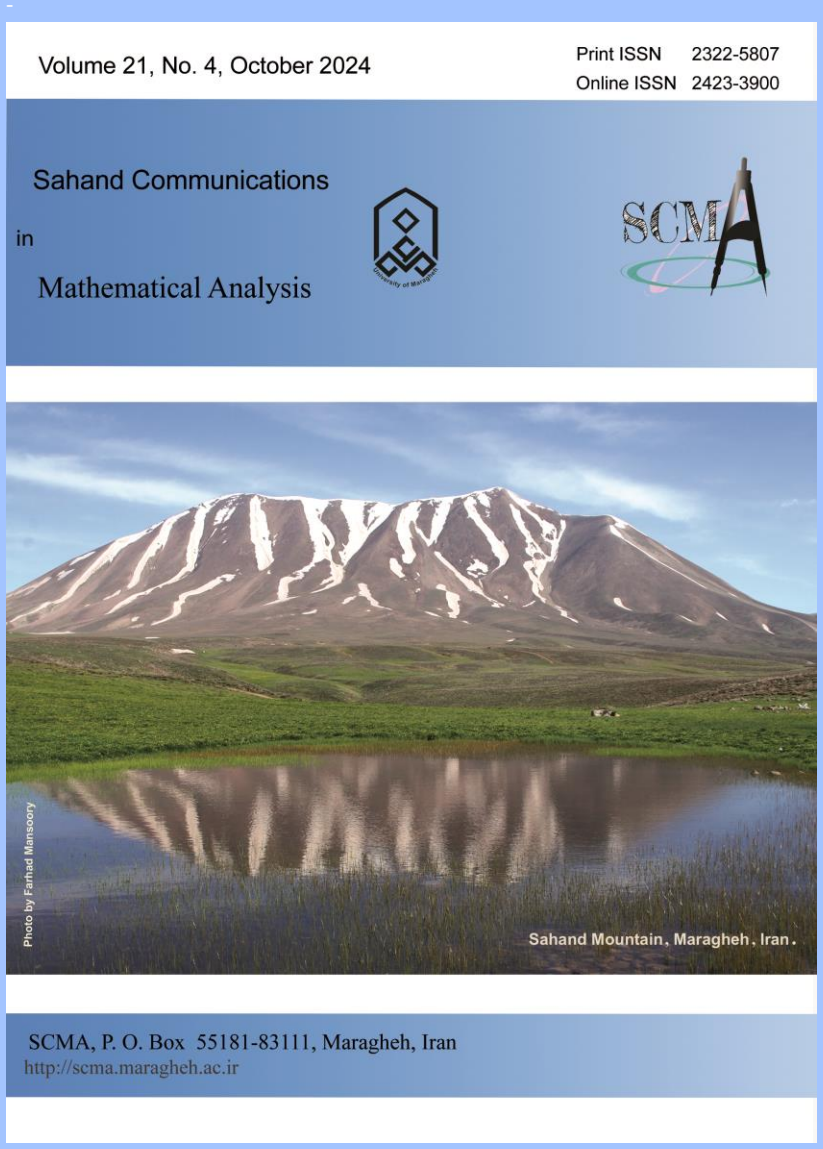
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Norm Retrievable Wavelet Systems

Mahdieh sadat Aghaei[1] and Mohammad Ali Hasankhani Fard[2*]

Abstract. This paper focuses on the norm retrieval problem for wavelet transform. The wavelet transform associated with an admissible wavelet is norm retrieval. However, this problem is difficult in special cases, i.e., for wavelet systems. We give some results of the norm retrieval for wavelet systems.

1. Introduction

The concept of frames in a Hilbert space $\mathcal{H}$ was originally introduced by Duffin and Schaeffer in the context of non-harmonic Fourier series [12]. In the last decade, various generalizations of the frames have been proposed such as fusion frames [9], G-frames [23], strong retro Banach frames [17], K-frames, and atomic systems [21], among others.

Given a separable Hilbert space $\mathcal{H}$ with the inner product $\langle .,. \rangle$, a sequence $\{f_k\}_{k=1}^{\infty}$ is called a frame for $\mathcal{H}$ if there exist constants $A > 0$ and $B < \infty$ such that for all $f \in \mathcal{H}$,

$$A \|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \|f\|^2 , \tag{1.1}$$

where A, B are the lower and upper frame bounds, respectively. The second inequality of the frame condition (1.1) is also known as the Bessel condition for the sequence $\{f_k\}_{k=1}^{\infty}$. The sequence $\{f_k\}_{k=1}^{\infty}$ is called a tight frame if $A = B$. A tight frame with frame bound 1 is called a Parseval frame. For more information on frames, refer to [10, 15, 24].

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The Fourier transform is the linear operator $\mathcal{F} : L_1(\mathbb{R}) \to C_0(\mathbb{R})$ is defined by

$$\begin{aligned}\mathcal{F}f(\xi) &= \hat{f}(\xi) \\ &:= \int_{-\infty}^{\infty} e^{-2\pi i\xi x} f(x)\, dx, \quad f \in L_1(\mathbb{R}),\ \xi \in \mathbb{R},\end{aligned}$$

where $C_0(\mathbb{R})$ is the space of all continuous functions on $\mathbb{R}$ which vanish at infinity [10, 11, 14].

