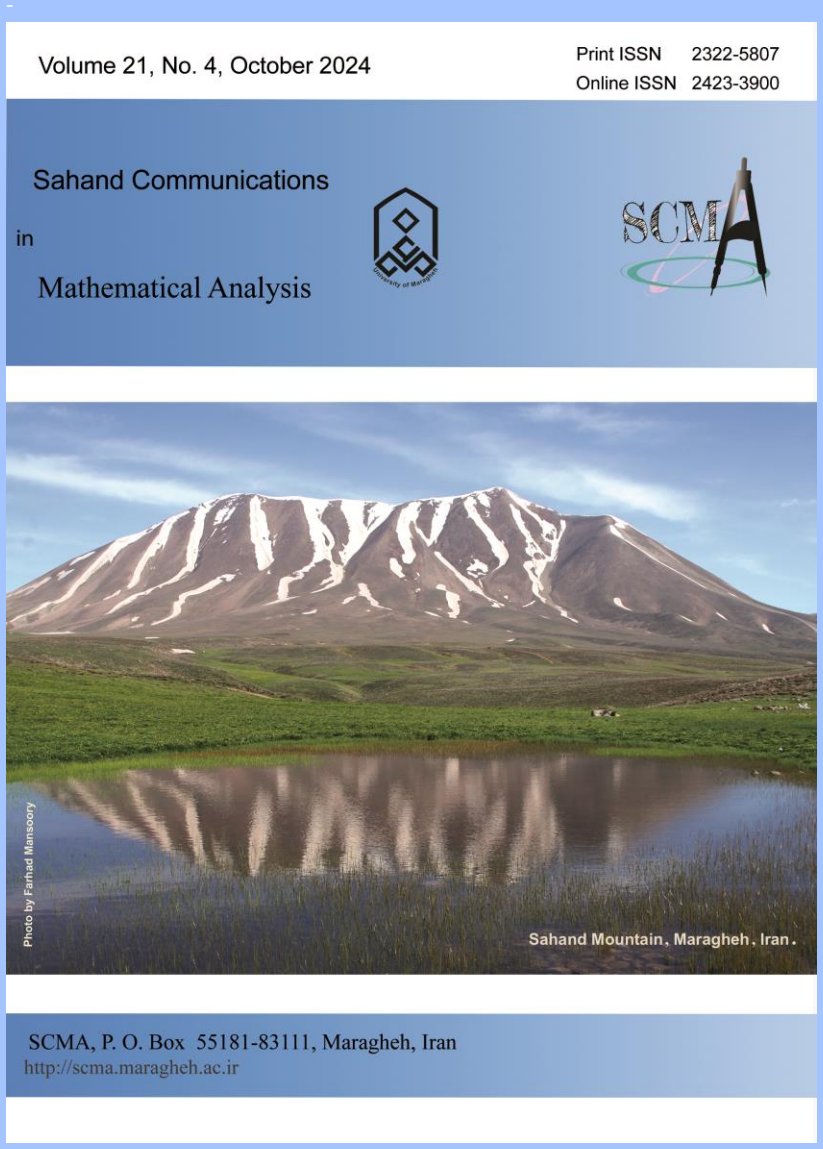
Analyzing Functions Mapping the Unit Disk to Limaçon Domain

Vali Soltani Masih and Shahram Najafzadeh

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 4
Pages: 223-239

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2024.2016346.1542



Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 4 (2024), 223-239
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2024.2016346.1542

Analyzing Functions Mapping the Unit Disk to Limaçon Domain

Vali Soltani Masih[1*] and Shahram Najafzadeh[2]

Abstract. This paper focuses on a particular type of starlike univalent function, namely $(1-sz)(1+z)$ with $-1/3 \leq s \leq 1/3$. This function maps the open unit disk $\mathbb{D}$ onto a Limaçon domain denoted by

$$\left[(u-1)^2+v^2-s^2\right]^2 < (1-s)^2\left[(u-1-s)^2+v^2\right].$$

The primary investigation concerns on analytic function families where zh'/h maps the unit disk to a subset of this domain. The paper provides the lower and upper bounds of the real and imaginary parts of these functions and highlights some distinguishing characteristics of these bounds. Furthermore, the study employed subordination theory to derive conclusions and corollaries for functions belonging to the aforementioned class.

1. Introduction

The class $\mathcal{A}$ consists of analytic functions $h(z)$ in $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ of the form

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n. \tag{1.1}$$

The subclass of $\mathcal{A}$ consisting of univalent functions h in $\mathbb{D}$ is denoted by $\mathcal{S}$. A function $h \in \mathcal{S}$ is said to be starlike (with respect to w_0) if $tw + (1-t)w_0 \in h(\mathbb{D})$ for all $w \in h(\mathbb{D})$ and all $t \in [0,1]$. We denote by $\mathcal{S}^*$ the class of all starlike functions in $\mathbb{D}$ with respect to $w_0 = 0$. In total

2020 *Mathematics Subject Classification.* 30C45, 30C80.
Key words and phrases. Univalent functions, Subordination, Starlike functions, Domain bounded by Limaçon.
Received: 23 November 2023, Accepted: 12 June 2024.
*Corresponding author.

expression, A function $h \in \mathcal{A}$ is said to be starlike of order $0 \leq \beta < 1$, written $h \in \mathcal{S}^*(\beta)$, if it satisfies

$$\Re \frac{zh'(z)}{h(z)} > \beta, \quad (z \in \mathbb{D}).$$

If F and H are analytic in $\mathbb{D}$, then we say that F is subordinate to H in $\mathbb{D}$, written as $F(z) \prec H(z)$, if there exists a self-map function $\omega(z)$, which is analytic in $\mathbb{D}$, satisfies $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $F(z) = H(\omega(z))$. If H is univalent in $\mathbb{D}$, then $F \prec H$ if and only if

$$F(\mathbb{D}) \subset H(\mathbb{D}), \qquad F(0) = H(0).$$

Further details on this property can be found in [4].

