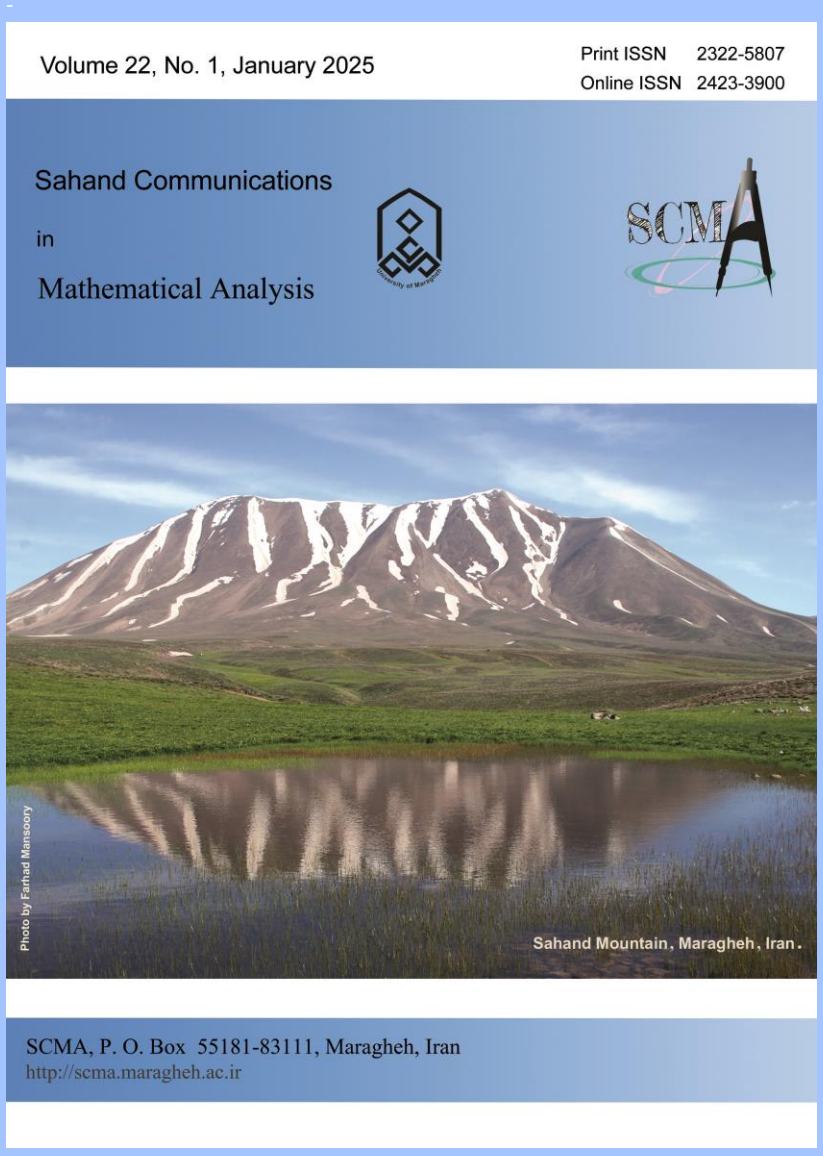
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Hyperbolic and Trigonometric Generators of Archimedean Copulas: Characteristics and Performance

Amirmasoud Kazemirad[1], Leila Golshani[2*], Vadood Najjari[3] and Mohsen Kokabinezhad[4]

Abstract. In this paper, hyperbolic and trigonometric functions, based on cosecant and cotangent as generator function for the family of Archimedean copula, are proposed. For these families, the dependence characteristics are compared. Also, to show the performance of these copulas, we investigate the stochastic frontier model based on them. Then we conclude that the copulas based on hyperbolic and trigonometric cosecant and cotangent are more suitable for modeling dependence structure.