The Fourier transform can be extended to a unitary operator on $L_2(\mathbb{R})$, satisfying in Plancherel's equation

$$\left\langle \hat{f}, \hat{g} \right\rangle = \langle f, g \rangle, \quad \forall f, g \in L_2(\mathbb{R}).$$

If $f, \hat{f} \in L_1(\mathbb{R})$, then

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i\xi x} \hat{f}(\xi)\, d\xi, \quad a.e.\ x \in \mathbb{R}.$$

For any $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$, the dilation operator D_a and the translation operator T_b on $L_2(\mathbb{R})$ are defined by

$$(D_a f)(x) = \frac{1}{\sqrt{|a|}} f\left(\frac{x}{a}\right), \qquad (T_b f)(x) = f(x-b), \quad \forall x \in \mathbb{R}.$$

In the following lemma, some properties of the dilation operator D_a and the translation operator T_b are given.

Lemma 1.1 ([10])**.** *The translation operators satisfy the following:*

(a) *T_b is unitary for all $b \in \mathbb{R}$.*
(b) *For each $f \in L_2(\mathbb{R})$, the mapping $y \to T_y f$ is continuous from $\mathbb{R}$ to $L_2(\mathbb{R})$.*

Similar statements hold for D_a, $a \in \mathbb{R} \setminus \{0\}$.

A nonzero function $\psi \in L_1(\mathbb{R})$ is called a wavelet if $\hat{\psi}(0) = 0$, which is equivalent to $\int_{-\infty}^{\infty} \psi(x)\, dx = 0$. A function $\psi \in L_2(\mathbb{R})$ is called admissible if

$$0 < C_\psi := \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\xi)\right|^2}{|\xi|} d\xi < \infty. \tag{1.2}$$

It is well known that if $\psi \in L_1(\mathbb{R})$, then $\hat{\psi}$ is continuous and the admissible condition (1.2) can only be satisfied if $\hat{\psi}(0) = 0$. Additionally, if $\hat{\psi}(0) = 0$ and $\int_{-\infty}^{\infty} (1+|x|)^{\alpha} |\psi(x)|\, dx < \infty$ for some $\alpha > 0$, then the admissibility condition (1.2) is satisfied [11].

For an admissible wavelet ψ, the (continuous) wavelet transform of any function $f \in L_2(\mathbb{R})$ is two variable function $W_\psi f$ which is defined by

$$W_\psi f(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} dx, \quad a \in \mathbb{R} \setminus \{0\}, \ b \in \mathbb{R}.$$

Using the convolution formula, we have

$$W_\psi f(a,b) = f * \tilde{\psi}_a(b), \quad a \in \mathbb{R} \setminus \{0\}, \ b \in \mathbb{R},$$

where $\tilde{\psi}_a := D_a \tilde{\psi}$ and $\tilde{\psi}(x) := \overline{\psi(-x)}$, for all $x \in \mathbb{R}$. A wavelet system associated with an admissible wavelet $\psi \in L_2(\mathbb{R})$, scaling parameter $a > 1$, and translation parameter $b > 0$ is the sequence $\{D_{a^j} T_{kb} \psi\}_{j,k \in \mathbb{Z}}$. A wavelet system $\{D_{a^j} T_{kb} \psi\}_{j,k \in \mathbb{Z}}$ is called a wavelet frame if it is a frame for $L_2(\mathbb{R})$, i.e., if there exist constants $0 < A \leq B < \infty$ such that for all $f \in L_2(\mathbb{R})$,

$$A\|f\|_2^2 \leq \sum_{j,k \in \mathbb{Z}} |\langle f, D_{a^j} T_{kb} \psi \rangle|^2 \leq B\|f\|_2^2, \tag{1.3}$$

where $\|.\|_2$ and $\langle ., . \rangle$ denote the standard norm and inner product on $L_2(\mathbb{R})$.

A wavelet basis is a wavelet system $\{D_{a^j} T_{kb} \psi\}_{j,k \in \mathbb{Z}}$ that forms an orthonormal basis for $L_2(\mathbb{R})$. Wavelets are used in signal processing and many other fields of science and technology. For more information on wavelets, refer to [16, 18, 19].

Signal reconstruction using phase retrievable frames has been an important and interesting subject in physics and engineering [5, 6, 13, 22].

The mathematical study of phase retrievable frames began in 2006 with a landmark paper by, Casazza, and Edidin [4] (see also [8] for the infinite dimensional case). Norm retrievable frames are defined in [3]. The authors obtained the relationship between phase retrievable frames and norm retrievable frames (see also [7] for the infinite dimensional case).

Definition 1.2 ([3, 4])**.**
a) A sequence $\{f_k\}_{k \in I}$ in $\mathcal{H}$ is phase retrieval if, for all $f, g \in \mathcal{H}$ satisfying $|\langle f, f_k \rangle| = |\langle g, f_k \rangle|$ for all $k \in I$, then $f = cg$ for some scalar c with $|c| = 1$.
b) A sequence $\{f_k\}_{k \in I}$ in $\mathcal{H}$ is norm retrieval if, for all $f, g \in \mathcal{H}$ satisfying $|\langle f, f_k \rangle| = |\langle g, f_k \rangle|$ for all $k \in I$, then $\|f\| = \|g\|$.

In the following phase retrieval for wavelet systems is defined.

