) \in I_0$, otherwise put

(4.2)

$$x^{\circ}(p) = f(p) \oplus \left[\frac{\lambda \prod_{i=1}^{3} h_i}{2^3} \otimes \sum_{i_x=0}^{x-1} \sum_{i_y=0}^{y-1} \sum_{i_z=0}^{z-1} \sum_{j_x, j_y, j_z=0}^{1} K(s, t_{i_x+j_x}, t_{i_y+j_y}, t_{i_z+j_z}, x(t_{i_x+j_x}, t_{i_y+j_y}, t_{i_z+j_z})) \right].$$

If we assume an initial value of $u_0 \in C_{\mathcal{F}}(I)$, we will demonstrate that the iterative process

$$u_k(p) = u_{k-1}^{\circ}(p), \quad p \in I_0, k \in \mathbb{N}, \tag{4.3}$$

offers an approximation to the solution of Eq. (1.1) in I_0.

4.1. **The Existence Result.** The following is crucial to obtain the next theorem.

Theorem 4.1. *Suppose*

(I)° $f \in C_{\mathcal{F}}(I)$,

(II)° K *is a continuous function from* $I \times I \times \mathbb{R}_{\mathcal{F}}$ *into* $\mathbb{R}_{\mathcal{F}}$ *so that*

$$D(K(s,t,u), K(s,t,v)) \leq k(s,t) D(u,v), \tag{4.4}$$

for every $u, v \in \mathbb{R}_{\mathcal{F}}, s, t \in I$, *where* $k \in C(I \times I)$ *and* $k(s, \cdot) \in C(I)$, *for all* $s \in I$.

(III)° *Given a real number* λ *so that* $\kappa := \lambda \sup_{s \in I} \|k(s, \cdot)\|_{L_1} < 1$.

In equation (1.1), *the integral operator* T *has a unique solution* $u^* \in C_{\mathcal{F}}(I)$. *Furthermore, the Picard sequence given by* $x_{r+1} = T(x_r)$ *with initial value* $x_0 := u_0$ *converges to* u^*, *for every* $u_0 \in C_{\mathcal{F}}(I)$ *and the subsequent error satisfies estimates*

$$\widehat{D}(x_r, u^*) \leq \frac{\kappa^r}{1 - \kappa} \widehat{D}(x_0, T(x_0)). \tag{4.5}$$

Proof. For every $x \in C_{\mathcal{F}}(I)$ we get

$$\omega_{\delta}(T(x)) \leq \sup\{D(f(s_1), f(s_2)) : s_1, s_2 \in I, d(s_1, s_2) \leq \delta\} + \lambda \otimes \sup \left\{ D\left((FH) \int_{I(s)} K(s_1, t, x(t)) dt, (FH) \int_{I(s)} K(s_2, t, x(t)) dt \right) : s_1, s_2 \in I, d(s_1, s_2) \leq \delta \right\}$$

$$\leq \omega_\delta(f) + \lambda \sup \left\{ (L) \int_{I(s)} D(K(s_1,t,x(t)) \right.$$
$$\left. , K(s_2,t,x(t))dt : s_1, s_2 \in I, d(s_1,s_2) \leq \delta \right\}$$
$$\leq \omega_\delta(f) + \lambda(L) \int_{I(s)} \omega_\delta^2(t)(K)dt.$$

Theorem 2-(II) and the Lebesgue's monotone convergence theorem imply that $\omega_\delta(T(x)) \to 0$ as $\delta \to 0$, so T maps $C_{\mathcal{F}}(I)$ into itself. Also,

$$\begin{aligned} & D(T(x)(s), T(y)(s)) \\ & = \lambda \otimes D \left((FH) \int_{I(s)} K(s,t,x(t)), (FH) \int_{I(s)} K(s,t,y(t))dt \right) \\ & \leq \lambda(L) \int_{I(s)} D(K(s,t,x(t)), K(s,t,y(t)))dt \\ & \leq \lambda(L) \int_{I(s)} k(s,t) D(x(t), y(t)))dt \\ & \leq \lambda \left\| k(s,\cdot) \right\|_{L_1} \widehat{D}(x,y), \end{aligned}$$

for all $s \in I, x, y \in C_{\mathcal{F}}(I)$, thus,

$$\widehat{D}(T(x), T(y)) \leq \lambda \sup_{s \in I} \left\| k(s,\cdot) \right\|_{L_1} \widehat{D}(x,y).$$