1. Introduction

Many statistical models are based on the assumption of independence of variables. However, the structure of some models is based on dependence. The concept of dependence was introduced by Sklar (1959) through functions that are useful for modeling dependence structure. One of the most important families of copulas is the Archimedean copula. The Archimedean copula possesses useful properties and numerous applications. This family is easy to construct and several copula families belong to it. Therefore, it is important to study and investigate the characteristics of Archimedean copula. Dependence measures are useful tools to show the structure of dependence between variables. One of the advantages of copula functions is that dependence measures can be introduced on their basis. Therefore, in this study by introducing new families of Archimedean copula, and their dependence measures are

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also investigated. Many studies have been done in relation to the generator function of the Archimedean copula. Some of them are related to the trigonometric and hyperbolic functions. For the first time, Bal and Najjari (2013) introduced hyperbolic cosecant generator function. Najjari et al. (2014) proposed hyperbolic cotangent function as a generator of Archimedean copula. After that, other forms of trigonometric and hyperbolic functions were introduced and their characteristics were investigated (Susam, 2020 and its references). Kazemi Rad et al. (2021) proposed several methods for constructing generators of Archimedean copula, where the previously introduced items are examples of them. In this paper, we concentrate on four Archimedean families with generator functions based on cosecant and cotangent trigonometric and hyperbolic functions, in which the cosecant functions are new and have not been investigated previously, and spearman's rho on the cotangent functions had not been obtained. One of the main instruments to obtain the technical efficiency is stochastic frontier analysis (SFA). For the first time, Smith (2008) considered the stochastic frontier model based on copula. After that, this model with different copulas was used (Najjari et al., 2016, Lutz et al., 2017 and Huaimo et al., 2018). In this study, to show the performance of the copula functions, standard SFA model and seven copula families with SFA are considered. This paper is organized as follows. In Section 2, preliminaries are introduced. In Section 3, new generator functions of the Archimedean copula are proposed and then their characteristics are obtained. The review on real data is in Section 4. Finally, conclusions are presented in Section 5.

2. Preliminaries

2.1. **Copulas and their Properties.** A bivariate copula is a function $C : [0,1]^2 \to [0,1]$ which satisfies:

(a) for every u, v in [0,1], $C(u,0) = 0 = C(0,v)$, $C(u,1) = u$ and $C(1,v) = v$;
(b) for every u_1, u_2, v_1, v_2 in [0,1] such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

The Archimedean copula is one of the most important families of copula. This family is characterized by its generator functions. The generator function φ is a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1) = 0$. Therefore,an Archimedean copula function $C : [0,1]^2 \to [0,1]$ with generator function φ is defined by

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)),$$

where $\varphi^{[-1]}$ is pseudo-inverse of φ, which is defined as

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{(-1)}(t), & 0 \leq t \leq \varphi(0) \\ 0, & \varphi(0) \leq t \leq \infty. \end{cases}$$

If $\varphi(0) = \infty$, then $\varphi^{[-1]} = \varphi^{(-1)}$. So, $C(u, v)$ is a strict Archimedean copula.

2.2. **Copulas in Stochastic Frontier Models.** The stochastic frontier models are (Aigner et al., 1977), (Meeusen and Vanden Broek, 1977)

$$\ln\, y_i = \beta_0 + \sum_n \beta_n \ln x_{ni} + \varepsilon_i.$$

where $\varepsilon_i = v_i - u_i$, $i = 1, \ldots, n$ and y_i is a scalar output, x_i is a vector of input, β is a vector of parameters, and v_i and u_i are error components. For each i, u_i's are independent (also for v_i's) and, u and v are dependent. Here v_i is the noise component and u_i is the technical inefficiency component which is non-negative. Suppose $u \sim F$ and $v \sim G$, then by considering the Sklar theorem (Nelsen, 2006), the joint distribution function is $H(u, v) = C(F(u), G(v))$. So its joint density function is as follows,

$$h(u, v) = f(u)g(v)c_\theta(F(u), G(v)). \tag{2.1}$$

for $\varepsilon = v - u$, the marginal function is

$$h(\varepsilon) = \int_0^{+\infty} f(u)g(u + \varepsilon)c_\theta(F(u), G(u + \varepsilon))du. \tag{2.2}$$

The technical efficiency based on copulas is defined as (Smith, 2008)

$$TE = \frac{1}{h(\varepsilon)} \int_{\mathbb{R}^+} \exp\{-u\}h(u, \varepsilon)du$$

where $h(u, \varepsilon)$ and $h(\varepsilon)$ are defined in (2.1) and (2.2).