Definition 1.3 ([1])**.** For an admissible wavelet ψ, the wavelet transform W_ψ is phase retrieval on $\mathcal{M} \subseteq L_2(\mathbb{R})$ with respect to $I \subseteq \mathbb{R}^+ \times \mathbb{R}$, if for

all $f, g \in \mathcal{M}$ satisfying $|W_\psi f(a,b)| = |W_\psi g(a,b)|$ for all $(a,b) \in I$, then $f = cg$ for some scalar c with $|c| = 1$. In particular, for an admissible wavelet ψ, scaling parameter $a > 1$ and translation parameter $b > 0$, the wavelet system $\{D_{a^j} T_{kb} \psi\}_{j,k\in\mathbb{Z}}$ is phase retrieval on $\mathcal{M} \subseteq L_2(\mathbb{R})$, if W_ψ is phase retrieval on $\mathcal{M}$ with respect to $I := \left\{\left(a^{-j}, kba^{-j}\right); \quad j,k \in \mathbb{Z}\right\}$.

The wavelet phase retrieval problem involves characterising the admissible wavelet ψ, the subset $\mathcal{M} \subseteq L_2(\mathbb{R})$ and $I \subseteq \mathbb{R}^+ \times \mathbb{R}$, for which the wavelet transform W_ψ is phase retrieval on $\mathcal{M} \subseteq L_2(\mathbb{R})$ with respect to I. The wavelet phase retrieval problem is difficult and has been solved in some special cases [1, 2, 20].

Wavelet phase retrieval involves reconstruction of a signal f by the absolute value of wavelet transform $W_\psi f$, where ψ is an admissible wavelet [1, 2, 20]. In this paper, we study the norm retrieval for wavelet transform W_ψ(Definition 2.1) and present results on norm retrieval for wavelet systems (Theorems 2.3, 2.4 and 2.12). In particular, we provide a condition for norm retrieval of wavelet system $\{D_{a^j} T_{kb} \psi\}_{j,k\in\mathbb{Z}}$, where $\psi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ is an admissible wavelet with a finite number of vanishing moments (Theorem 2.13).

2. Main Results

In this section, we study wavelet norm retrievability, which we define as follows:

Definition 2.1. For an admissible wavelet ψ, the wavelet transform W_ψ is norm retrieval on $\mathcal{M} \subseteq L_2(\mathbb{R})$ with respect to $I \subseteq \mathbb{R}^+ \times \mathbb{R}$, if for all $f, g \in \mathcal{M}$ satisfying $|W_\psi f(a,b)| = |W_\psi g(a,b)|$ for all $(a,b) \in I$, it follows that $\|f\|_2 = \|g\|_2$. In particular, for an admissible wavelet ψ, scaling parameter $a > 1$ and translation parameter $b > 0$, the wavelet system $\{D_{a^j} T_{kb} \psi\}_{j,k\in\mathbb{Z}}$ is norm retrieval on $\mathcal{M} \subseteq L_2(\mathbb{R})$, if W_ψ is norm retrieval on $\mathcal{M}$ with respect to $I := \left\{\left(a^{-j}, kba^{-j}\right); \quad j,k \in \mathbb{Z}\right\}$.

In the following, an example of the norm retrievability of the wavelet transform is provided. Another example is given in Example 2.15.

Example 2.2. Consider the Haar function $\psi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ is defined by

$$\psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2}, \\ -1 & \frac{1}{2} \le x < 1, \\ o & \text{otherwise.} \end{cases}$$

Then, the wavelet transform W_ψ is norm retrieval on $L_2(\mathbb{R})$ with respect to $I := \left\{\left(a, 2^{-j}k\right) : \ a \in \mathbb{R}^+, j,k \in \mathbb{Z}\right\} \subseteq \mathbb{R}^+ \times \mathbb{R}$. Indeed, if $f, g \in L_2(\mathbb{R})$ satisfy $\left|W_\psi f\left(a, 2^{-j}k\right)\right| = \left|W_\psi g\left(a, 2^{-j}k\right)\right)\right|$ for all $a \in \mathbb{R}^+$ and $j,k \in \mathbb{Z}$,

then $\left|W_\psi f\left(2^{-j}, 2^{-j}k\right)\right| = \left|W_\psi g\left(2^{-j}, 2^{-j}k\right)\right|$ for all $j, k \in \mathbb{Z}$. Using that $\left\{2^{\frac{j}{2}}\psi\left(2^j . - k\right)\right\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L_2(\mathbb{R})$ [10, 11], we have

$$\begin{aligned}
\|f\|_2^2 &= \sum_{j,k\in\mathbb{Z}} \left|\left\langle f, 2^{\frac{j}{2}}\psi\left(2^j . - k\right)\right\rangle\right|^2 \\
&= \sum_{j,k\in\mathbb{Z}} \left|W_\psi f\left(2^{-j}, 2^{-j}k\right)\right|^2 \\
&= \sum_{j,k\in\mathbb{Z}} \left|W_\psi g\left(2^{-j}, 2^{-j}k\right)\right|^2 \\
&= \sum_{j,k\in\mathbb{Z}} \left|\left\langle g, 2^{\frac{j}{2}}\psi\left(2^j . - k\right)\right\rangle\right|^2 \\
&= \|g\|_2^2
\end{aligned}$$

which implies $\|f\|_2 = \|g\|_2$.

The following theorem demonstrates that the norm retrievability of wavelet transform for admissible wavelets is achieved on $\mathcal{M} = L_2(\mathbb{R})$ with respect to $I = \mathbb{R}^+ \times \mathbb{R}$.