Therefore, T is a contraction in the complete metric space $\left(C_{\mathcal{F}}(I), \widehat{D} \right)$ from condition (III)°. Assume that the initial value, $x_0 \in C_{\mathcal{F}}(I)$ and that $x_n = T(x_{n-1}), n \in \mathbb{N}$. The Banach contraction principle states that integral equation (1.1) has a unique solution and that (4.5) is valid. □

4.2. **Convergence Analysis and Estimation Error.** Here, with the given integral Eq. (1.1), we get an estimation error between the exact and approximate solutions. Prior to presenting this section's major finding, it is necessary to understand the following concepts: Suppose $k : I \times I \to \mathbb{R}$ be a bounded function. Let us define

$$\omega_\delta^1(s)\big(Ku\big) := \sup_{t_1,t_2 \in I} \left\{ D(K(s,t_1,u(t_1)), K(s,t_2,u(t_2)) \ ; \ d(t_1,t_2) \leq \delta \right\},$$
$$\omega_\delta^{1,3}(s,r)\big(K\big) := \sup_{t_1,t_2 \in I} \left\{ D(K(s,t_1,r), K(s,t_2,r)) \ ; \ d(t_1,t_2) \leq \delta \right\},$$

where $s, r \in \mathbb{R}$. Note that from (4.4) we have

(4.6)

$$\begin{aligned}
\omega_\delta^1(s)\big(Kx\big) &= \sup_{t_1,t_2\in I}\{D(K(s,t_1,x(t_1)),K(s,t_2,x(t_2)) \ ; \ d(t_1,t_2)\le\delta\}\\
&\le \sup_{t_1,t_2\in I}\{D(K(s,t_1,x(t_1)),K(s,t_1,x(t_2)) \ ; \ d(t_1,t_2)\le\delta\}\\
&\quad + \sup_{t_1,t_2\in I}\{D(K(s,t_1,x(t_2),K(s,t_2,x(t_2)) \ ; \ d(t_1,t_2)\le\delta\}\\
&\le \sup_{t_1,t_2\in I}\{k(s,t_1)D(x(t_1),x(t_2)) \ ; \ d(t_1,t_2)\le\delta\}\\
&\quad + \sup_{|t|\le\|x\|}\sup_{t_1,t_2\in I}\{D(K(s,t_1,t),K(s,t_2,t)) \ ; \ d(t_1,t_2)\le\delta\}\\
&\le M_K\omega_\delta(x) + \sup_{s\in I,|u|\le\|x\|}\omega_\delta^{1,3}(s,t,u)(K),
\end{aligned}$$

where $M_K := \sup_{s,t\in I}|k(s,t)|$.

Theorem 4.2. *Let f and K fulfill Theorem 4.1's requirements. If $u_0 \in C_{\mathcal{F}}(I)$ is an initial value and λ be a real integer such that $R = \lambda|I|M_K < 1$, then the iterative process* (4.3) *converges to the unique solution u^* of* (1.1)*. Furthermore, the following error estimate is valid:*

$$\widehat{D}(u^*,u_r) \le \frac{\kappa^r}{1-\kappa}\widehat{D}(u_0,B(u_0)) + \frac{\zeta(\delta)}{1-R}. \tag{4.7}$$

where

$$\zeta(\delta) = \lambda|I|\left(\sup_r M_K\omega_\delta(x_r) + \sup_{s\in I,|u|\le M_0}\omega_\delta^{1,3}(s,t,u)(K)\right),$$

and $M_0 = \sup_{r\in\mathbb{N}}\|x_r\|_u$.