Ma and Minda [15] studied analytic functions Θ in the open unit disk $\mathbb{D}$ that have a positive real part, map $\mathbb{D}$ onto regions starlike with respect to $\Theta(0) = 1$, symmetric about the real axis, and satisfy the normalization conditions $\Theta(0) = 1, \Theta'(0) > 0$. They introduced classes based on such functions, including $\mathcal{ST}(\Theta)$, which

$$\mathcal{ST}(\Theta) = \left\{ h \in \mathcal{A} \colon \frac{zh'(z)}{h(z)} \prec \Theta(z), \quad z \in \mathbb{D} \right\}.$$

For $-1 \leq B < A \leq 1$, the subclass $\mathcal{ST}[A, B]$ of $\mathcal{S}$ consists of Janowski starlike functions [6], defined as:

$$\mathcal{ST}[A, B] := \mathcal{ST}\left(\frac{1 + Az}{1 + Bz}\right).$$

The special case $\mathcal{S}^*(\beta) := \mathcal{ST}[1-2\beta, -1]$ is known as the class of starlike functions of order $0 \leq \beta < 1$, and when $\beta = 0$, i.e., $A = 1$ and $B = -1$, it reduces to the usual class $\mathcal{S}^*$ of starlike functions. The subclass $\mathcal{SST}(\gamma) := \mathcal{ST}(((1 + z)/(1 - z))^\gamma)$ consists of strongly starlike functions of order $0 < \gamma \leq 1$, and the subclass $\mathcal{S}^*_C := \mathcal{ST}(\Theta_1)$ with

$$\Theta_1(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$$

is related to a cardioid and has been investigated by Sharma et al. [24]. The subclasses $\mathcal{S}^*_L$ and $\mathcal{ST}_{1/2}$ were introduced by Sokół [27] and Kanas et al. [10], respectively. Several functions such as

$$(1 + z)^s, \quad (0 < |s| \leq 1),$$
$$z + \sqrt{1 + z^2},$$
$$e^z,$$
$$\frac{2}{1 + e^{-z}},$$

$$\sqrt{2}-\left(\sqrt{2}-1\right)\left(\frac{1-z}{1+2(\sqrt{2}-1)z}\right)^{1/2},$$

were considered in place of $\Theta(z)$ in [5, 16, 19, 20, 23, 26]. Other Ma-Minda type classes and its properties can be found in papers such as [7–9, 11–14, 18, 28, 29, 33].

The class $\mathcal{N}$ is defined as the set of analytic functions h in $\mathcal{A}$ such that

$$\Re\frac{zh'(z)}{h(z)}>-\frac{1}{2},$$

for z in the open unit disk $\mathbb{D}$. It has been shown in [22] that $\mathcal{N}$ consists of univalent functions that are starlike in one direction.

Ma-Minda functions are a class of functions that have important properties in the theory of geometric functions theory. These properties include the ability to obtain coefficients, logarithmic coefficients and the q-derivative. These functions have been extensively studied in the literature and there are many papers which provide detailed information on their properties and applications such as [1, 2, 30–32].

A limaçon is a curve that can be described in polar coordinates by the equation

$$r=b-a\cos\theta, \tag{1.2}$$

where a and b are real numbers and θ lies in the interval $[0,2\pi)$. When converted to Cartesian coordinates, equation (1.2) becomes (see, e.g. [17])

$$\left(x^2+y^2+ax\right)^2=b^2\left(x^2+y^2\right) \tag{1.3}$$

which can be parametrized as

$$\begin{cases} x=(b-a\cos\theta)\cos\theta, \\ y=(b-a\cos\theta)\sin\theta. \end{cases} \tag{1.4}$$

The purpose of this study is to explore the geometric properties of functions in the class $\mathcal{ST}_s$. The paper introduces a family of analytic functions and discusses their properties. It also determines various sharp radius constants for the class $\mathcal{ST}_s$. Moreover, necessary and sufficient conditions for certain special members of $\mathcal{A}$ to be in the class $\mathcal{ST}_s$ are derived. Finally, using subordination theory, the paper derives conclusions and corollaries for functions belonging to the class investigated in this study.

In [3, 17] domains bounded by the curve limaçon have been investigated and researched for various functions. This paper investigates functions given by the relation (2.1) on their domain, which is a type of Limaçon.

2. A Family of Analytic Functions

The family of analytic functions

$$\mathscr{H}_s(z) = (1 - sz)(1 + z) \quad \text{for} \quad z \in \mathbb{D} \tag{2.1}$$

is considered for $-1 \leq s < 1$ and $z \in \mathbb{D}$. From the fact that the function $z+a_2z^2$ is univalent and starlike if and only if $|a_2| \leq 1/2$ [4], the following lemma is concluded:

Lemma 2.1. *Let*

$$g(z) := \frac{\mathscr{H}_s(z) - 1}{1 - s} = z - \frac{s}{1 - s}z^2 \quad \textit{for} \quad -1 \leq s < 1,$$

where the function $\mathscr{H}_s(\cdot)$ defined by (2.1)*. Then*

$$g(z) \in \mathcal{S}^* \quad \Longleftrightarrow \quad -1 \leq s \leq \frac{1}{3}.$$

The function $\mathscr{H}_s(\cdot)$ defined by equation (2.1) is an univalent and starlike function with respect to $\mathscr{H}_s(0) = 1$ for $-1 \leq s \leq 1/3$.

The behavior of $\mathscr{H}_s$ is examined by analyzing its real and imaginary parts. The main findings are summarized in the following theorem.