Theorem 2.3. *If ψ is an admissible wavelet, then the wavelet transform W_ψ is norm retrieval on $L_2(\mathbb{R})$ with respect to $\mathbb{R}^+ \times \mathbb{R}$.*

Proof. Using [10, Proposition 11.1.1], for all $f, g \in L_2(\mathbb{R})$ satisfying $|W_\psi f(a,b)| = |W_\psi g(a,b)|$ for all $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$, we have

$$\begin{aligned}
C_\psi \|f\|_2^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |W_\psi f(a,b)|^2 \frac{da\,db}{a^2} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |W_\psi g(a,b)|^2 \frac{da\,db}{a^2} \\
&= C_\psi \|g\|_2^2
\end{aligned}$$

and hence $\|f\|_2 = \|g\|_2$. □

Norm retrieval with respect to $I \subseteq \mathbb{R}^+ \times \mathbb{R}$ implies norm retrieval with respect to dense subsets of I.

Theorem 2.4. *If ψ is an admissible wavelet such that the wavelet transform W_ψ is norm retrieval on $L_2(\mathbb{R})$ with respect to $I \subseteq \mathbb{R}^+ \times \mathbb{R}$, then W_ψ is norm retrieval on $L_2(\mathbb{R})$ with respect to any dense subset J of I.*

Proof. Let $(a,b) \in I$. There exists a sequence $\{(a_n, b_n)\}_{n=1}^{\infty} \subseteq J$ such that $a_n \to a$ and $b_n \to b$, as $n \to \infty$. Using Lemma 1.1, for any $f \in L_2(\mathbb{R})$, we have

$$\Big|\, |W_\psi f(a_n, b_n)| - |W_\psi f(a,b)| \,\Big|$$

$$
\begin{aligned}
&\leq |W_\psi f(a_n, b_n) - W_\psi f(a, b)| \\
&= \left| \left\langle f, D_{a_n} T_{\frac{b_n}{a_n}} \psi \right\rangle - \left\langle f, D_a T_{\frac{b}{a}} \psi \right\rangle \right| \\
&= \left| \left\langle f, \left(D_{a_n} T_{\frac{b_n}{a_n}} - D_a T_{\frac{b}{a}} \right) \psi \right\rangle \right| \\
&\leq \|f\|_2 \left\| \left(D_{a_n} T_{\frac{b_n}{a_n}} - D_a T_{\frac{b}{a}} \right) \psi \right\| \\
&\leq \|f\|_2 \left\| (D_{a_n} - D_a) \left(T_{\frac{b_n}{a_n}} \psi \right) \right\| + \|D_a\| \left\| T_{\frac{b_n}{a_n}} \psi - T_{\frac{b}{a}} \psi \right\| \\
&= \|f\|_2 \left\| (D_{a_n} - D_a) \left(T_{\frac{b_n}{a_n}} \psi \right) \right\| + \left\| T_{\frac{b_n}{a_n}} \psi - T_{\frac{b}{a}} \psi \right\| \to 0,
\end{aligned}
$$

as $n \to \infty$ and hence

$$
|W_\psi f(a, b)| = \lim_{n \to \infty} |W_\psi f(a_n, b_n)|. \tag{2.1}
$$

Now let $f, g \in L_2(\mathbb{R})$ satisfying $|W_\psi f(c, d)| = |W_\psi g(c, d)|$ for all $(c, d) \in J$. Let $(a, b) \in I$ and choose $\{(a_n, b_n)\}_{n=1}^{\infty} \subseteq J$ such that $a_n \to a$ and $b_n \to b$, as $n \to \infty$. Using equation (2.1), we have

$$
\begin{aligned}
|W_\psi f(a, b)| &= \lim_{n \to \infty} |W_\psi f(a_n, b_n)| \\
&= \lim_{n \to \infty} |W_\psi g(a_n, b_n)| \\
&= |W_\psi g(a, b)|.
\end{aligned}
$$

Hence by norm retrievability of W_ψ on $L_2(\mathbb{R})$ with respect to I, we have $\|f\|_2 = \|g\|_2$. □

In the rest of paper, we study the norm retrieval problem for wavelet systems. In general, phase retrieval implies norm retrieval. Thus, in all cases wavelet phase retrieval problem which is solved, the results also apply to the wavelet norm retrieval problem. For example, if ψ is a wavelet with a finite number of vanishing moments, then the wavelet system is phase retrieval (and hence norm retrieval) for the set $\mathcal{M}$ of real valued bandlimited functions in $L_2(\mathbb{R})$ [1]. Additionally, in general, tight frames are norm retrieval, and thus tight wavelet frames, and especially wavelet bases, are norm retrieval on $L_2(\mathbb{R})$.

Definition 2.5 ([1])**.** A function $f \in L_1(\mathbb{R})$ has n vanishing moments, where $n \in \mathbb{N}$, if $\int_{-\infty}^{\infty} x^m f(x) = 0$, for $m = 0, 1, \ldots, n$.

It is well known that a function f has n vanishing moments if and only if $\lim_{\xi \to 0} \frac{\hat{f}(\xi)}{\xi^n} = 0$, where $n \in \mathbb{N}$ and $f \in L_1(\mathbb{R})$ such that $x^n f(x) \in L_1(\mathbb{R})$ [16, Lemma 6.0.4].