3. Generator Functions and Dependence Measures

Suppose the Archimedean generator function has the following form: (Kazemi Rad et al., 2021)

$$\varphi(t) = g(\theta t) - g(\theta). \tag{3.1}$$

We consider the function $g(\theta)$ as trigonometric and hyperbolic functions of cosecant and cotangent, where all of them have two continuous derivatives on $(0, 1)$ and they satisfy all properties of generator functions for

the entire range of the copula parameter. These copula generators are as follows:

$$\begin{aligned}
&\varphi_{\cot(t)} = \cot(\theta t) - \cot(\theta), && \theta \in (0, \pi/2] \\
&\varphi_{\coth(t)} = \coth(\theta t) - \coth(\theta), && \theta \in (0, \infty) \\
&\varphi_{\csc(t)} = \csc(\theta t) - \csc(\theta), && \theta \in (0, \pi/2] \\
&\varphi_{\csch(t)} = \csch(\theta t) - \csch(\theta), && \theta \in (0, \infty)
\end{aligned}$$

For all the generator functions, $\lim_{t\to 0^+} \phi(t) = \infty$. Thus, $\varphi^{[-1]} = \varphi^{(-1)}$ and the inverse of them exists. So, the inverse of generator functions and the copula functions in the general form (3.1) have in the following form:

$$\varphi^{(-1)}(t) = \frac{1}{\theta} g^{-1}(t + g(\theta))$$

and

$$\begin{aligned}
C(u, v) &= \varphi^{[-1]}(\varphi(u) + \varphi(v)) \\
&= \frac{1}{\theta} g^{-1}(g(u\theta) + g(v\theta) - g(\theta))
\end{aligned}$$

Table 1, shows the copula functions of trigonometric and hyperbolic of cosecant and cotangent.

TABLE 1. Copulas and θ intervals

Family	$C(u, v)$	θ interval
Cot-copula	$\frac{1}{\theta}\text{arccot}(\cot(\theta u) + \cot(\theta v) - \cot(\theta))$	$(0, \frac{\pi}{2}]$
Coth-copula	$\frac{1}{\theta}\text{arccoth}(\coth(\theta u) + \coth(\theta v) - \coth(\theta))$	$(0, \infty)$
Csc-copula	$\frac{1}{\theta}\text{arccsc}(\csc(\theta u) + \csc(\theta v) - \csc(\theta))$	$(0, \frac{\pi}{2}]$
Csch-copula	$\frac{1}{\theta}\text{arccsch}(\csch(\theta u) + \csch(\theta v) - \csch(\theta))$	$(0, \infty)$

For dependence measures, we consider Kendall's tau, Spearman's rho and tail dependence. First, we obtain Kendall's tau according to the copula parameter, using the following relation:

$$\tau_c = 1 + 4 \int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt$$

where $\varphi'(t)$ is the derivative of the generator function. For copula functions in Table 1, Kendall's tau was obtained in a closed form. By using L'Hopital's rule, τ intervals were obtained. The results are shown in Table 2.

TABLE 2. Kendall's tau and τ interval

Family	τ	τ interval
Cot-copula	$1-\frac{2}{\theta^2}+\frac{2}{\theta}\cot(\theta)$	$\left(1-\frac{8}{\pi^2},\frac{1}{3}\right)$
Coth-copula	$1+\frac{2}{\theta^2}-\frac{2}{\theta}\coth(\theta)$	$\left(\frac{1}{3},1\right)$
Csc-copula	$1-\frac{4}{\theta^2}+\frac{4}{\theta^2}\ln(\cos\theta)+\frac{4}{\theta^2}\csc\theta\ln(\sec\theta+\tan\theta)$	$\left(\frac{1}{3},1\right)$
Csch-copula	$1+\frac{4}{\theta^2}-\frac{4}{\theta^2}\ln(\cosh\theta)-\frac{4}{\theta^2}\text{csch}\theta\arctan(\sinh\theta)$	$\left(-1,\frac{1}{3}\right)$

The cot-copula family shows low dependence between variables, and the coth-copula family is used to model positive dependence between variables. For the family of hyperbolic and trigonometric cosecant, the range of dependence is wide. The csc-copula family has a positive dependence, while csch-copula family has a limited range of positive dependence and a wide range of negative dependence for modeling. For tail dependence measures, upper and lower tail dependence coefficients are defined by

$$\begin{aligned}\lambda_U &= \lim_{t\to 1^-} P\{F(X)>t|G(X)>t\}\\ &= \lim_{t\to 1^-} 2-\frac{1+C(t,t)}{1-t}\end{aligned}$$

and

$$\begin{aligned}\lambda_L &= \lim_{t\to 0^+} P\{F(X)\le t|G(X)\le t\}\\ &= \lim_{t\to 0^+} \frac{C(t,t)}{t}\end{aligned}$$