Theorem 2.2. *Let $\mathscr{H}_s(\cdot)$ $(-1 \leq s \leq 1/3)$ be a function defined by* (2.1)*. Then*

$$m(s) := \min_{|z|\leq 1} \Re\left\{\mathscr{H}_s\left(z\right)\right\} = \begin{cases} \dfrac{(3s+1)^2}{8s}, & -1 \leq s \leq -\dfrac{1}{3}, \\ 0, & -\dfrac{1}{3} \leq s \leq \dfrac{1}{3}, \end{cases}$$

$$M(s) := \max_{|z|\leq 1} \Re\left\{\mathscr{H}_s\left(z\right)\right\} = \begin{cases} 2 - 2s, & -1 \leq s \leq \dfrac{1}{5}, \\ \dfrac{(3s+1)^2}{8s}, & \dfrac{1}{5} \leq s \leq \dfrac{1}{3}, \end{cases}$$

and

$$\left|\Im\left\{\mathscr{H}_s\left(z\right)\right\}\right| < \begin{cases} \dfrac{\left(3(1-s) + \sqrt{\Delta}\right)^{3/2}\left(s - 1 + \sqrt{\Delta}\right)^{1/2}}{32|s|}, & s \in \left[-1, \dfrac{1}{3}\right] \setminus \{0\}, \\ 1, & s = 0, \end{cases}$$

where

$$\Delta = (1 - s)^2 + 32s^2. \tag{2.2}$$

Proof. For parametric equations of the Limaçon $w = u + iv = \mathscr{H}_s(z)$ we put $z = \mathrm{e}^{\mathrm{i}\theta}$ with $0 \leq \theta < 2\pi$. The equations are derived by expressing

$\mathscr{H}_s\left(e^{i\theta}\right)$ in terms of $\cos\theta$ and $\sin\theta$, by equations

$$\begin{cases} u(\theta) &= 1 + (1-s)\cos\theta - s\cos 2\theta \\ v(\theta) &= (1-s)\sin\theta - s\sin 2\theta. \end{cases} \tag{2.3}$$

Taking a parametrization

$$\begin{cases} u(\theta) - 1 - s = (1 - s - 2s\cos\theta)\cos\theta \\ v(\theta) = (1 - s - 2s\cos\theta)\sin\theta. \end{cases} \tag{2.4}$$

We can determine the curve generated by applying the function $\mathscr{H}_s(z) = u+iv$ to the unit circle $|z| = 1$, as stated in equation (1.3). The resulting curve is represented by the equation:

$$\left[(u-1)^2 + v^2 - s^2\right]^2 = (1-s)^2\left[(u-1-s)^2 + v^2\right]. \tag{2.5}$$

We note that the minimum or maximum of $u(\theta)$ for this limaçon are attained at the critical points of the function and are given by equation

$$u'(\theta) = [4s\cos\theta + s - 1]\sin\theta = 0. \tag{2.6}$$

(i) Assuming $s = 0$, equation (2.6) only has critical points at $\theta = 0$ and $\theta = \pi$. Therefore, the Limaçon $\mathscr{H}_0(z)$, where $z = \mathrm{e}^{\mathrm{i}\theta}$ with $0 \le \theta < 2\pi$, has a minimum value of 0 and a maximum value of 2 for $u(\theta)$ over $0 \le \theta < 2\pi$.

(ii) Assuming $s \neq 0$, equation (2.6) has critical points at $\theta = 0$, $\theta = \pi$, and the solution to

$$\cos\theta_1 = \frac{1-s}{4s}.$$

- When $1 - s \le 4|s|$, the minimum and maximum values of $u(\theta)$ over $0 \le \theta < 2\pi$ are given by specific formulas, depending on the value of s. Specifically, when $-1 \le s \le -1/3$, the minimum is $(3s+1)^2/(8s)$, and the maximum is $2 - 2s$. When $1/5 \le s \le 1/3$, the minimum is zero, and the maximum is $(3s+1)^2/(8s)$.
- When $1 - s > 4|s|$, equation (2.6) has critical points only at $\theta = 0$, $\theta = \pi$, and the minimum and maximum values for $u(\theta)$ over $0 \le \theta < 2\pi$ are given by 0 and $2-2s$, respectively, when $-1/3 < s < 1/5$.

We obtain the minimum or maximum values of $v(\theta)$ by solving the equation

$$v'(\theta) = (1-s)\cos\theta - 2s\cos 2\theta = 0.$$

(i) When $s = 0$, $v'(\theta) = 0$ if and only if $\theta = \pi/2$ or $3\pi/2$. As a result, we have $|v(\theta)| \le 1$.

(ii) When $s \neq 0$, $v'(\theta) = 0$ if and only if

$$\cos\theta_{1,2} = \frac{1 - s \pm \sqrt{\Delta}}{8s},$$

where Δ is given by (2.2) (see Figure 1).

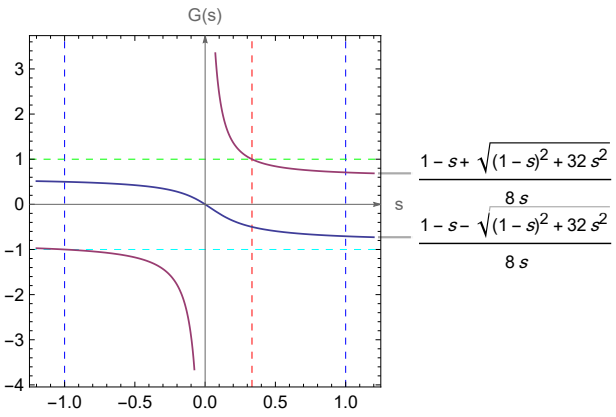


FIGURE 1. The functions $G(s) = \frac{1-s\pm\sqrt{\Delta}}{8s}$.