Definition 2.6 ([1])**.** A function $f \in L_1(\mathbb{R})$ has a finite number of vanishing moments if $\lim_{\xi\to 0} \frac{\hat{f}(\xi)}{\xi^m}$ is a nonzero complex number for some $m \in \mathbb{N}$.

Example 2.7. Consider the function $\psi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ defined by

$$\psi(x) = \begin{cases} e^{-x} - 2e^{-2x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Then $\int_{-\infty}^{\infty} \psi(x)\, dx = 0$ and $\int_{-\infty}^{\infty} (1+|x|)\,|\psi(x)|\, dx = \frac{7}{2}$. Thus, ψ is an admissible wavelet. Moreover, it is easy to show that

$$\hat{\psi}(\xi) = \frac{-\pi i \xi}{(1+\pi i\xi)(1+2\pi i\xi)}$$

for any $\xi \in \mathbb{R}$, and hence $\lim_{\xi\to 0} \frac{\hat{\psi}(\xi)}{\xi} = -\pi i$.

Example 2.8. Consider the function $\psi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ defined by

$$\psi(x) = \begin{cases} \cos x & -\pi \leq x \leq \pi, \\ o & \text{otherwise.} \end{cases}$$

Then $\int_{-\infty}^{\infty} \psi(x)\, dx = \int_{-\infty}^{\infty} x\psi(x)\, dx = 0$, $\int_{-\infty}^{\infty} x^2\psi(x)\, dx = -4\pi$ and $\int_{-\infty}^{\infty} (1+|x|)\,|\psi(x)|\, dx = 2\pi + 4$. Thus, ψ is an admissible wavelet which has vanishing moments 2. It is easy to show that

$$\hat{\psi}(\xi) = \frac{4\pi\xi \sin 2\pi^2\xi}{1 - 4\pi^2\xi^2}$$

for any $\xi \neq \pm\frac{1}{2\pi}$, and hence $\lim_{\xi\to 0} \frac{\hat{\psi}(\xi)}{\xi^2} = 8\pi^3$.

Example 2.9. Let $m \in \mathbb{N}$. Consider the function $\psi_m \in L_2(\mathbb{R})$ whose Fourier transform is given by

$$\hat{\psi}_m(\xi) = \begin{cases} \xi^m & -1 \leq \xi \leq 1, \\ o & \text{otherwise.} \end{cases}$$

Then $\hat{\psi}_m(0) = 0$, $C_{\psi_m} = \int_{-\infty}^{\infty} \frac{|\hat{\psi}_m(\xi)|^2}{|\xi|} d\xi = \frac{1}{m}$, and $\lim_{\xi\to 0} \frac{\hat{\psi}_m(\xi)}{\xi^m} = 1$. Thus, ψ_m is an admissible wavelet and has a finite number of vanishing moments.

For any $m \in \mathbb{N}$ and $\lambda > 0$, we define the subset $\mathcal{M}_\lambda^m$ of $L_2(\mathbb{R})$ by

$$\mathcal{M}_\lambda^m := \left\{ f \in L_2(\mathbb{R}) \cap C^m(\mathbb{R});\ \left\| f^{(m)} \right\|_2 = \lambda \|f\|_2 \right\},$$

where $C^m(\mathbb{R})$ is the space of m times differentiable functions with continuous m-th derivatives.

Example 2.10. Let $\gamma > 0$ be given. For the function $f(x) = e^{-\gamma x^2}$, we have

$$\begin{aligned}
\left\|f'\right\|_2^2 &= \int_{-\infty}^{\infty} \left|f'(x)\right|^2 dx \\
&= 4\gamma^2 \int_{-\infty}^{\infty} x^2 e^{-2\gamma x^2} dx \\
&= 4\gamma^2 \left(\lim_{u\to\infty} \frac{-xe^{-2\gamma x^2}}{4\gamma}\Big|_{-u}^{u} + \frac{1}{4\gamma} \int_{-\infty}^{\infty} e^{-2\gamma x^2} dx \right) \\
&= \gamma \int_{-\infty}^{\infty} e^{-2\gamma x^2} dx \\
&= \gamma \left\|f\right\|_2^2
\end{aligned}$$

and

$$\begin{aligned}
\left\|f''\right\|_2^2 &= \int_{-\infty}^{\infty} \left|f''(x)\right|^2 dx \\
&= 4\gamma^2 \int_{-\infty}^{\infty} \left(2\gamma x^2 - 1\right)^2 e^{-2\gamma x^2} dx \\
&= 4\gamma^2 \left(4\gamma^2 \int_{-\infty}^{\infty} x^4 e^{-2\gamma x^2} dx - 4\gamma \int_{-\infty}^{\infty} x^2 e^{-2\gamma x^2} dx + \int_{-\infty}^{\infty} e^{-2\gamma x^2} dx \right) \\
&= 4\gamma^2 \left(3\gamma \int_{-\infty}^{\infty} x^2 e^{-2\gamma x^2} dx - 4\gamma \int_{-\infty}^{\infty} x^2 e^{-2\gamma x^2} dx + \int_{-\infty}^{\infty} e^{-2\gamma x^2} dx \right) \\
&= 4\gamma^2 \left(\int_{-\infty}^{\infty} e^{-2\gamma x^2} dx - \gamma \int_{-\infty}^{\infty} x^2 e^{-2\gamma x^2} dx \right) \\
&= 3\gamma^2 \int_{-\infty}^{\infty} e^{-2\gamma x^2} dx \\
&= 3\gamma^2 \left\|f\right\|_2^2.
\end{aligned}$$

Thus $\left\|f'\right\|_2 = \sqrt{\gamma}\left\|f\right\|_2$ and $\left\|f''\right\|_2 = \sqrt{3}\gamma\left\|f\right\|_2$, which implies that $f \in \mathcal{M}^1_{\sqrt{\gamma}} \cap \mathcal{M}^2_{\sqrt{3}\gamma}$.

