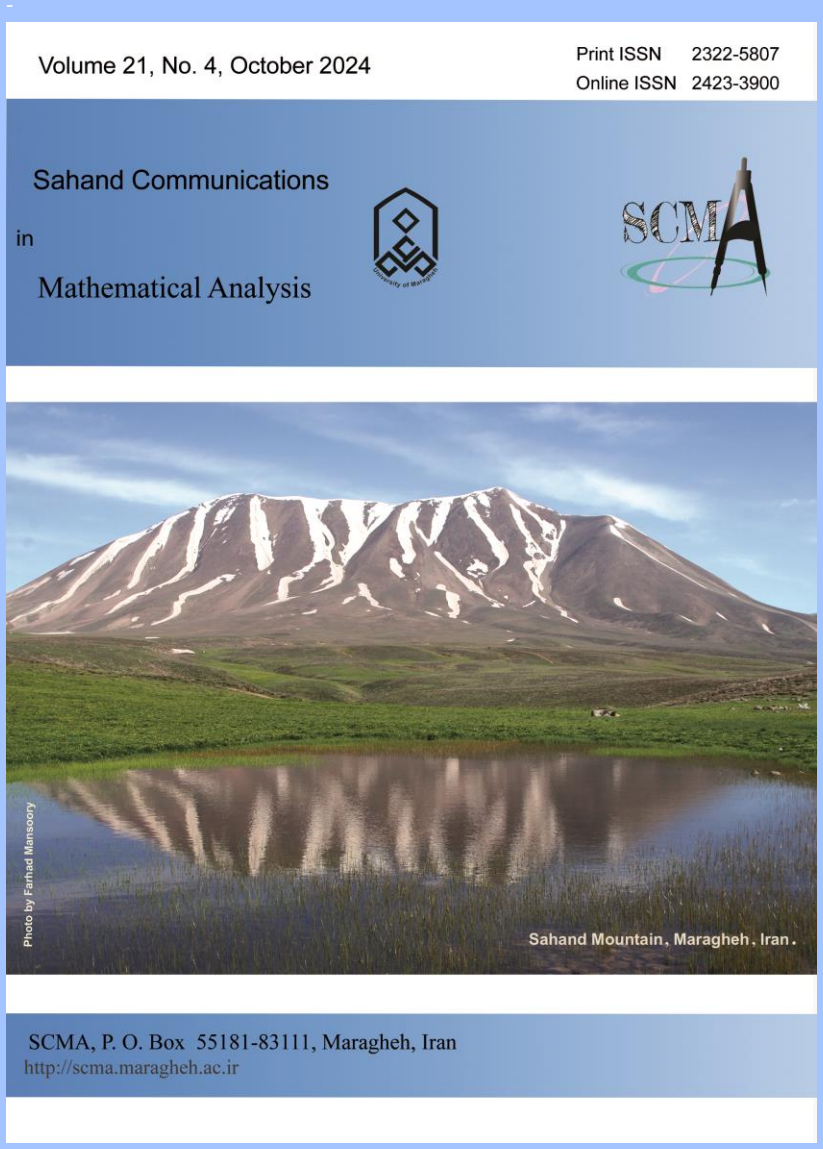
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Chebyshev Weighted Type Integral Inequality in Fuzzy and Abstract Spaces

Bayaz Daraby[1*] and Zahra Vaezi[2]

Abstract. In this paper, we express and prove Chebyshev weighted type inequality for fuzzy integrals and in abstract spaces where the functions are strictly monotone functions. Furthermore, we have shown our results for n-th strictly monotone functions. The theorems are illustrated with several examples.

1. Introduction

The fuzzy concept was first introduced by by Zadeh [16] in 1965, which soon spread to all branches of science. In 1974, a scientist named Sugeno [13] introduced the fuzzy integral using the fuzzy concept, which is also known as the Sugeno integral. The Sugeno integral is similar to the Lebesgue integral, with the difference that addition and multiplication are replaced by the supremum and infimum. Two main classes of fuzzy integral are Choquet and pseudo integrals both of which are idempotent, continuous and monotonic operators. Fuzzy integral inequalities have many applications in decision theory, optimization, information retrieval, solving differential equations, statistics and probability including concepts like the arithmetic mean and geometric mean. The features and applications of fuzzy measures and fuzzy integral have been investigated and studied by many reseachers around the world.

Ralesco and Adams [11] have provided several equivalent definitions for fuzzy integral. Wang and Kiler [15] have a general approach for fuzzy measurability and fuzzy integrals. Agahi and et al.[1] developed the Chebyshev type integral inequalities for the fuzzy integral. Later,

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* Corresponding author.

Ouyang and Fang [9] generalized main results for Sugeno integral of monotone functions based on Lebesgue measure, Subsequently Ouyang [6] presented a generalization of the fuzzy Chebyshev-type inequalities to the kind of arbitrary fuzzy measures. Recently, Daraby and his colleagues [3] proved generalizations of related Fritz Carlson type and some important inequalities for Sugeno integral [2, 4, 5].