- If $\cos\theta_1 = (1 - s - \sqrt{\Delta})/(8s)$, then for $s \in [-1, 1)\backslash\{0\}$, we have $|\cos\theta_1| \leq 1$. By carrying out some straightforward calculations, we can obtain an upper bound for $|v(\theta)|$ over $0 \leq \theta < 2\pi$. In particular, we have

$$|v(\theta)| \leq \frac{\left(3(1-s) + \sqrt{\Delta}\right)^{3/2}\left(s - 1 + \sqrt{\Delta}\right)^{1/2}}{32|s|}.$$

- If $\cos\theta_2 = (1 - s + \sqrt{\Delta})/(8s)$, then we have $|\cos\theta_2| \leq 1$ for $s = -1, 1/3$. In these cases, we can derive upper bounds for $\left|\Im\left\{\mathscr{H}_s\left(e^{i\theta}\right)\right\}\right|$. Specifically, we have

$$\left|\Im\left\{\mathscr{H}_{-1}\left(e^{i\theta}\right)\right\}\right| \leq \frac{3\sqrt{3}}{2} \quad \text{and} \quad \left|\Im\left\{\mathscr{H}_{1/3}\left(e^{i\theta}\right)\right\}\right| \leq \frac{\sqrt{3}}{2}.$$

□

3. The Class $\mathcal{ST}_s$ and its Properties

Definition 3.1. Let $\mathcal{ST}_s$ denote the subfamily of $\mathcal{A}$ consisting of the functions h, satisfying the condition

$$\frac{zh'(z)}{h(z)} \prec \mathscr{H}_s(z) \quad \text{for} \quad z \in \mathbb{D}, \quad -1 \leq s \leq \frac{1}{3}. \tag{3.1}$$

From Theorem 2.2, it follows that if $h \in \mathcal{ST}_s$, then

$$\Re\left\{\frac{zh'(z)}{h(z)}\right\} > m(s) \quad \text{for} \quad z \in \mathbb{D}.$$

From a geometric perspective, the condition in equation (3.1) means that the expression $zh'(z)/h(z)$ lies within a domain bounded by a Limaçon curve given by (2.5). If h belongs to the family of functions $\mathcal{ST}_s$, then using Theorem 2.2, we can conclude that:

$$\min_{|z|=1} \Re\left\{\mathscr{H}_s(z)\right\} < \Re\left\{\frac{zh'(z)}{h(z)}\right\} < \max_{|z|=1} \Re\left\{\mathscr{H}_s(z)\right\},$$

which can be written as

$$m(s) < \Re\left\{\frac{zh'(z)}{h(z)}\right\} < M(s),$$

where $m(s)$ and $M(s)$ are given by Theorem 2.2. Therefore, we can deduce the following corollary.

Corollary 3.2. *The class $\mathcal{ST}_s$ satisfies the following inclusion:*

(i) *For $-1/3 \leq s \leq 1/3$, $\mathcal{ST}_s \subset \mathcal{S}^*$,*
(ii) *For $-1 \leq s \leq -1/9$ or $0 < s \leq 1/3$, $\mathcal{ST}_s \subset \mathcal{N}$.*

The theorem presented in this section employs a similar approach as in [27] to establish the upper and lower bounds for $|h_n|$ in the class $\mathcal{ST}_s$.

Theorem 3.3. *Suppose $h(z) = \sum_{n=1}^{\infty} h_n z^n \in \mathcal{ST}_s$ with $h_1 = 1$. Then the inequality*

$$\sum_{n=2}^{\infty} \left(n^2 - (2 - s + |s|)^2\right) |h_n|^2 \leq (2 - s + |s|)^2 - 1$$

holds, which means that a certain expression involving the coefficients of $h(z)$ is less than or equal to a constant that depends on s.

Proof. Suppose $h(z) = \sum_{n=1}^{\infty} h_n z^n \in \mathcal{ST}_s$ with $h_1 = 1$ and let $\omega(z)$ be an analytic self-map of the unit disk $\mathbb{D}$ with $\omega(0) = 0$. Then for $z = re^{i\theta}$ with $0 < r < 1$ and $0 \leq \theta < 2\pi$ we have the expression

$$h(z) = \frac{zh'(z)}{(1 + (1 - s)\,\omega(z) - s\,\omega(z)^2)\,h(z)}.$$

Moreover, we obtain the inequality

$$\begin{aligned} 2\pi \sum_{n=1}^{\infty} |h_n|^2 r^{2n} &= \int_0^{2\pi} \left|h\left(re^{i\theta}\right)\right|^2 d\theta \\ &= \int_0^{2\pi} \frac{\left|re^{i\theta} h'\left(re^{i\theta}\right)\right|^2}{\left|1 + (1 - s)\,\omega\left(re^{i\theta}\right) - s\,\omega^2\left(re^{i\theta}\right)\right|^2} d\theta \end{aligned}$$

$$\geq \frac{1}{(2-s+|s|)^2}\int_0^{2\pi}\left|re^{i\theta}h'\left(re^{i\theta}\right)\right|^2 d\theta$$
$$= \frac{2\pi}{(2-s+|s|)^2}\sum_{n=1}^{\infty} n^2\,|h_n|^2\,r^{2n}.$$

By sending $r \to 1^-$, we obtain the desired result. □

Corollary 3.4. *Let $h(z) = z+\sum_{n=2}^{\infty} h_n z^n \in \mathcal{ST}_s$. Then for $n = 3, 4, \ldots$*

$$|h_n| \leq \begin{cases} \sqrt{\dfrac{4(1-s)^2-1}{n^2-4(1-s)^2}}, & -\dfrac{1}{3} \leq s \leq 0, \\ \sqrt{\dfrac{3}{n^2-4}}, & 0 \leq s \leq \dfrac{1}{3}. \end{cases}$$

Assume that the function h of the form (1.1) belongs to the class $\mathcal{ST}_s$. The functions $\mathscr{H}_s(\cdot)$ given by (2.1) belongs to the class

$$\mathbf{H} := \left\{\sum_{n=1}^{\infty} a_n z^n \colon \sum_{n=1}^{\infty} |a_n|^2 < \infty\right\},$$

the Hardy class of analytic functions in $\mathbb{D}$. As a consequence of some result in [15], we know that $|h_n| = O(1/n)$ in terms of the sharp growth order.