To study norm retrieval for wavelet transform on $\mathcal{M}^m_\lambda$ with respect to $\left\{a^{-j};\ j \in \mathbb{N}\right\} \times \mathbb{R}$, where $a > 1$, we need the following lemma:

Lemma 2.11. *If $\psi \in L_1(\mathbb{R})$ is an admissible wavelet such that $c := \lim_{\xi\to 0} \frac{\hat{\psi}(\xi)}{\xi^m} \neq 0$ and $f \in \mathcal{M}^m_\lambda$ for some $\lambda > 0$, then*

$$\left\|f^{(m)}\right\|_2 = \frac{(2\pi)^m}{|c|} \lim_{a\to 0^+} \frac{\left\|f * \tilde{\psi}_a\right\|_2}{a^{m+\frac{1}{2}}}.$$

Proof. For any $f \in \mathcal{M}_\lambda^m$, $f * \tilde{\psi}_a \in L_2(\mathbb{R})$ by [14, Proposition 2.39]. Using $\lim_{\xi\to 0} \frac{\hat{\psi}(\xi)}{\xi^m} = c$, there exists $\delta > 0$ such that $\left|c - \frac{\hat{\psi}(\xi)}{\xi^m}\right| < 1$ for all $\xi \in (-\delta, \delta)$. Also using $\hat{\psi} \in C_0(\mathbb{R})$, there exists $M > \delta$ such that $\left|\hat{\psi}(\xi)\right| < 1$ for all $\xi \in I_M := (-\infty, -M) \cup (M, \infty)$ and hence $\left|c - \frac{\hat{\psi}(\xi)}{\xi^m}\right| \leq |c| + \frac{1}{M^m}$ for all $\xi \in I_M$. The continuity of the function $\xi \to \left|c - \frac{\hat{\psi}(a\xi)}{(a\xi)^m}\right|^2$ on compact set $J_M := [-M, -\delta] \cup [\delta, M]$ implies that $M' := \sup_{\xi\in J_M} \left|c - \frac{\hat{\psi}(\xi)}{\xi^m}\right|^2 < \infty$. Using $a\mathbb{R} := \{a\xi : \xi \in \mathbb{R}\} = \mathbb{R}$ and $\mathbb{R} = I_M \cup (-\delta, \delta) \cup J_M$, we have

$$\begin{aligned}
\mu :&= \sup_{\xi\in\mathbb{R}} \left|c - \frac{\hat{\psi}(a\xi)}{(a\xi)^m}\right|^2 \\
&= \sup_{\xi\in a\mathbb{R}} \left|c - \frac{\hat{\psi}(\xi)}{\xi^m}\right|^2 \\
&= \sup_{\xi\in\mathbb{R}} \left|c - \frac{\hat{\psi}(\xi)}{\xi^m}\right|^2 \\
&\leq \max\left\{1, \left(|c| + \frac{1}{M^m}\right)^2, M'\right\} \\
&< \infty.
\end{aligned}$$

Additionally, for any $f \in \mathcal{M}_\lambda^m$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \left|\xi^m \hat{f}(\xi)\right|^2 &= \frac{1}{(2\pi)^m} \left\|\widehat{f^{(m)}}\right\|_2^2 \\
&= \frac{\lambda^2}{(2\pi)^m} \|f\|_2^2 \\
&< \infty.
\end{aligned}$$

Using Plancherel's theorem and Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\left|\left\|\frac{c}{(2\pi i)^m} f^{(m)}\right\|_2 - a^{-\left(m+\frac{1}{2}\right)} \left\|f * \tilde{\psi}_a\right\|_2\right|^2 \\
&\quad\leq \left\|\frac{\bar{c}}{(2\pi i)^m} f^{(m)} - a^{-\left(m+\frac{1}{2}\right)} f * \tilde{\psi}_a\right\|_2^2 \\
&\quad= \int_{-\infty}^{\infty} \left|\frac{\bar{c}}{(2\pi i)^m} \widehat{f^{(m)}}(\xi) - a^{-\left(m+\frac{1}{2}\right)} \hat{f}(\xi)\, \hat{\tilde{\psi}}_a(\xi)\right|^2 d\xi
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \left| \frac{\bar{c}}{(2\pi i)^m} (2\pi i\xi)^m \hat{f}(\xi) - a^{-m} \hat{f}(\xi) \overline{\hat{\psi}_a(a\xi)} \right|^2 d\xi$$
$$= \int_{-\infty}^{\infty} \left| \xi^m \hat{f}(\xi) \right|^2 \left| c - \frac{\hat{\psi}(a\xi)}{(a\xi)^m} \right|^2 d\xi \to 0 \text{ as } a \to 0^+.$$

Hence,

$$\left\| f^{(m)} \right\|_2 = \frac{(2\pi)^m}{|c|} \lim_{a\to 0^+} \frac{\left\| f * \tilde{\psi}_a \right\|_2}{a^{m+\frac{1}{2}}}. \qquad \square$$

In the next theorem, a sufficient condition for norm retrieval of wavelet transform with respect to $\left\{a^{-j};\ j \in \mathbb{N}\right\} \times \mathbb{R}$, where $a > 1$ is given.