Proof. Let x_r be defined by the Picard sequence $x_r = T(x_{r-1})$, where $x_0 := u_0 \in \mathbb{R}_{\mathcal{F}}$ is the initial value. Consider iterative procedure (4.3) and using Theorems 3.3, 2.4, equation (4.4) and the relations

$$\begin{aligned}
&\prod_{i=1}^{3} h_i = (x_{j_x+1} - x_{j_x})(y_{j_y+1} - y_{j_y})(z_{j_z+1} - z_{j_z}),\\
&\sum_{i_x,i_y,i_z=0}^{1} = 2^3,\\
&\sum_{j_x=0}^{x-1}\sum_{j_y=0}^{y-1}\sum_{j_z=0}^{z-1}\prod_{i=1}^{3} h_i \le |I|,
\end{aligned}$$

then for all $r \in \mathbb{N}\cup\{0\}, p = (s_x,t_y,u_z) \in I_0$, we get

$D(x_{r+1}(p), u_{r+1}(p))$

$$
\begin{aligned}
&\leq D(x_{r+1}(p), {x_r}^{\circ}(p)) + D\left(x_r^{\circ}(p), u_{r+1}(p)\right) \\
&\leq \frac{\lambda}{2^3} \sum_{i_x,i_y,i_z=0}^{1} D\left((FR) \int_I K(p,t,x_r(t))dt, \sum_{j_x=0}^{x-1}\sum_{j_y=0}^{y-1}\sum_{j_z=0}^{z-1} \right. \\
&\left. \prod_{i=1}^{3} h_i(K(p, t_{j_x+i_x}, t_{j_y+i_y}, t_{j_z+i_z}, x_r(t_{j_x+i_x}, t_{j_y+i_y}, t_{j_z+i_z})) \right) \\
&\quad + \sum_{j_x=0}^{x-1}\sum_{j_y=0}^{y-1}\sum_{j_z=0}^{z-1}\prod_{i=1}^{3} h_i D\big(K(p, t_{j_x+i_x}, t_{j_y+i_y}, t_{j_z+i_z}, x_r(t_{j_x+i_x} \\
&, t_{j_y+i_y}, t_{j_z+i_z})), K(p, t_{j_x+i_x}, t_{j_y+i_y}, t_{j_z+i_z}, u_r(t_{j_x+i_x}, t_{j_y+i_y}, t_{j_z+i_z}))\big) \\
&\leq \frac{\lambda}{2^3} \sum_{i_x,i_y,i_z=0}^{1} \left(|I|\omega_\delta^1(p)(Kx_r) + M_K \sum_{j_x=0}^{x-1}\sum_{j_y=0}^{y-1}\sum_{j_z=0}^{z-1} \right. \\
&\left. \prod_{i=1}^{3} h_i D\left(x_r(t_{j_x+i_x}, t_{j_y+i_y}, t_{j_z+i_z}), u_r(t_{j_x+i_x}, t_{j_y+i_y}, t_{j_z+i_z})\right) \right) \\
&\leq \lambda|I| \left(\omega_\delta^1(p)(Kx_r) + M_K \widehat{D}(x_r, u_r) \right).
\end{aligned}
$$

Observe that M_0 exits because x_r is a convergent sequence. Also, from (4.6) we get

$$\sup_{p\in I} \omega_\delta^1(p)(Kx_r) \leq M_K \omega_\delta(x_r) + \sup_{p\in I, |u|\leq M_0} \omega_\delta^{1,3}(p,t)(K).$$

right. Thus, from inequality (4.6) we get

$$\widehat{D}(x_{r+1}, u_{r+1}) \leq \zeta(\delta) + R\widehat{D}(x_r, u_r). \tag{4.8}$$

Let $\limsup_r \widehat{D}(x_{r+1}, u_{r+1}) = C$. If $C > 0$ then take $\limsup$ as $\delta \to 0, r \to +\infty$ from both sides of (4.8), it is concluded that $R \geq 1$, a contradiction, thus $\lim_r \widehat{D}(x_{r+1}, u_{r+1}) = 0$.