According to Theorem 3.2, for $-1/3 \leq s \leq 1/3$, the function $\mathscr{H}_s(\cdot)$ is univalent and starlike in the unit disk $\mathbb{D}$ with respect to $\mathscr{H}_s(0) = 1$ and $\mathscr{H}_s'(0) = 1 - s > 0$.

Several problems over the class $\mathcal{ST}_s$ have extremal solutions given by functions in $\mathcal{ST}_s$, and we will consider some of those functions. A function h belongs to $\mathcal{ST}_s$ if and only if there exists an analytic function q such that q satisfies $q \prec \mathscr{H}_s$ and

$$h(z) = z\exp\left(\int_0^z \frac{q(\xi)-1}{\xi}\,\mathrm{d}\xi\right) \quad \text{for some } q \text{ with } q \prec \mathscr{H}_s. \tag{3.2}$$

Example 3.5. By choosing $q(z) = \mathscr{H}_s(z)$ in equation (3.2), we obtain a function given by

$$h_s(z) = z\exp\left(\int_0^z \frac{\mathscr{H}_s(\xi)-1}{\xi}\,\mathrm{d}\xi\right) = z\exp\left((1-s)z - \frac{s}{2}z^2\right),$$

which belongs to $\mathcal{ST}_s$. The function $h_s(z)$ is extremal for various problems in $\mathcal{ST}_s$.

The purpose of the following lemma is to determine the size of the largest or smallest disk centered at the fixed point $(1, 0)$ that is either entirely contained within or contains the Limaçon domain $\mathscr{H}_s(\mathbb{D})$.

Theorem 3.6. *Let $R_1(s)$ and $R_2(s)$ be real numbers given by*

$$R_1(s) = \begin{cases} 1, & -\frac{1}{3} \leq s \leq 0, \\ 1-2s, & 0 \leq s \leq \frac{1}{3} \end{cases}$$

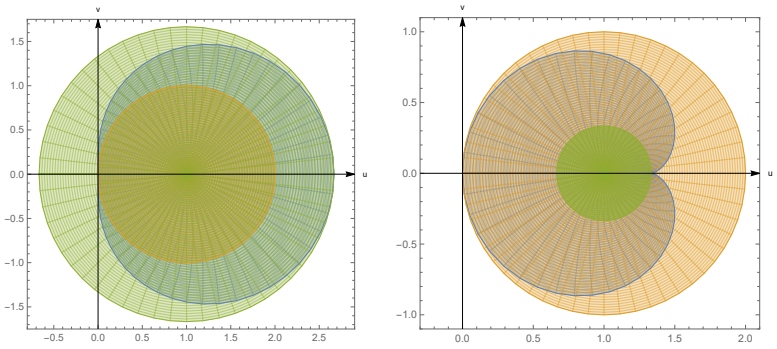
and

$$R_2(s) = \begin{cases} 1-2s, & -\frac{1}{3} \leq s \leq 0, \\ 1, & 0 \leq s \leq \frac{1}{3}. \end{cases}$$

Then

$$\{w \,:\, |w-1| < R_1(s)\} \subseteq \mathscr{H}_s(\mathbb{D}) \subseteq \{w \,:\, |w-1| < R_2(s)\}.$$

This relationship can be visualized in Figure 2.



(A) Location of areas $\mathscr{H}_{-1/3}(z)$, $\frac{5}{3}z+1$ and $z+1$.

(B) Location of areas $\mathscr{H}_{1/3}(z)$, $\frac{1}{3}z+1$ and $z+1$.

FIGURE 2. The image of $\mathbb{D}$ under $\mathscr{H}_s(z)$, $R_1(s)\cdot z+1$ and $R_2(s)\cdot z+1$.

Proof. The square of distance between the point $(1,0)$ and any point on the limaçon $\mathscr{H}_s(\cdot)$ can be calculated using the formula $z(\theta) = (1-u(\theta))^2+(v(\theta))^2$. The functions $u(\theta)$ and $v(\theta)$ are determined by equation (2.3). The derivative of $z(\theta)$ is provided as $z'(\theta) = 2s(1-s)\sin\theta$, which reveals that $z(\theta)$ has critical points at $\theta = 0$ and $\theta = \pi$.

- Let $-1/3 \leq s \leq 0$. In this case, it is shown that

$$\sqrt{z(\pi)} = 1 \quad \text{and} \quad \sqrt{z(0)} = 1-2s$$

are the minimum and maximum values of $\sqrt{z(\theta)}$, respectively. Moreover, it is shown that the minimum value is achieved at $\theta = \pi$, while the maximum is achieved at $\theta = 0$.

- Let $0 \leq s \leq 1/3$. In this case, it is shown that

$$\sqrt{z(\pi)} = 1 \quad \text{and} \quad \sqrt{z(0)} = 1 - 2s$$

are still the minimum and maximum values of $\sqrt{z(\theta)}$, respectively. However, the minimum value is now achieved at $\theta = 0$, while the maximum is achieved at $\theta = \pi$.

□

Given Theorem 3.6, we present a result for the class $\mathcal{ST}_s$ that is contained in the following corollary.

Corollary 3.7. *Let $h \in \mathcal{A}$. Then*

$$\left| \frac{zh'(z)}{h(z)} - 1 \right| < L \quad \Longrightarrow \quad h \in \mathcal{ST}_s,$$

where L is an arbitrary real number such that $0 < L \leq R_1(s)$ and $R_1(s)$ is given by Theorem 3.6.

Next we determine the necessary and sufficient conditions for the membership of some special members in $\mathcal{ST}_c$.