The Chebyshev type integral inequality is one of the most important inequalities that has many applications in mathematics, such as normal distribution (or any arbitrary distribution) except for cases where the mean and variance are unknown and prove the weak law of large numbers. In [14] Tong stated and proved the Chebyshev type integral inequality as follows:

Theorem 1.1. *Let $f, h \in L[0,1]$ be two real-valued, continuous and conomonotone functions and let μ be the Lebesgue measure on $\mathbb{R}$. Then*

$$\int_0^1 f(x)h(x)\ d\mu \geq \int_0^1 f(x)\ d\mu \int_0^1 h(x)\ d\mu$$

holds.

In fact Chebyshev type integral inequality is special case of the weighted type integral inequlity which is mentioned as follows:

Theorem 1.2 ([8])**.** *Let $f, h \in L[0,1]$ be integrable functions, both increasing or both decreasing. Furthermore, let w from $[0,1]$ onto $\mathbb{R}^+$ be an integrable function. Then*

$$\int_a^b w(x)f(x)h(x)\ dx \int_a^b w(x)\ dx \geq \int_a^b w(x)f(x)\ dx \int_a^b w(x)h(x)\ dx$$

holds.

If one of the functions is increasing and the other decreasing, the reverse inequality holds.

The Chebyshev weighted type integral inequality's application along with the Littlewood type integral inequality is the Grusse-Barens and Borel type integral inequality's [7] proof in classic context.

The main purpose of this context is to prove the Chebyshev weighted-type inequality for the Sugeno integral, which is organized as follows: In the second part, we provide basic definitions and theorems that are used in the subsequent sections. In the third part, we present the main theorems along with examples.

2. Preliminaries

The basic definitions and theorems of Sugeno integral which will need in the next, are as follows:

Definition 2.1 ([15])**.** Let $\sum$ be a σ-algebra of subsets of X and let $\mu : \sum \to [0, \infty]$ be a non-negative, extended real-valued set function, we call that μ is a fuzzy measure if and only if:

(FM1) $\mu(\emptyset) = 0$;
(FM2) $M, N \in \sum$ and $M \subseteq N$ supply $\mu(M) \leq \mu(N)$(monotonicity feature);
(FM3) $(M_n) \subseteq \sum$, $M_1 \subseteq M_2 \subseteq \cdots$ supply $\lim_{n\to\infty} \mu(M_n) = \mu\left(\bigcup_{i=1}^{\infty} M_n\right)$ (continuity the lower);
(FM4) $(M_n) \subseteq \sum$, $M_1 \supseteq M_2 \supseteq \cdots$ and $\mu(M_1) < \infty$ supply $\lim_{n\to\infty} \mu(M_n) = \mu\left(\bigcap_{i=1}^{\infty} M_n\right)$ (continuity the upper).

If f is a function on X with conditons of previous definition, we shall symbolized $F_\alpha = \{x \in X \mid f(x) \geq \alpha\} = \{f \geq \alpha\}$.

It is interesting to note that: if $\alpha \leq \beta$, then $\{f \geq \beta\} \subseteq \{f \geq \alpha\}$.

we denote a set $\mathfrak{F}^\mu$ with fuzzy measure μ as below:

$$\mathfrak{F}^\mu(X) = \{f : X \to [0, \infty) | \quad \text{f is } \mu - \text{measurable}\}.$$

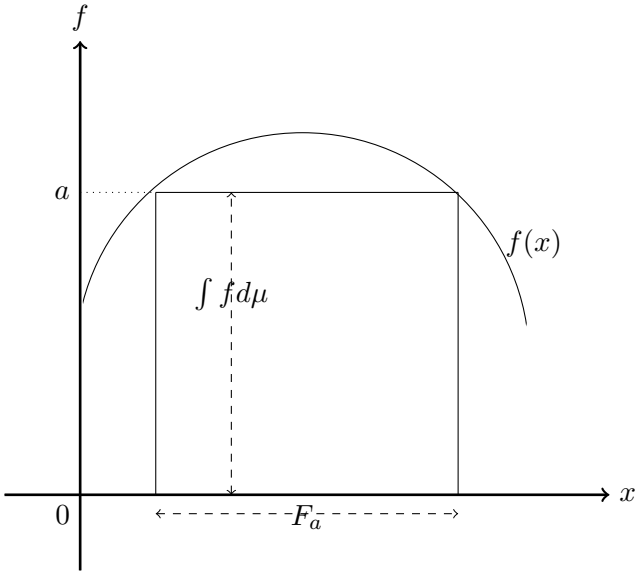
Definition 2.2 ([2])**.** Le μ be a fuzzy measure on $(X, \sum)$. If $f \in \mathfrak{F}^\mu(X)$ and $A \in \sum$, then the Sugeno integral of f on A, with considering to the fuzzy measure μ, is denoted as:

$$\oint_A f \, d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)),$$

where $\vee$, $\wedge$ symbolize the operations *sup* and *inf* on $[0, \infty)$. Specifically, if $A = X$ then:

$$\begin{aligned} \oint_X f \, d\mu &= \oint f \, d\mu \\ &= \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)). \end{aligned}$$

If $X = (-\infty, \infty)$, $\mathfrak{F}$ is the Borel filed $\mathfrak{B}$, μ is the Lebesgue measure and $f : X \to [0, \infty)$ is a unimodal continuous function, then the geometric of $\oint f \, d\mu$ is the edge's length of the largest square between the curve of $f(x)$ and the $x-$axis (see the following figure).