Theorem 2.12. *Suppose $m \in \mathbb{N}$, $\lambda > 0$, and $a > 1$. If $\psi \in L_1(\mathbb{R})$ is an admissible wavelet such that $c := \lim_{\xi\to 0} \frac{\hat{\psi}(\xi)}{\xi^m} \neq 0$, then the wavelet transform W_ψ is norm retrieval on $\mathcal{M}_\lambda^m$ with respect to $\left\{a^{-j};\ j \in \mathbb{N}\right\} \times \mathbb{R}$.*

Proof. Let $f, g \in \mathcal{M}_\lambda^m$ satisfy $\left|W_\psi f\left(a^{-j}, b\right)\right| = \left|W_\psi g\left(a^{-j}, b\right)\right|$ for all $j \in \mathbb{N}$ and for all $b \in \mathbb{R}$. Using lemma 2.11, we have

$$\begin{aligned}
\left\| f^{(m)} \right\|_2 &= \frac{(2\pi)^m}{|c|} \lim_{j\to\infty} \frac{\left\| f * \tilde{\psi}_{a^{-j}} \right\|_2}{a^{-j\left(m+\frac{1}{2}\right)}} \\
&= \frac{(2\pi)^m}{|c|} \lim_{j\to\infty} \frac{\left(\int_{-\infty}^{\infty} \left|W_\psi f\left(a^{-j}, b\right)\right|^2 db \right)^{\frac{1}{2}}}{a^{-j\left(m+\frac{1}{2}\right)}} \\
&= \frac{(2\pi)^m}{|c|} \lim_{j\to\infty} \frac{\left(\int_{-\infty}^{\infty} \left|W_\psi g\left(a^{-j}, b\right)\right|^2 db \right)^{\frac{1}{2}}}{a^{-j\left(m+\frac{1}{2}\right)}} \\
&= \frac{(2\pi)^m}{|c|} \lim_{j\to\infty} \frac{\left\| g * \tilde{\psi}_{a^{-j}} \right\|_2}{a^{-j\left(m+\frac{1}{2}\right)}} \\
&= \left\| g^{(m)} \right\|_2.
\end{aligned}$$

Hence,

$$\|f\|_2 = \|g\|_2. \qquad \square$$

In the next theorem, a sufficient condition for norm retrieval of a wavelet system is provided.

Theorem 2.13. *Suppose $m \in \mathbb{N}$, $\lambda > 0$, $a > 1$, and $b > 0$. If $\psi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ is an admissible wavelet such that $c := \lim\limits_{\xi\to 0} \frac{\hat{\psi}(\xi)}{\xi^m} \neq 0$, then the wavelet system $\{D_{a^j} T_{kb} \psi\}_{j,k\in\mathbb{Z}}$ is norm retrieval on the set $\mathcal{M}$*

of all $f \in \mathcal{M}_{\lambda}^{m}$ *such that for any* $i, j \in \mathbb{N}, k \in \mathbb{Z}$, $\left|W_{\psi} f\left(a^{-i}, kba^{-j}\right)\right| = \lambda_{ij}\left|W_{\psi} f\left(a^{-j}, kba^{-j}\right)\right|$, *for some* $\lambda_{ij} > 0$ *which is independent of* f.

Proof. Let $f, g \in \mathcal{M}$ satisfy $\left|W_{\psi} f\left(a^{-j}, kba^{-j}\right)\right| = \left|W_{\psi} g\left(a^{-j}, kba^{-j}\right)\right|$ for all $j, k \in \mathbb{Z}$. Thus,

$$\begin{aligned}\left|W_{\psi} f\left(a^{-i}, kba^{-j}\right)\right| &= \lambda_{ij}\left|W_{\psi} f\left(a^{-j}, kba^{-j}\right)\right| \\ &= \lambda_{ij}\left|W_{\psi} g\left(a^{-j}, kba^{-j}\right)\right| \\ &= \left|W_{\psi} g\left(a^{-i}, kba^{-j}\right)\right|\end{aligned}$$

for any $i, j \in \mathbb{N}$ and $k \in \mathbb{Z}$. Thus,

$$\left|W_{\psi} f\left(a^{-i}, kba^{-j}\right)\right| = \left|W_{\psi} g\left(a^{-i}, kba^{-j}\right)\right| \tag{2.2}$$

for any $i, j \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Let $\beta \in \mathbb{R}$ and $j \in \mathbb{N}$. Set $k_j := \left\lfloor \frac{a^j \beta}{b} \right\rfloor \in \mathbb{Z}$. Now we have $k_j \leq \frac{a^j \beta}{b} < k_j + 1$. Hence, $0 \leq \left|\beta - k_j b a^{-j}\right| < b a^{-j} \to 0$ when $j \to \infty$. Thus, $\{k_j\}_{j=1}^{\infty} \subset \mathbb{Z}$ and $\beta = \lim_{j \to \infty} k_j a^{-j}$. For any $i \in \mathbb{N}$, we have

$$\left|k_j b a^{i-j} - \beta a^i\right| = \left|a^i\right|\left|k_j b a^{-j} - \beta\right| \to 0, \text{ as } j \to \infty.$$