Considering that $\kappa \leq R < 1$, we may infer that u_r converge to u^* from Theorem 4.1 and $\widehat{D}(u^*, u_r) \leq \widehat{D}(u^*, x_r) + \widehat{D}(x_r, u_r)$. Also, we see that

$$
\begin{aligned}
\widehat{D}(x_r, u_r) &\leq \zeta(\delta) + R\widehat{D}(x_{r-1}, u_{r-1}) \\
&\leq \zeta(\delta) + \zeta(\delta)R^1 + R^2\widehat{D}(x_{r-2}, u_{r-2}) \\
&\ \ \vdots \\
&\leq \zeta(\delta)(1 + R^1 + \cdots + R^r) \\
&\leq \zeta(\delta)\frac{1}{1-R}.
\end{aligned}
$$

So from (4.5) the desired estimate (4.7) is obtained. □

5. Numerical Examples

In the following examples, the suggested iterative method was evaluated to demonstrate the correctness of the theoretical conclusions and to demonstrate the method's accuracy and convergence. In these examples, we assumed that $I = [0,1] \times [0,1] \times [0,1]$, $\lambda = 1$ and

(5.1)

$$F(s,t,u,r) = f(s,t,u,r) \oplus \left[\lambda \otimes (FH) \int_0^s \int_0^t \int_0^u k(s,t,u,x,y,z) \otimes F^2(x,y,z,r)dx\,dy\,dz\right],$$

where $s,t,u \in [0,1]$. To obtain a numerical solution, let $n_1 = n_2 = n_3 = n$ be fixed and $I_0 = \{(s,t,u) : s,t,u = 0, 1/n, \ldots, 1\}$, a Matlab code for iterative procedure (4.3) with the initial value $u_0 = 0$ was used and the condition to exit the loop was $\max_{s\in I_0}\{|u_{k+1}(s) - u_k(s)|\} < 10^{-30}$. Denote $\underline{E}(s) := |\underline{F}^*(s) - \underline{F}_k(s)|, s \in I_0$ and $\overline{E}(s) := |\overline{F}^*(s) - \overline{F}_k(s)|, s \in I_0$ for the absolute error functions, $\underline{E} := \max\{\underline{E}(s), s \in I_0\}$ and $\overline{E} := \max\{\overline{E}(s), s \in I_0\}$ for the maximum absolute errors (MAEs).

Example 5.1. Consider $f = \left(\underline{f}, \overline{f}\right)$ and

$$\underline{f}(s,t,u,r) = \left(r^2+r\right)^2 st^4u^4(\cos(s)\left(s^2-2\right) - 2s\sin(s) + 2)/9 + \left(r^2+r\right)stu,$$
$$\overline{f}(s,t,u,r) = (r-2)^2 st^4u^4(\cos(s)\left(s^2-2\right) - 2s\sin(s) + 2)/9 + (2-r)stu,$$
$$k(s,t,u,x,y,z) = stu\sin(x).$$

The exact solution is

$$F^* = \left(\underline{F}^*, \overline{F}^*\right),\ \underline{F}^*(s,t,u,r) = \left(r^2+r\right)stu,\ \overline{F}^*(s,t,u,r) = (2-r)stu.$$