For all copula functions in Table 1, $\lambda_L=0$ and $\lambda_U=1/2$ were obtained for the range of copula parameter. The lower and upper tail dependence for all of them is constant. The third measure of dependence under consideration is Spearman's rho, which is defined by

$$\rho_C = 12\int_0^1\int_0^1 C(u,v)dudv-3$$

For the copula functions in Table 1, we could not obtain Spearman's rho in terms of the parameter in a closed form. Therefore, we derived an approximate relation for them. For this purpose, 1000 values for θ were randomly selected from the range of the copula parameter and then we obtained Spearman's rho by numerical methods. After that, various types of nonlinear regression models were fitted, using SPSS 28 software

and based on R-square (see Table 3), appropriate model was obtained for four copula functions as follows:

$$\rho_C = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3$$

TABLE 3. R-square for copulas

Family	R-square
Cot-copula	1
Coth-copula	0.992
Csc-copula	0.998
Csch-copula	0.98

4. Application

In this section, we evaluate the efficiency of hydroelectric power plants, using the standard stochastic frontier model (which is denoted by SFA) and the stochastic frontier model with A12, Gumbel, Clayton and trigonometric and hyperbolic cotangent and cosecant copulas (referred to by copulas names). Details of Clayton, A12 and Gumbel copulas are given in Table 4.

TABLE 4. Details of A12, Gumbel and Clayton copulas

Family	Generator	Kendall's tau	λ_L	λ_U	θ interval
Clayton	$\frac{1}{\theta}\left(\frac{1}{t^\theta}-1\right)$	$\frac{\theta}{\theta+2}$	$2^{\frac{-1}{\theta}}$	0	$(0,\infty)$
Gumbel	$(-\ln t)^\theta$	$\frac{\theta-1}{\theta}$	0	$2-2^{\frac{1}{\theta}}$	$[1,\infty)$
A12	$\left(\frac{1}{t}-1\right)^\theta$	$1-\frac{2}{3\theta}$	$2^{\frac{-1}{\theta}}$	$2-2^{\frac{1}{\theta}}$	$[1,\infty)$

Note: A12 family is numbered as 4.2.12 in Table 4.1 Nelsen's book (2006)

The analyzed data includes 29 dams in Iran with an average practical power of 10 MW (Available online at Moe.gov.ir). In this analysis, two inputs and one output are considered as follows:

Input 1: Number of employees.
Input 2: Average practical power of the power plant.
Output: Utilization percentage of power plant capacity.

The analyzed data are shown in Table 5.

TABLE 5. The estimated parameters of the SFA models

	Powerhouse	Utilized power (MW)	Average practical power (MW)	Number of employees
1	Aras	21	22	45
2	Sardasht	150	150	59
3	Mughan	10	13	27
4	Zayandehrud	42	56	36
5	Caron 4	795	1000	68
6	Kohrang	9	39	15
7	Vaforghan	8	10	49
8	Rudbar	405	450	42
9	Amir-Kabir	22	90	48
10	Taleghan	12	18	43
11	Klan-lar	52	116	40
12	Latyan	28	45	59
13	Levark	41	44	39
14	Dez	358	520	224
15	Abbaspur	1269	2000	351
16	Suleiman Mosque	1684	2000	267
17	Karkheh	401	400	223
18	Caron 3	1842	2000	280
19	Maroon	141	150	194
20	Getund	1000	1000	136
21	Simre	120	480	124
22	Kurdistan	10	10	43
23	Darian	210	210	79
24	Mulla- Sadra	60	100	15
25	Darudzan	7	10	9
26	Jiroft	30	32	35
27	Sefidroud	86	88	31
28	Takam	12	15	12
29	Sieh Bisheh	301	1040	98

Calculations have been done using Matlab software, and in this regard, the "fminsearchbnd" function has been used. In this review, v has a half normal distribution and u has a half-normal distribution as follows:

$$u \sim N\left(\sigma_u\sqrt{\frac{2}{\pi}}, \sigma_u^2\left(1-\frac{2}{\pi}\right)\right), \quad v \sim N\left(0, \sigma_v^2\right)$$

Also, maximum likelihood (ML) method was used to estimate the parameters of the stochastic frontier models.