Remark 2.3. Assume that $F(\alpha) = \mu(A \cap F_\alpha)$, if $\mu(A) < \infty$, with utilize these features: $\fint_A f \ d\mu \geq \alpha \Leftrightarrow \mu(A \cap \{f \geq \alpha\}) \geq \alpha$ and $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow \fint_A f \ d\mu \leq \alpha$, it is interesting to note that

$$F(\alpha) = \alpha \quad \Rightarrow \quad \fint_A f \ d\mu = \alpha.$$

Thus, in terms of numerical calculation, the Sugeno integral can be estimated by solving the equation $F(\alpha) = \alpha$.

It is well known that Sugeno integral is a type of nonlinear integral [13], i.e, in general case:

$$\fint (af + bg) \ d\mu = a \fint f \ d\mu + b \fint g \ d\mu,$$

doesn't hold.

Roman-Flores and Aores-Franulic have demonstrated optimal Sugeno integral inequalities for monotone functions in [12], as follows:

Theorem 2.4 ([12])**.** *Let μ be the Lebesgue measure on $\mathbb{R}$ and $g : [0, \infty] \to [0, \infty]$ be a continuous and strictly increasing function. If $\fint_0^a g \ d\mu = p$, then*

$$g(a - p) \geq \fint_0^1 g \ d\mu = p, \tag{2.1}$$

for all $a \geq 0$. Additionally, if $0 < p < a$, then $g(a - p) = p$.

Theorem 2.5 ([12]). *Let μ be the Lebesgue measure on $\mathbb{R}$ and $g : [0,\infty] \to [0,\infty]$ be a continuous and strictly decreasing function. If $\oint_0^a g\ d\mu = p$, then*

$$g(p) \geq \oint_0^1 g\ d\mu = p,$$

for all $a \geq 0$. Additionally, if $0 < p < a$, then $g(p) = p$.

Ouyang [10] has studied comonotonic $-\star-$ property for Sugeno integral as follows:

Theorem 2.6. *Let $f, g \in \mathfrak{F}^{\mu}(X)$ and μ be an arbitrary fuzzy measure such that $\oint f\ d\mu$ and $\oint g\ d\mu$ are finite. And let $\star : [0,\infty]^2 \to [0,\infty]$ be continuous and non-decreasing in both arguments and bounded from above by mimimum. If f, g are comonotone, then the inequality*

$$\oint_A f \star g\ d\mu \geq \left(\oint_A f\ d\mu\right) \star \left(\oint_A g\ d\mu\right) \tag{2.2}$$

holds.

In view of the fact that

$$\oint_A f \star g\ d\mu \leq \left(\oint_A f\ d\mu\right) \star \left(\oint_A g\ d\mu\right)$$

holds for comonotone functions $f, g \in \mathfrak{F}^{\mu}(X)$ whenever $\star \geq max$.

It is of great interest to determine the operator $\star$ such that

$$\oint_A f \star g\ d\mu = \left(\oint_A f\ d\mu\right) \star \left(\oint_A g\ d\mu\right) \tag{2.3}$$

holds for any comonotone functions f, g and for fuzzy measure μ and any measurable set A.

Definition 2.7 ([3, 10]). Let $\star : [0,\infty]^2 \to [0,\infty]$ is a binary operation. We say that the Sugeno integral possesses comonotonic $-\star-$ property, if the equality (2.3) holds for any fuzzy measure space $(X, \sum, \mu)$, any measurable set A and for any comotone function $f, g : X \to [0,\infty]$.

3. Main Results

In this section, we state and prove the fuzzy Chebyshev weighted type integral inequality for strictly monotone functions.

Theorem 3.1. *Let f, h and $w \in L[0,1]$ be real-valued, measurable and strictly increasing functions where μ be the Lebesgue measure on $\mathbb{R}$, then the inequality*

$$\left(\oint_0^1 w.f.h\ d\mu\right)\left(\oint_0^1 w\ d\mu\right) \geq \left(\oint_0^1 w.f\ d\mu\right)\left(\oint_0^1 w.h\ d\mu\right) \tag{3.1}$$

holds.

Proof. Suppose that

$$\oint_0^1 w.f.h\ d\mu = r, \quad \oint_0^1 w\ d\mu = s, \quad \oint_0^1 w.f\ d\mu = p, \quad \oint_0^1 w.g\ d\mu = q,$$

we will show that,

$$rs \geq pq.$$

The proof will be divided into three parts:

(A) According this prposition: $\oint_A f\ d\mu \leq \mu(A)$, we conclude r, s, p and $q \in [0, 1]$.
    If r or s or p or $q = 0$, then at least one of the functions will be almost every where zero, hence (3.1) inequality holds.