Hence $\left\|\left(T_{k_j b a^{i-j}} - T_{\beta a^i}\right)\psi\right\| \to 0$ as $j \to \infty$ by Lemma 1.1. Also $\left\|D_{a^{-i}}\right\| = 1$ by Lemma 1.1. Now for any $f \in \mathcal{M}$, we have

$$\begin{aligned}&\left|\left|W_{\psi} f\left(a^{-i}, k_j b a^{-j}\right)\right| - \left|W_{\psi} f\left(a^{-i}, \beta\right)\right|\right| \\ &\qquad \leq \left|W_{\psi} f\left(a^{-i}, k_j b a^{-j}\right) - W_{\psi} f\left(a^{-i}, \beta\right)\right| \\ &\qquad = \left|\left\langle f, D_{a^{-i}} T_{k_j b a^{i-j}} \psi\right\rangle - \left\langle f, D_{a^{-i}} T_{\beta a^i} \psi\right\rangle\right| \\ &\qquad = \left|\left\langle f, D_{a^{-i}} T_{k_j b a^{i-j}} \psi - D_{a^{-i}} T_{\beta a^i} \psi\right\rangle\right| \\ &\qquad = \left|\left\langle f, \left(D_{a^{-i}} T_{k_j b a^{i-j}} - D_{a^{-i}} T_{\beta a^i}\right)\psi\right\rangle\right| \\ &\qquad = \left|\left\langle f, D_{a^{-i}}\left(T_{k_j b a^{i-j}} - T_{\beta a^i}\right)\psi\right\rangle\right| \\ &\qquad \leq \|f\|_2 \left\|D_{a^{-i}}\right\| \left\|\left(T_{k_j b a^{i-j}} - T_{\beta a^i}\right)\psi\right\| \\ &\qquad = \|f\|_2 \left\|\left(T_{k_j b a^{i-j}} - T_{\beta a^i}\right)\psi\right\| \to 0, \text{ as } j \to \infty\end{aligned}$$

by Cauchy Schwarz inequality. Additionally, for any $i \in \mathbb{N}$, we have

$$\left|W_{\psi} f\left(a^{-i}, \beta\right)\right| = \lim_{j \to \infty}\left|W_{\psi} f\left(a^{-i}, k_j b a^{-j}\right)\right|. \tag{2.3}$$

Using equations (2.2) and (2.3), we have

$$\left|W_{\psi} f\left(a^{-i}, \beta\right)\right| = \left|W_{\psi} g\left(a^{-i}, \beta\right)\right|$$

for any $i \in \mathbb{N}$. Hence $\|f\|_2 = \|g\|_2$ by Theorem 2.12. □

Corollary 2.14. *Suppose $m \in \mathbb{N}$, $\lambda > 0$, $a > 1$, and $b > 0$. If $\psi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ is an admissible wavelet such that $c := \lim_{\xi \to 0} \frac{\hat{\psi}(\xi)}{\xi^m} \neq 0$, then the wavelet system $\{D_{a^j} T_{kb} \psi\}_{j,k \in \mathbb{Z}}$ is norm retrieval on*

$$\mathcal{M}_\lambda^m \cap \overline{Span} \left\{ \psi \left(a^j . - kba^l \right) ;\ j \in \mathbb{N},\ k, l \in \mathbb{Z} \right\}^\perp .$$

Proof. For any $i, j \in \mathbb{N}$ and $k \in \mathbb{Z}$, it is easy to show that

$$\left| W_\psi f \left(a^{-i}, kba^{-j} \right) \right| = 0 = \left| W_\psi f \left(a^{-j}, kba^{-j} \right) \right|$$

for any $f \in \mathcal{M}_\lambda^m \cap \overline{Span} \left\{ \psi \left(a^j . - kba^l \right) ;\ j \in \mathbb{N},\ k, l \in \mathbb{Z} \right\}^\perp$. Now the result is obtained from Theorem 2.13. □

Example 2.15. Using Corollary 2.14, for given $\lambda > 0$, the wavelet system $\{D_{a^j} T_{kb} \psi\}_{j,k \in \mathbb{Z}}$, where ψ is the wavelet introduced in Example 2.7, is norm retrieval on the set $\mathcal{M}$ of all $f \in \mathcal{M}_\lambda^1$ which

$$\int_0^\infty f \left(a^{-j} x + kba^{l-j} \right) \left(e^{-x} - 2e^{-2x} \right) dx = 0; \quad \forall j \in \mathbb{N},\ \forall k, l \in \mathbb{Z}.$$

Also using Corollary 2.14, for given $\lambda > 0$, the wavelet system $\{D_{a^j} T_{kb} \psi\}_{j,k \in \mathbb{Z}}$, where ψ is the wavelet introduced in Example 2.8, is norm retrieval on the set $\mathcal{M}$ of all $f \in \mathcal{M}_\lambda^2$ which

$$\int_{-\pi}^{\pi} f \left(a^{-j} x + kba^{l-j} \right) \cos x dx = 0; \quad \forall j \in \mathbb{N},\ \forall k, l \in \mathbb{Z}.$$

Remark 2.16. In this paper, we study the special case of the norm retrieval problem for wavelet transforms, i.e., for admissible wavelets with a finite number of vanishing moments. The general problem remains an open problem.

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