The estimation of parameters for SFA model and SFA with copulas are shown in Table 6. The results show that the dependence parameter θ, has different values for copula families. These values of parameter show the dependence between u and v which are shown by Kendall's tau. The maximum dependence is shown in csc-copula, where it is $\tau = 0.7206$ for

$\theta = 1.5247$. So, there is 72.06% dependence between u and v. Then, Kendall's tau indicates that there is a positive correlation between u and v. Therefore, it has reduced the variance of $\epsilon = u - v$, and the estimated model has the least error and more accurate technical efficiency between copula function. Also, the minimum dependence belongs to the Gumbel with $\tau = 0.0559$.

TABLE 6. The estimated parameters of the SFA models

Family	σ_u	σ_v	θ	β_0	β_1	β_2	τ
Clayton	2.706	0.2159	1.0011	3.2028	0.4551	0.2565	0.3335
Gumbel	2.7174	0.2524	1.0593	3.192	0.4633	0.2608	0.0559
A12	3.4934	0.2472	1.1662	3.2707	0.4262	0.2459	0.4283
Cot-copula	2.873	0.2441	1.7164	3.2011	0.4559	0.2569	0.5283
Coth-copula	1.8018	0.1189	2.0995	3.1354	0.4793	0.2614	0.5853
Csc-copula	0.2836	0.2992	1.5247	2.9922	0.5993	0.3428	0.7206
Csch-copula	0.2388	0.2988	1.5166	3.0505	0.6531	0.3218	0.5135
SFA	0.7494	0.5743	-	3.9225	0.159	0.1502	-

Table 7, Consists of efficiency scores (ES) and the ranks of 29 dams by the models, for which their estimated parameters are given in Table 6. The results show that Gatund dam is an efficient power plant for most of the copulas, while Mughan is only effective with cot and Clayton copulas, and Sefidroud power plant is shown by Gumbel and Coth copulas. Now, to compare the results of the estimated models, we obtain the correlation between the ranks of power plants with each model. The results are shown in Table 8.

According to the results of Table 8, the correlation between the ranks in the models is high. So, the obtained results show that the SFA model and SFA-copulas models are expected to produce complementary. Among the copula models, the model which uses A12 has the highest correlation 0.97 by the SFA model, and Csch model has the lowest correlation 0.82 with the SFA Model in this study.

5. Conclusion

In this paper, we concentrate on four Archimedean families, for which generator functions are based on cotangent and cosecant trigonometric and hyperbolic functions and their dependence measures are investigated. Also, we showed the performance of these copula functions with the stochastic frontier model, and concluded that the stochastic frontier model with the Csc-copula has a more accurate technical efficiency. Finally, we concluded that there is a high dependence between the outcomes of the SFA model and SFA-copulas models.

TABLE 7. Efficiency scores (ES) and ranks of 29 dams by several methods

Powerhouse	Clayton ES	Clayton rank	Gumbel ES	Gumbel rank	A12 ES	A12 rank	Cot ES	Cot rank
Aras	0.54	14	0.51	15	0.53	14	0.54	14
Sardasht	0.9	5	0.89	5	0.89	5	0.9	5
Mughan	1	2	0.96	3	0.99	2	1	2
Zayandehrud	0.82	6	0.81	8	0.82	6	0.82	6
Caron 4	0.52	15	0.54	14	0.52	15	0.52	15
Kohrang	0.35	21	0.36	21	0.35	21	0.36	21
Vaforghan	0.3	26	0.28	26	0.3	26	0.3	26
Rudbar	0.59	12	0.61	12	0.58	12	0.59	12
Amir-Kabir	0.49	17	0.48	17	0.49	17	0.49	17
Taleghan	0.58	13	0.55	13	0.58	13	0.58	13
Klan-lar	0.45	19	0.45	18	0.45	19	0.45	19
Latyan	0.33	25	0.31	25	0.33	24	0.33	25
Levark	0.66	11	0.64	11	0.65	11	0.66	11
Dez	0.8	7	0.77	9	0.81	7	0.8	9
Abbaspur	0.34	22	0.33	24	0.34	22	0.34	22
Suleiman Mosque	0.42	20	0.41	20	0.42	20	0.42	20
Karkheh	0.51	16	0.49	16	0.52	16	0.51	16
Caron 3	0.33	23	0.33	23	0.34	23	0.33	23
Maroon	0.26	27	0.24	27	0.26	27	0.26	27
Getund	1	1	1	2	1	1	1	1
Simre	0.23	28	0.22	28	0.23	28	0.23	28
Kurdistan	0.74	10	0.69	10	0.74	10	0.74	10
Darian	0.45	18	0.44	19	0.45	18	0.45	18
Mulla- Sadra	0.04	29	0.04	29	0.04	29	0.04	29
Darudzan	0.81	9	0.81	7	0.79	9	0.81	8
Jiroft	0.98	4	0.96	4	0.98	4	0.98	4
Sefidroud	0.99	3	1	1	0.98	3	0.99	3
Takam	0.81	8	0.82	6	0.8	8	0.82	7
Sieh Bisheh	0.33	24	0.34	22	0.33	25	0.33	24