(B) By the geometric significance of fuzzy integral and with the characteristic of strictly increasing functions, we must have r, s, p and $q \neq 1$.

(C) We show the proof for r, s, p and $q \in (0, 1)$.

Let r, s, p and $q \in (0, 1)$, we know that, $rs < r$, $rs < s$, $pq < p$ and $pq < q$.

Thus we have,

$$1 - rs > 1 - r, \qquad 1 - rs > 1 - s, \tag{3.2}$$

$$1 - pq > 1 - p, \qquad 1 - pq > 1 - q. \tag{3.3}$$

From f, h and w are strictly increasing functions, we see that $w.f.h$ is strictly increasing function. By (2.1) (Theorem 2.5) and (3.2) we obtain,

$$w.f.h(1 - rs) > w.f.h(1 - r) = r, \tag{3.4}$$

since w is strictly increasing function. By (2.1) (Theorem 2.5) and (3.2) we have,

$$w(1 - rs) > w(1 - s) = s, \tag{3.5}$$

from (3.4) and (3.5) we obtain,

$$w.f.h(1 - rs).w(1 - rs) > rs. \tag{3.6}$$

As f and w are strictly increasing functions, we see $w.f$ is strictly increasing function. From (2.1) (Theorem 2.5) and (3.3) we give,

$$w.f(1 - pq) > w.f(1 - p) = p, \tag{3.7}$$

also h and w are strictly increasing functions, we know $w.h$ is strictly increasing function. By (2.1) (Theorem 2.5) and (3.3) we get,

$$w.h(1 - pq) > w.h(1 - q) = q. \tag{3.8}$$

If we suppose,

$$rs < pq, \tag{3.9}$$

then we must have,

$$1 - rs > 1 - pq.$$

Since $w.f$ and $w.h$ is strictly increasing functions, from (2.1) (Theorem (2.6)), we have,

$$w.f(1 - rs) > w.f(1 - pq), \tag{3.10}$$

$$w.h(1 - rs) > w.h(1 - pq), \tag{3.11}$$

comparing (3.7) with (3.10) we obtain,

$$w.f(1 - rs) > w.f(1 - pq) > w.f(1 - p) = p, \tag{3.12}$$

again comparing (3.8) with (3.11) we have,

$$w.h(1 - rs) > w.h(1 - pq) > w.h(1 - q) = q. \tag{3.13}$$

Multiply (3.12) by (3.13) we obtain,

$$w.f.h(1 - rs).w(1 - rs) > pq. \tag{3.14}$$

So from (3.6) we conclude (3.14), hence we have,

$$rs > pq, \tag{3.15}$$

However, (3.15) contradicts (3.9), in fact,

$$rs \geq pq.$$

The proof is completed. □

We illustrate the Theorem 3.1 with the following example:

Example 3.2. Let $f(x) = x^3$, $h(x) = \sqrt{x^3}$ and $w(x) = \sqrt[3]{x}$ be strictly increasing functions on $[0, 1]$ and μ be the Lebesgue measure, we have:

$$\fint_0^1 w.f.h \; d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(w.f.h \geq \alpha)] = 0.24960,$$

$$\fint_0^1 w \; d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(w \geq \alpha)] = 0.68232,$$

$$\fint_0^1 w.f \; d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(w.f \geq \alpha)] = 0.30185,$$

$$\fint_0^1 w.h \; d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(w.h \geq \alpha)] = 0.39636.$$

Therefore,

$$0.17030 = \left(\fint_0^1 w.f.h \; d\mu\right)\left(\fint_0^1 w \; d\mu\right)$$

$$> \left(\fint_0^1 w.f \, d\mu\right)\left(\fint_0^1 w.h \, d\mu\right)$$
$$= 0.11964.$$

Theorem 3.3. *Let f, h and $w \in L[0,1]$ be real-valued measurable and strictly decreasing functions and μ be the Lebesgue measure on $\mathbb{R}$, then the inequality*

$$\left(\fint_0^1 w.f.h \, d\mu\right)\left(\fint_0^1 w \, d\mu\right) \geq \left(\fint_0^1 w.f \, d\mu\right)\left(\fint_0^1 w.h \, d\mu\right)$$

holds.

Proof. Applying Thoerem 2.6, the proof is similar to Thoerem 3.1. □

Remark 3.4. When one of the functions is strictly monotone in opposite direction, the reverse fuzzy Chebyshev weighted type inequality holds.