Theorem 3.8. (i) *A function $h(z) = z + h_n z^n$ $(n = 2, 3, \ldots)$ belongs to class $\mathcal{ST}_s$ if*

$$|h_n| \leq \begin{cases} \dfrac{1}{n}, & -\dfrac{1}{3} \leq s \leq 0, \\ \dfrac{1-2s}{n-2s}, & 0 \leq s \leq \dfrac{1}{3}. \end{cases}$$

(ii) *The function $z/(1 - Az)^2 \in \mathcal{ST}_s$ if*

$$|A| \leq \begin{cases} \dfrac{1}{3}, & -\dfrac{1}{3} \leq s \leq 0, \\ \dfrac{1-2s}{3-2s}, & 0 \leq s \leq \dfrac{1}{3}. \end{cases}$$

Proof. (i) Since $\mathcal{ST}_s \subset \mathcal{S}^*$, we have $|h_n| \leq 1/n$ for all $h \in \mathcal{ST}_s$. Moreover

$$\begin{aligned} \left| \frac{zh'(z)}{h(z)} - 1 \right| &= \left| \frac{(n-1)h_n z^{n-1}}{1 + h_n z^{n-1}} \right| \\ &< \frac{(n-1)|h_n|}{1 - |h_n|}. \end{aligned}$$

We can then apply Corollary 3.7 to obtain a bound for $(n-1)|h_n|/(1-|h_n|)$, so that

$$\frac{(n-1)|h_n|}{1-|h_n|} \leq R_1(s),$$

where $R_1(s)$ is given by Theorem 3.6. This inequality gives us the desired result.

(ii) It is clear that the Koebe function $z/(1-z)^2$ is not in the class $\mathcal{ST}_s$. However, if we define $h(z) = z/(1-Az)^2$, where $|A| \neq 1$, then

$$\begin{aligned}\left|\frac{zh'(z)}{h(z)} - 1\right| &= \left|\frac{2Az}{1-Az}\right| \\ &< \frac{2|A|}{1-|A|}.\end{aligned}$$

Applying Corollary 3.7, we can choose $R_1(s)$ such that

$$\frac{2|A|}{1-|A|} \leq R_1(s),$$

as given by Theorem 3.6. This inequality provides us with the desired result.

□

In [21], Miller and Mocanu explored differential subordination and derived several results, which are summarized below.

Lemma 3.9 ([21])**.** *Suppose that $\mathfrak{q}$ is an analytic and univalent function in the open unit disk $\mathbb{D}$. Let Ψ and Θ be analytic functions defined in a domain Ω that contains the image of $\mathfrak{q}$ on $\mathbb{D}$, with $\Psi(w)$ non-zero whenever w is in the image of $\mathfrak{q}$ on $\mathbb{D}$. Define $Q(z) = z\mathfrak{q}'(z)\cdot\Psi(\mathfrak{q}(z))$ and $h(z) = \Theta(\mathfrak{q}(z)) + Q(z)$, and suppose that either (i) h is a convex function or (ii) Q is a starlike function. In addition, assume that (iii)*

$$\begin{aligned}\Re\left\{\frac{zh'(z)}{Q(z)}\right\} &= \Re\left\{\frac{\Theta'(\mathfrak{q}(z))}{\Psi(\mathfrak{q}(z))} + \frac{zQ'(z)}{Q(z)}\right\} \\ &> 0.\end{aligned}$$

If $\mathfrak{p}(z)$ is an analytic function defined in $\mathbb{D}$ with $\mathfrak{p}(0) = \mathfrak{q}(0)$ and $\mathfrak{p}(\mathbb{D})$ contained in Ω, and if $\mathfrak{p}(z)$ satisfies inequality:

$$\Theta(\mathfrak{p}(z)) + z\mathfrak{p}'(z)\cdot\Psi(\mathfrak{p}(z)) \prec \Theta(\mathfrak{q}(z)) + z\mathfrak{q}'(z)\cdot\Psi(\mathfrak{q}(z)), \tag{3.3}$$

then $\mathfrak{p}(z) \prec \mathfrak{q}(z)$, and $\mathfrak{q}(z)$ is the best dominant.

Lemma 3.10 ([21])**.** *Let $h(z)$ be an analytic function in the open unit disk $\mathbb{D}$, and let $\Phi(w)$ be a function analytic in a domain D that contains the image of h on $\mathbb{D}$ such that $\Re\{\Phi(h(z))\} > 0$. Additionally, assume*

that either (i) h is a convex function, or (ii) $G(z) = zh'(z) \cdot \Phi(h(z))$ is a starlike function.

Suppose that p is an analytic function defined in $\mathbb{D}$ with $p(0) = h(0)$ and $p(\mathbb{D})$ contained in D, and that satisfies the inequality

$$\mathfrak{p}(z) + z\mathfrak{p}'(z) \cdot \Phi(h(z)) \prec h(z).$$

Then we conclude that $\mathfrak{p}(z) \prec h(z)$.

In the following, we introduces a theorem that applies a similar method as in [25] for differentiating functions that belong to the class $\mathcal{ST}_s$.

Theorem 3.11. *Suppose that $\mathfrak{p}(z)$ is analytic in the open unit disk $\mathbb{D}$ with $\mathfrak{p}(0) = 1$, $\beta > 0$, and $z \in \mathbb{D}$. If*

$$\mathfrak{p}(z) + \beta \cdot \frac{z\mathfrak{p}'(z)}{\mathfrak{p}(z)} \prec \mathscr{H}_s(z) \quad \textit{for} \quad -\frac{1}{3} \leq s \leq \frac{1}{3},$$

then it follows that $\mathfrak{p}(z) \prec \mathscr{H}_s(z)$ and $\mathscr{H}_s(z)$ is the best dominant.

Proof. Consider the function $\Phi(z) = \beta/z$ and let h be the function $\mathscr{H}_s$ defined in equation (2.1). The domain $D = \mathbb{C}\backslash\{0\}$ contains the image of $\mathscr{H}_s(\mathbb{D})$ under $\Phi(z)$.