Powerhouse	Coth ES	Coth rank	Csc ES	Csc rank	Csch ES	Csch rank	SFA ES	SFA rank
Aras	0.53	14	0.5	16	0.36	15	0.58	14
Sardasht	0.9	5	0.95	5	0.65	7	0.86	6
Mughan	0.99	3	0.96	4	0.72	2	0.98	3
Zayandehrud	0.82	6	0.86	8	0.61	8	0.75	8
Caron 4	0.53	15	0.64	13	0.41	13	0.43	19
Kohrang	0.36	21	0.41	21	0.31	19	0.26	27
Vaforghan	0.3	26	0.26	26	0.19	26	0.36	23
Rudbar	0.59	12	0.72	10	0.49	9	0.45	18
Amir-Kabir	0.49	17	0.51	15	0.36	16	0.47	16
Taleghan	0.58	13	0.53	14	0.39	14	0.63	11
Klan-lar	0.45	18	0.49	17	0.34	17	0.39	21
Latyan	0.33	25	0.31	25	0.22	23	0.36	24
Levark	0.66	11	0.66	11	0.48	12	0.63	10
Dez	0.8	9	0.78	9	0.48	10	1	2
Abbaspur	0.33	22	0.34	24	0.2	25	0.41	20
Suleiman Mosque	0.42	20	0.44	20	0.26	21	0.47	15
Karkheh	0.51	16	0.49	18	0.3	20	0.66	9
Caron 3	0.33	23	0.35	23	0.21	24	0.38	22
Maroon	0.25	27	0.23	28	0.15	28	0.35	25
Getund	1	1	1	1	0.69	6	1	1
Simre	0.22	28	0.24	27	0.15	27	0.24	28
Kurdistan	0.74	10	0.65	12	0.48	11	0.86	5
Darian	0.45	19	0.47	19	0.31	18	0.45	17
Mulla- Sadra	0.04	29	0.05	29	0.04	29	0.03	29
Darudzan	0.81	8	0.89	7	0.71	4	0.59	13
Jiroft	0.98	4	0.98	3	0.71	3	0.95	4
Sefidroud	1	2	1	2	0.79	1	0.83	7
Takam	0.82	7	0.89	6	0.69	5	0.62	12
Sieh Bisheh	0.33	24	0.38	22	0.24	22	0.3	26

TABLE 8. Correlation between several models by ranks of power plants

	Clayton	Gumbel	A12	Cot	Coth	Csc	Csch	SFA
Clayton	1	0.995	0.997	1	0.998	0.987	0.972	0.916
Gumbel	0.995	1	0.992	0.996	0.998	0.995	0.978	0.902
A12	0.997	0.992	1	0.997	0.995	0.983	0.96	0.933
Cot	1	0.996	0.997	1	0.998	0.988	0.972	0.916
Coth	0.998	0.998	0.995	0.998	1	0.991	0.976	0.909
Csc	0.987	0.995	0.983	0.988	0.991	1	0.985	0.875
Csch	0.972	0.978	0.96	0.972	0.976	0.985	1	0.82
SFA	0.916	0.902	0.933	0.916	0.909	0.875	0.82	1

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