At first, let us investigate the conditions for existence and uniqueness as stated in Theorem 4.1. The equation (5.1) is the form of (1.1) for $K(s,t,u,x,y,z,u) = k(s,t,u,x,y,z) \otimes u^2$, where $s,t,u,x,y,z \in [0,1], u \in \mathbb{R}_{\mathcal{F}}$. Condition (I)° is clear. Condition (II)° follows from Theorem 2.4-(iii) for a closed subset of $C_{\mathcal{F}}(I)$. Condition (III)° can be easily calculated as

$$\begin{aligned}\kappa &:= \lambda \sup_{r\in I} \|k(r,\cdot)\|_{L_1} \\ &= \left(\sup_{s,t,u\in[0.1]} stu\right) \int_I \sin(x)dx\,dy\,dz \\ &\simeq 0.4597.\end{aligned}$$

To compare MAEs, see Figures 1, 2 and Table 1.

TABLE 1. The MAEs results for the level sets

	$n=5$		$n=10$	
r	E	$\overline{E}$	E	$\overline{E}$
0.25	1.8242E-4	0.0058	4.4738E-5	0.0014
0.50	0.0011	0.0043	2.5879E-4	0.0010
0.75	0.0033	0.0031	7.9691E-4	7.2239E-4
	$n=20$		$n=40$	
r	E	$\overline{E}$	E	$\overline{E}$
0.25	1.1131E-5	3.5373E-4	2.7794E-6	8.8310E-5
0.50	6.4374E-5	2.5928E-4	1.6073E-5	6.4733E-5
0.75	1.9817E-4	1.7964E-4	4.9477E-5	4.4852E-5

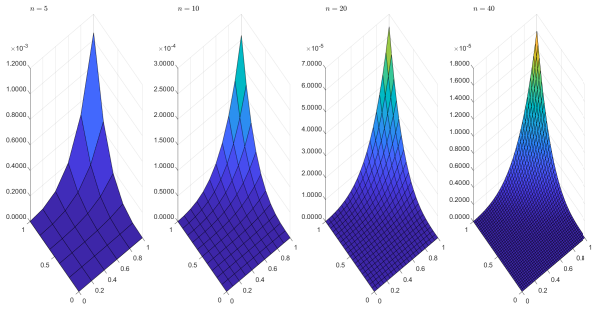


FIGURE 1. The error-function graphs $\underline{E}$ at nodes $(s,t,1) \in I_0$ for $r = 0.5$ and $n = 5, 10, 20, 40$, respectively

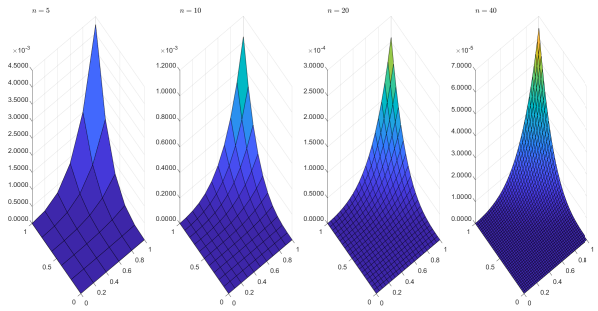


FIGURE 2. The error-function graphs $\overline{E}$ in nodes $(s,t,1) \in I_0$ for $r = 0.5$ and $n = 5, 10, 20, 40$, respectively

Example 5.2. Consider

$$f = \left(\underline{f}, \overline{f}\right),$$

$$\underline{f}(s,t,u,r) = r^2 s_1 s_2 s_3 - \frac{r^4 {s_1}^4 {s_2}^4 {s_3}^4 \left(s_1 + s_2 + s_3\right)}{108},$$

$$\overline{f}(s,t,u,r) = r s_1 s_2 s_3 - \frac{r^2 {s_1}^4 {s_2}^4 {s_3}^4 \left(s_1 + s_2 + s_3\right)}{108},$$

$$k(s,t,u,x,y,z) = \frac{1}{3} stu(x+y+z).$$

The exact solution of this example is as follows:

$$F^* = \left(\underline{F}^*, \overline{F}^*\right), \qquad \underline{F}^*(s,t,u,r) = r^2 stu, \qquad \overline{F}^*(s,t,u,r) = rstu.$$