Using Theorem 2.2, we can show that $\Re\{\Phi(\mathscr{H}_s(z))\} > 0$ for $-1/3 \leq s \leq 1/3$. Specifically,

$$\begin{aligned} \Re\left\{\Phi(\mathscr{H}_s(z))\right\} &= \beta \cdot \Re\left\{\frac{1}{\mathscr{H}_s(z)}\right\} \\ &> \frac{\beta}{2} \cdot \frac{1+3s}{(1+s)^2} \\ &\geq 0. \end{aligned}$$

Furthermore, using Lemma 2.1, we can conclude that $z\mathfrak{p}'(z) \cdot \Phi(\mathfrak{p}(z))$ is starlike, because

$$z\mathfrak{p}'(z) \cdot \Phi(\mathfrak{p}(z)) = \beta \frac{z\mathscr{H}_s'(z)}{\mathscr{H}_s(z)}.$$

Hence, by applying Lemma 3.10, we can deduce the desired assertion. □

If we substitute $\mathfrak{p}(z) = zf'(z)/f(z)$ into Theorem 3.11, we obtain the following conclusion.

Corollary 3.12. *If $h \in \mathcal{A}$ satisfy*

$$\frac{zh'(z)}{h(z)} + \beta\left(1 + \frac{zh''(z)}{h'(z)} - \frac{zh'(z)}{h(z)}\right) \prec \mathscr{H}_s(z) \quad \textit{for} \quad z \in \mathbb{D},\ \beta > 0,$$

then $h \in \mathcal{ST}_s$.

Theorem 3.13. *Suppose that $\mathfrak{p}(z)$ is an analytic function defined in the open unit disk $\mathbb{D}$ with $\mathfrak{p}(0) = 1$. Assume that $-1 \leq B < A \leq 1$ and the inequality*

$$1 + \beta \cdot z\mathfrak{p}'(z) \prec \mathscr{H}_s(z) \quad \textit{for} \quad \beta \in \mathbb{R}\backslash\{0\} \quad \textit{and} \quad -\frac{1}{3} \leq s \leq \frac{1}{3}$$

holds for all $\beta \in \mathbb{R}\backslash\{0\}$. If

$$|\beta| \geq \frac{(1+|B|)^2}{A-B} \cdot R_2(s),$$

where $R_2(s)$ is given by Theorem 3.6, then we can conclude that

$$\mathfrak{p}(z) \prec \frac{1+Az}{1+Bz}.$$

Proof. Let $\mathfrak{q}(z)$ be a function defined as $\mathfrak{q}(z) = \frac{1+Az}{1+Bz}$, where $-1 \leq B < A \leq 1$. It can be shown that $\mathfrak{q}(z)$ is a convex univalent function in the open unit disk $\mathbb{D}$. Consider the functions $\Theta(w) = 1$ and $\Psi(w) = \beta$ for $\beta \in \mathbb{R}\backslash\{0\}$, which are analytic in a domain containing $\mathfrak{q}(\mathbb{D})$. Define $Q(z) = z\mathfrak{q}'(z) \cdot \Psi(\mathfrak{q}) = \beta \cdot z\mathfrak{q}'(z)$ and $h(z) = \Theta(\mathfrak{q}) + Q(z) = 1 + \beta \cdot z\mathfrak{q}'(z)$. It follows that

$$\begin{aligned} \Re\left\{\frac{zQ'(z)}{Q(z)}\right\} &= \Re\left\{\frac{zh'(z)}{Q(z)}\right\} \\ &= \Re\left\{1 + \frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)}\right\} \\ &> 0. \end{aligned}$$

We observe that $Q(z)$ is starlike and univalent, and using Lemma 3.9, we can show that if $1 + \beta \cdot z\mathfrak{p}'(z) \prec 1 + \beta \cdot z\mathfrak{q}'(z)$, then $\mathfrak{p}(z) \prec \mathfrak{q}(z)$.

To prove the main result, we need to find β such that

$$\begin{aligned} \mathscr{H}_s(z) &\prec 1 + \beta \cdot z\mathfrak{q}'(z) \\ &= H(z). \end{aligned} \tag{3.4}$$

Using Theorem 3.6, we can show that

$$|H(\xi) - 1| \geq R_2(s) \quad \text{for} \quad |\xi| = 1.$$

Therefore, for $|\xi| = 1$, we have $|1 + B\xi| \leq 1 + |B|$, which implies that

$$\frac{|\beta|(A-B)}{|1+B\xi|^2} \geq \frac{|\beta|(A-B)}{(1+|B|)^2}.$$

Thus, we can derive the desired result from equation (3.4) and the relation $R_2(s)$:

$$\frac{|\beta|(A-B)}{(1+|B|)^2} \geq R_2(s) \quad \Longrightarrow \quad |\beta| \geq \frac{(1+|B|)^2}{A-B} \cdot R_2(s).$$

□

If we substitute $A = -B = 1$ and $0 \leq s \leq 1/3$ into Theorem 3.11, we obtain the following conclusion.

Corollary 3.14. *Suppose that $\mathfrak{p}(z)$ is an analytic function defined in the open unit disk $\mathbb{D}$ with $\mathfrak{p}(0) = 1$. Assume that $0 \leq s \leq 1/3$ and the inequality*

$$1 + \beta \cdot z\mathfrak{p}'(z) \prec \mathscr{H}_s(z)$$

holds for all $\beta \in \mathbb{R}\backslash\{0\}$. If $|\beta| \geq 2$, then we can conclude that

$$\Re\{\mathfrak{p}(z)\} > 0.$$

Example 3.15. If we let $\mathfrak{p}(z) = 1 + \frac{1-s}{2}z - \frac{s}{4}z^2$ and $0 \leq s \leq 1/3$, then from Corollary 3.14, we deduce

$$1 + 2 \cdot z\mathfrak{p}'(z) \prec \mathscr{H}_s(z) \quad \Longrightarrow \quad \Re\{\mathfrak{p}(z)\} > 0.$$

4. Conclusions

The paper presented a thorough examination of the Limaçon curve, exploring various parameters and their effects. The author investigated families of starlike functions that were situated within the regions bounded by the curve, providing examples and highlighting the properties of extremal functions within these families. The study also established both upper and lower bounds for the real and imaginary components of these functions, identifying key features of these bounds. Furthermore, the paper employed subordination theory to draw conclusions and corollaries for functions belonging to this class. Overall, this research provided valuable insights into the nature of the Limaçon and its associated function.