We show this in the following example:

Example 3.5. Let $f(x) = 1 + x$ be strictly increasing function and $h(x) = 1 - x$ and $w(x) = \frac{1}{1+x}$ be strictly decreasing functions on $[0,1]$, we obtain:

$$\fint_0^1 w.f.h \, d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(w.f.h \geq \alpha)] = 0.50000,$$
$$\fint_0^1 w \, d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(w \geq \alpha)] = 0.618040,$$
$$\fint_0^1 w.f \, d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(w.f \geq \alpha)] = 1,$$
$$\fint_0^1 w.h \, d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(w.h \geq \alpha)] = 0.50000.$$

These follows that,

$$0.30902 = \left(\fint_0^1 w.f.h \, d\mu\right)\left(\fint_0^1 w \, d\mu\right)$$
$$< \left(\fint_0^1 w.f \, d\mu\right)\left(\fint_0^1 w.h \, d\mu\right)$$
$$= 0.50000.$$

Corollary 3.6. *Let $f_1, f_2, \ldots, f_n$ and w be such that any two of them are strictly monotone functions and conditions of Thoerem 3.1 are the*

same. Then

$$\left(\oint_0^1 w.\left(f_1.f_2.\ldots.f_n\right)\ d\mu\right)\left(\oint_0^1 w\ d\mu\right) \geq \left(\oint_0^1 w.f_1\ d\mu\right)\left(\oint_0^1 w.f_2\ d\mu\right)\cdots\left(\oint_0^1 w.f_n\ d\mu\right)$$

holds.

Proof. Induction assumption: Since f_1 and f_2 are strictly monotone functions, from Theorem 3.1 for $n = 2$, we have,

$$\left(\oint_0^1 w.(f_1.f_2)\ d\mu\right)\left(\oint_0^1 w\ d\mu\right) \geq \left(\oint_0^1 w.f_1\ d\mu\right)\left(\oint_0^1 w.f_2\ d\mu\right).$$

Rule of induction: If we let for every $n \geq 2$, P_n, the following inequality,

$$\left(\oint_0^1 w.\left(f_1.f_2.\ldots.f_n\right)d\mu\right)\left(\oint_0^1 w\ d\mu\right) \geq \left(\oint_0^1 w.f_1\ d\mu\right)\left(\oint_0^1 w.f_2\ d\mu\right)\cdots\left(\oint_0^1 w.f_n\ d\mu\right),$$

holds, then from the Theorems 3.1 and assumption of induction we obtain the following inequality

$$\left(\oint_0^1 \left(w.\left(f_1.f_2.\ldots.f_n\right).f_{n+1}\right)\ d\mu\right)\left(\oint_0^1 w\ d\mu\right) \geq \left(\oint_0^1 w.f_1\ d\mu\right)\left(\oint_0^1 w.f_2\ d\mu\right)\cdots\left(\oint_0^1 w.f_n\ d\mu\right)\left(\oint_0^1 w.f_{n+1}\ d\mu\right).$$

Therefore for every $n \geq 2$, corollary is true. □

In the following, we state and prove the Chebyshev weighted type inequality in abstract spaces.

Theorem 3.7. *Let f, h and $w \in L[0,1]$ be strictly increasing functions and μ be an arbitrary measure and let $\star : [0,\infty)^2 \to [0,\infty)$, then the inequality*

$$\left(\oint_0^1 \left(w\left(x\right)\star f\left(x\right)\star h\left(x\right)\right)\ d\mu\left(x\right)\right)\star\left(\oint_0^1 w(x)\ d\mu\left(x\right)\right) \geq \left(\oint_0^1 \left(w(x)\star f(x)\right)\ d\mu(x)\right)\star\left(\oint_0^1 \left(w(x)\star h(x)\right)\ d\mu(x)\right) \tag{3.16}$$

holds.

Proof. From (2.2) (Theorem 2.6), we obtain,

$$\left(\oint_0^1 \left(\left(w\left(x\right)\star f\left(x\right)\right)\star h\left(x\right)\right)\ d\mu\left(x\right)\right)\star\left(\oint_0^1 w(x)\ d\mu(x)\right)$$

$$\geq \left(\fint_0^1 \left(\left(w\left(x\right) \star f\left(x\right) \right) d\mu\left(x\right) \right) \star \left(\fint_0^1 w(x)\ d\mu\left(x\right) \right) \right.$$
$$\left. \star \left(\fint_0^1 h(x)\ d\mu\left(x\right) \right) . \right.$$

According to the comonotone $-\star-$ property for comonotone functions (Definition 2.7), we have,

$$\left(\fint_0^1 \left(w\left(x\right) \star f\left(x\right) \star h\left(x\right) \right)\ d\mu\left(x\right) \right) \star \left(\fint_0^1 w(x)\ d\mu\left(x\right) \right)$$
$$\geq \left(\fint_0^1 \left(w(x) \star f(x) \right)\ d\mu(x) \right) \star \left(\fint_0^1 \left(w(x) \star h(x) \right)\ d\mu(x) \right) .$$

The proof is completed. □

We illustrate the Theorem (3.6) with some examples:

Example 3.8. Let $f(x) = x^4$, $h(x) = x^3$ and $w(x) = x^2$ be strictly increasing functions on $[0, 1]$, $\mu = \sqrt{m}$ be the Lebesgue measure and $\star \in \{\vee, \wedge, PL, PF\}$.