At first, let us investigate the conditions for existence and uniqueness in Theorem 4.1. The equation (5.1) is the form of (1.1) for $K(s,t,u,x,y,z,u) = k(s,t,u,x,y,z) \otimes u^2$, where $s,t,u,x,y,z \in [0,1], u \in \mathbb{R}_{\mathcal{F}}$. Condition (I)° is clear. Condition (II)° follows from Theorem 2.4-(iii). Condition (III)° can be easily calculated as

$$\begin{aligned} \kappa &= \lambda \sup_{r \in I} \|k(r,\cdot)\|_{L_1} \\ &= \frac{1}{3} \left(\sup_{s,t,u \in [0.1]} stu \right) \int_I (x+y+z) dx\, dy\, dz \simeq 0.5. \end{aligned}$$

To compare MAEs, one can refer to Figures 3, 4 and Table 2.

TABLE 2. The MAEs results for level sets

	$n=5$		$n=10$	
r	E	E	E	E
0.25	8.9066E-6	1.824E-4	2.1857E-6	3.5032E-5
0.50	1.4286E-4	5.7331E-4	3.5032E-5	1.4047E-4
0.75	7.2620E-4	0.0013	1.7793E-4	3.1695E-4
	$n=20$		$n=40$	
r	E	E	E	E
0.25	5.4372E-7	8.7157E-6	1.3577E-7	2.7794E-6
0.50	8.7157E-6	3.4950E-5	2.1763E-6	1.6073E-5
0.75	4.4260E-5	7.883E-5	1.1051E-5	4.9478E-5

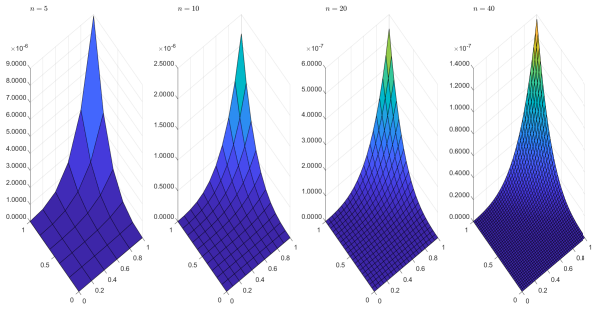


FIGURE 3. The error-function graphs $\underline{E}$ at nodes $(s,t,1) \in I_0$ for $r = 0.25$ and $n = 5, 10, 20, 40$, respectively

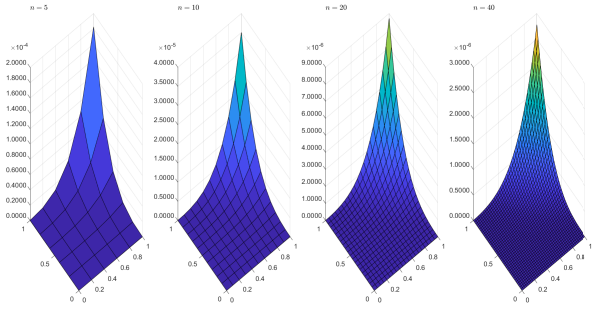


FIGURE 4. The error-function graphs $\overline{E}$ in nodes $(s,t,1) \in I_0$ for $r = 0.25$ and $n = 5, 10, 20, 40$, respectively

6. Conclusions

In this paper, we defined 3-D fuzzy integrals and presented 3-D fuzzy mappings. We proved the existence and unique solution of 3DNFVIE by applying Banach's fixed point principle, which we established as a theorem. Furthermore, we presented an iterative approach based on the successive approximation method to estimate the 3DNFVIE solution. The iterative method's convergence to the unique solution is examined. The advantages of the proposed method include its precision, accuracy and ease of use. Section 6 reports on a numerical experiment that verifies the method's strong applicability to 3DNFVIE.

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