1) If $\star = \vee$, we have:

$$\left(\fint_0^1 \left(x^2 \vee x^4 \vee x^3 \right) d\mu(x) \right) \vee \left(\fint_0^1 x^2 d\mu(x) \right) = \fint_0^1 x^2\ d\mu(x)$$
$$= 0.524889,$$

and

$$\left(\fint_0^1 (x^2 \vee x^4) d\mu(x) \right) \vee \left(\fint_0^1 (x^2 \vee x^3) d\mu(x) \right) = 0.471074.$$

From this, we conclude that:

$$0.524889 = \left(\fint_0^1 w \star f \star h\ d\mu \right) \star \left(\fint_0^1 w\ d\mu \right)$$
$$> \left(\fint_0^1 w \star f\ d\mu \right) \star \left(\fint_0^1 w \star h\ d\mu \right)$$
$$= 0.471074.$$

2) If $\star = \wedge$, we get:

$$\left(\fint_0^1 \left(x^2 \wedge x^4 \wedge x^3 \right) d\mu(x) \right) \wedge \left(\fint_0^1 x^2\ d\mu(x) \right) = \fint_0^1 x^2 d\mu(x)$$
$$= 0.524889,$$

and

$$\left(\fint_0^1 (x^2 \wedge x^4) d\mu(x) \right) \wedge \left(\fint_0^1 (x^2 \wedge x^3) d\mu(x) \right) = 0.524889.$$

This shows that:

$$\begin{aligned} 0.524889 &= \left(\oint_0^1 w \star f \star h \ d\mu\right) \star \left(\oint_0^1 w \ d\mu\right) \\ &= \left(\oint_0^1 w \star f \ d\mu\right) \star \left(\oint_0^1 w \star h \ d\mu\right) \\ &= 0.524889. \end{aligned}$$

3) If $\star = PF$, we calculate:

$$\begin{aligned} PF\left(\oint_0^1 PF(x^2, x^4, x^3) \ d\mu(x), \oint_0^1 x^2 \ d\mu(x)\right) &= \oint_0^1 x^2 d\mu(x) \\ &= 0.524889 \end{aligned}$$

and

$$PF\left(\oint_0^1 PF(x^2, x^4) d\mu(x), \oint_0^1 PF(x^2, x^3) d\mu(x)\right) = 0.524889.$$

This shows that:

$$\begin{aligned} 0.524889 &= \left(\oint_0^1 w \star f \star h \ d\mu\right) \star \left(\oint_0^1 w \ d\mu\right) \\ &= \left(\oint_0^1 w \star f \ d\mu\right) \star \left(\oint_0^1 w \star h \ d\mu\right) \\ &= 0.524889. \end{aligned}$$

4) If $\star = PL$, we obtain:

$$\begin{aligned} PL\left(\oint_0^1 PL(x^2, x^4, x^3) \ d\mu(x), \oint_0^1 x^2 \ d\mu(x)\right) &= \oint_0^1 x^2 d\mu(x) \\ &= 0.524889 \end{aligned}$$

and

$$PL\left(\oint_0^1 PL(x^2, x^4) d\mu(x), \oint_0^1 PL(x^2, x^3) d\mu(x)\right) = 0.471074.$$

This shows that:

$$\begin{aligned} 0.524889 &= \left(\oint_0^1 w \star f \star h \ d\mu\right) \star \left(\oint_0^1 w \ d\mu\right) \\ &> \left(\oint_0^1 w \star f \ d\mu\right) \star \left(\oint_0^1 w \star h \ d\mu\right) \\ &= 0.471074. \end{aligned}$$

Example 3.9. Suppose

$$f(x) = \begin{cases} \frac{1}{3}x^2 & \text{if } x \in [0, \frac{1}{2}] \\ \sqrt{x} & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$$

$$w(x) = \begin{cases} x^3 & \text{if } x \in [0, \frac{1}{2}] \\ 6\sqrt{x} & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$$

and $h(x) = 3\sqrt{x}$ are strictly increasing functions on $[0, 1]$ and $\mu = \dfrac{m}{2}$ and $\star \in \{\vee, \wedge, PL, PF\}$, we have,

1) If $\star = \vee$, we obtain:

$$\left(\fint_{\frac{1}{2}}^{1} (6\sqrt{x} \vee \sqrt{x} \vee 3\sqrt{x})\ d\mu(x)\right) \vee \left(\fint_{\frac{1}{2}}^{1} 6\sqrt{x}\ d\mu(x)\right) = \left(\fint_{\frac{1}{2}}^{1} (6\sqrt{x}\ d\mu(x)\right) = 0.496600,$$

and

$$\left(\fint_{\frac{1}{2}}^{1} (6\sqrt{x} \vee \sqrt{x}) d\mu(x)\right) \vee \left(\fint_{\frac{1}{2}}^{1} (6\sqrt{x} \vee 3\sqrt{x}) d\mu(x)\right) = 0.496600.$$

This shows that:

$$\begin{aligned} 0.496600 &= \left(\fint_{\frac{1}{2}}^{1} w \star f \star h\ d\mu\right) \star \left(\fint_{\frac{1}{2}}^{1} w\ d\mu\right) \\ &= \left(\fint_{\frac{1}{2}}^{1} w \star f\ d\mu\right) \star \left(\fint_{\frac{1}{2}}^{1} w \star h\ d\mu\right) \\ &= 0.496600. \end{aligned}$$

2) If $\star = \wedge$, we get:

$$\left(\fint_{0}^{\frac{1}{2}} (x^3 \wedge \frac{1}{3}x^2 \wedge 3\sqrt{x})\ d\mu(x)\right) \wedge \left(\fint_{0}^{\frac{1}{2}} x^3\ d\mu(x)\right) = \fint_{0}^{\frac{1}{2}} x^3 d\mu(x) = 0.57270,$$

and

$$\left(\fint_{0}^{\frac{1}{2}} (x^3 \wedge \frac{1}{3}x^2) d\mu(x)\right) \wedge \left(\fint_{0}^{\frac{1}{2}} (x^3 \wedge 3\sqrt{x}) d\mu(x)\right) = 0.52179.$$

This shows that:

$$\begin{aligned} 0.57270 &= \left(\oint_0^{\frac{1}{2}} w \star f \star h \; d\mu \right) \star \left(\oint_0^{\frac{1}{2}} w \; d\mu \right) \\ &> \left(\oint_0^{\frac{1}{2}} w \star f \; d\mu \right) \star \left(\oint_0^{\frac{1}{2}} w \star h \; d\mu \right) \\ &= 0.52179. \end{aligned}$$

3) If $\star = PF$, we calculate:

$$\begin{aligned} PF\left(\oint_{\frac{1}{2}}^1 PF((6\sqrt{x}, \sqrt{x}, 3\sqrt{x})) \; d\mu(x), \oint_{\frac{1}{2}}^1 6\sqrt{x} d\mu(x) \right) &= \oint_{\frac{1}{2}}^1 \left(6\sqrt{x} d\mu(x)\right) \\ &= 0.496600, \end{aligned}$$

and

$$PF\left(\oint_{\frac{1}{2}}^1 PF(6\sqrt{x}, \sqrt{x}) d\mu(x), \oint_{\frac{1}{2}}^{\frac{1}{2}} PF(6\sqrt{x}, 3\sqrt{x}) d\mu(x) \right) = 0.496600.$$

This shows that:

$$\begin{aligned} 0.496600 &= \left(\oint_{\frac{1}{2}}^1 w \star f \star h \; d\mu \right) \star \left(\oint_{\frac{1}{2}}^1 w \; d\mu \right) \\ &= \left(\oint_{\frac{1}{2}}^1 w \star f \; d\mu \right) \star \left(\oint_{\frac{1}{2}}^1 w \star h \; d\mu \right) \\ &= 0.496600. \end{aligned}$$

4) If $\star = PL$, we calculate:

$$\begin{aligned} PL\left(\oint_{\frac{1}{2}}^1 PL((6\sqrt{x}, \sqrt{x}, 3\sqrt{x})) d\mu(x), \oint_{\frac{1}{2}}^1 6\sqrt{x} \; d\mu(x) \right) &= \oint_{\frac{1}{2}}^1 \left(6\sqrt{x} \; d\mu(x)\right) \\ &= 0.496600, \end{aligned}$$

and

$$PL\left(\oint_{\frac{1}{2}}^1 PL(6\sqrt{x}, \sqrt{x}) d\mu(x), \oint_{\frac{1}{2}}^{\frac{1}{2}} PL(6\sqrt{x}, 3\sqrt{x}) d\mu(x) \right) = 0.24662.$$

This shows that:

$$\begin{aligned} 0.496600 &= \left(\fint_{\frac{1}{2}}^{1} w \star f \star h \; d\mu \right) \star \left(\fint_{\frac{1}{2}}^{1} w \; d\mu \right) \\ &> \left(\fint_{\frac{1}{2}}^{1} w \star f \; d\mu \right) \star \left(\fint_{\frac{1}{2}}^{1} w \star h \; d\mu \right) \\ &= 0.24662. \end{aligned}$$

Theorem 3.10. *Let f, h and $w \in L[0,1]$ be strictly decreasing functions and μ be the Lebesgue measure and let $\star : [0,\infty)^2 \to [0,\infty)$, then the inequality*

$$\begin{aligned} &\left(\fint_0^1 \left(w\left(x\right) \star f\left(x\right) \star h\left(x\right)\right) \; d\mu\left(x\right) \right) \star \left(\fint_0^1 w(x) \; d\mu\left(x\right) \right) \\ &\quad \geq \left(\fint_0^1 (w(x) \star f(x)) \; d\mu(x) \right) \star \left(\fint_0^1 (w(x) \star h(x)) \; d\mu(x) \right) \end{aligned}$$

holds.

Proof. By using Thoerem 2.6, the proof is similar to Thoerem (3.6). □

Remark 3.11. When one of the functions is strictly monotone in opposite direction, the reverse fuzzy Chebeyshev weighted inequality for abstract spaces holds.

We illustrate it with the following example:

Example 3.12. Let $f(x) = 2x$, $h(x) = 1 - x$ and $w(x) = \frac{x}{2}$ which $f(x)$ and $w(x)$ are strictly increasing and $h(x)$ be strictly decreasing functions on $\left[0, \frac{1}{2}\right]$ and $\mu = m^2$ be the Lebesgue measure and $\star = \wedge$, we obtain:

$$\begin{aligned} \left(\fint_0^{\frac{1}{2}} \left(\frac{x}{2} \wedge 2x \wedge 1 - x \right) \; d\mu(x) \right) \wedge \left(\fint_0^{\frac{1}{2}} \frac{x}{2} \; d\mu(x) \right) &= \fint_0^{\frac{1}{2}} \frac{x}{2} \; d\mu(x) \\ &= 0.095401, \end{aligned}$$

and

$$\left(\fint_0^{\frac{1}{2}} \left(\frac{x}{2} \wedge 2x \right) d\mu(x) \right) \wedge \left(\fint_0^{\frac{1}{2}} (\frac{x}{2} \wedge 1 - x) d\mu(x) \right) = 0.133975,$$

This shows that:

$$\begin{aligned} 0.095401 &= \left(\int\!\!\!\!\!\!- _0^{\frac{1}{2}} w \star f \star h \; d\mu \right) \star \left(\int\!\!\!\!\!\!- _0^{\frac{1}{2}} w \; d\mu \right) \\ &< \left(\int\!\!\!\!\!\!- _0^{\frac{1}{2}} w \star f \; d\mu \right) \star \left(\int\!\!\!\!\!\!- _0^{1} w \star h \; d\mu \right) \\ &= 0.133975. \end{aligned}$$

Corollary 3.13. *Let $f_1, f_2, \ldots, f_n$ and w be such that any two of them are strictly monotone functions and the conditions of Theorem* (3.6) *are satisfied. Then*

$$\begin{aligned} &\left(\int\!\!\!\!\!\!- _0^{1} w \star (f_1 \star f_2 \star \cdots \star f_n) \; d\mu \right) \star \left(\int\!\!\!\!\!\!- _0^{1} w \; d\mu \right) \\ &\quad \geq \left(\int\!\!\!\!\!\!- _0^{1} w \star f_1 \; d\mu \right) \star \left(\int\!\!\!\!\!\!- _0^{1} w \star f_2 \; d\mu \right) \star \cdots \star \left(\int\!\!\!\!\!\!- _0^{1} w \star f_n \; d\mu \right) \end{aligned}$$

holds.

Proof. Induction assumption: Since f_1 and f_2 are strictly monotone functions, from Theorem (3.6) for $n = 2$, we have,

$$\begin{aligned} &\left(\int\!\!\!\!\!\!- _0^{1} w \star (f_1 \star f_2) \; d\mu \right) \star \left(\int\!\!\!\!\!\!- _0^{1} w \; d\mu \right) \\ &\quad \geq \left(\int\!\!\!\!\!\!- _0^{1} w \star f_1 \; d\mu \right) \star \left(\int\!\!\!\!\!\!- _0^{1} w \star f_2 \; d\mu \right). \end{aligned}$$

Rule of induction: If we let for every $n \geq 2$ the following inequality

(3.17)

$$\begin{aligned} &\left(\int\!\!\!\!\!\!- _0^{1} w \star (f_1 \star f_2 \star \cdots \star f_n) \; d\mu \right) \star \left(\int\!\!\!\!\!\!- _0^{1} w \; d\mu \right) \\ &\quad \geq \left(\int\!\!\!\!\!\!- _0^{1} w \star f_1 \; d\mu \right) \star \left(\int\!\!\!\!\!\!- _0^{1} w \star f_2 \; d\mu \right) \star \cdots \star \left(\int\!\!\!\!\!\!- _0^{1} w \star f_n \; d\mu \right). \end{aligned}$$

holds, then from the Theorem (3.6) and assumption of the induction, we obtaine the following inequality

$$\begin{aligned} &\left(\int\!\!\!\!\!\!- _0^{1} (w \star (f_1 \star f_2 \star \cdots \star f_n) \star f_{n+1}) \; d\mu \right) \star \left(\int\!\!\!\!\!\!- _0^{1} w \; d\mu \right) \\ &\quad \geq \left(\int\!\!\!\!\!\!- _0^{1} w \star f_1 \; d\mu \right) \star \left(\int\!\!\!\!\!\!- _0^{1} w \star f_2 \; d\mu \right) \star \cdots \end{aligned}$$

$$\star\left(\int_0^1 w\star f_n\ d\mu\right)\star\left(\int_0^1 w\star f_{n+1}\ d\mu\right).$$

Therefore for every $n\geq 2$, corollary is true. □

4. Conclusion

We introduced a general version of Chebyshev type inequality for the Sugeno integral on fuzzy measure space and abstract spaces. For further investigations we propose to consider the Chebyshev type inequality for Choquet, pseudo and quantum integrals.

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[1] Department of Mathematics, University of Maragheh, P. O. Box 55136-553, Maragheh, Iran.
Email address: bdaraby@maragheh.ac.ir

[2] Department of Mathematics, University of Maragheh, P. O. Box 55136-553, Maragheh, Iran.
Email address: zahraavaezii@gmail.com