5. Data Availability Statement

Not applicable.

6. Conflict of Interest

The author declare no conflict of interest.

Acknowledgment. The author thank the editor and the anonymous referees for constructive and pertinent suggestions.

References

1. E. Amini, M. Fardi, S. Al-Omari and R. Saadeh, *Certain differential subordination results for univalent functions associated with q-Salagean operators[J]*, AIMS Math., 8 (2023), pp. 15892-15906.

2. E. Amini, S. Al-Omari, M. Fardi and K. Nonlaopon, *Duality on q-Starlike Functions Associated with Fractional q-Integral Operators and Applications*, Symmetry, 14 (2022), pp. 2076.
3. K. Bano and M. Raza, *Starlikness associated with limacon*, Filomat, 37 (2023), pp. 851–862.
4. P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften. New York, NY, USA: SpringerVerlag, 1983.
5. P. Goel, S.S. Kumar, *Certain class of starlike functions associated with modified sigmoid function*, Bull. Malays. Math. Sci. Soc., 43 (2020), pp. 957-991.
6. W. Janowski, *Extremal problems for a family of functions with positive real part and for some related families*, Ann. Pol. Math., 23 (1970), pp. 159-177.
7. R. Kargar, A. Ebadian and J. Sokół, *On Booth lemniscate and starlike functions*, Anal. Math. Phys., 9 (2019), pp. 143-54.
8. S.S Kumar and G. Kamaljeet, *A cardioid domain and starlike functions*, Anal. Math. Phys., 11 (2021), pp. 1-34.
9. S. Kanas and V.S. Masih, *On the behaviour of analytic representation of the generalized Pascal snail*, Anal. Math. Phys., 11 (2021), pp. 1-27.
10. S. Kanas, V.S. Masih and A. Ebadian, *Relations of a planar domains bounded by hyperbolas with families of holomorphic functions*, J. Inequal. Appl., 2019 (2019), pp. 1-14.
11. K. Kuroki and S. Owa, *Notes on new class for certain analytic functions*, Adv. Math.: Sci. J., 1 (2012), pp. 127-31.
12. S. Kanas and H. M. Srivastava, *Linear operators associated with k-uniformly convex functions*, Integral Transforms Spec. Funct., 9 (2000), pp. 121–132.
13. S. Kanas and A. Wiśniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math., 105 (1999), pp. 327-36.
14. S. Kanas and A. Wiśniowska, *Conic regions k-uniform convexity, II.*, Zeszyty Nauk. Poslitech. Rzeszowskiej Mat., 22 (1998), pp. 65-78.
15. W.C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, In: Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Conf. Proc. Lecture Notes Anal., I Int. Press, Cambridge, pp. 157–169.
16. V.S. Masih, A. Ebadian and J. Sokół, *On strongly starlike functions related to the Bernoulli lemniscate*, Tamkang J. Math., 53 (2022), pp. 187-99.
17. V.S. Masih and S. Kanas, *Subclasses of starlike and convex functions associated with the limaçon domain*, Symmetry, 12 (2020),

pp. 942.
18. V.S. Masih, A. Ebadian and S. Yalçin, *Some properties associated to a certain class of starlike functions*, Math. Slovaca, 69 (2019), pp. 1329-1340
19. R. Mendiratta, S. Nagpal and V. Ravichandran, *A subclass of starlike functions associated with left-half of the lemniscate of Bernoulli*, Internat. J. Math., 25 (2014), pp. 1450090.
20. R. Mendiratta, S. Nagpal and V. Ravichandran, *On a subclass of strongly starlike functions associated with exponential function*,Bull. Malays. Math. Sci. Soc., 38 (2015), pp. 365-386.
21. S.S. Miller and P.T Mocanu, *Differential Subordinations,Theory and Applications*, Series of Monographs and Textbooks in Pure and Applied Mathematics, vol. 225. Marcel Dekker Inc. NewYork/Basel, 2000.
22. S. Ozaki, *On the theory of multivalent functions. II.*, Sci. rep. Tokyo Bunrika Daigaku. sect. A, 4 (1941), pp. 45-87.
23. R.K. Raina and J. Sokół, *Some properties related to a certain class of starlike functions*, C.R. Math., Ser. I, 353 (2015), pp. 973-978.
24. K. Sharma, N.K. Jain and V. Ravichandran, *Starlike functions associated with a cardioid*, Afrika Mat., 27 (2016), pp. 923-939.
25. P. Sharma, R.K. Raina and J. Sokół, *Certain Ma–Minda type classes of analytic functions associated with the crescent-shaped region*, Anal. Math. Phys., 9 (2019), pp. 1887-1903.
26. L. Shi, H. M. Srivastava, M. G. Khan, N. Khan, B. Ahmad, B. Khan and W. K. Mashwani, *Certain subclasses of analytic multivalent functions associated with petal-shape domain*, Axioms, 10 (2021), pp. 1-14.
27. J. Sokół, *Coefficient Estimates in a Class of Strongly Starlike Functions*, Kyungpook Math. J., 49 (2015), pp. 349–353.
28. J. Sokół, *On some subclass of strongly starlike functions*, Demonstratio Math., 31 (1998), pp. 81-86.
29. H. M. Srivastava, S. Hussain, I. Ahmad and A. Syed Ghoos, *Coefficient bounds for analytic and bi-univalent functions associated with some conic domains*, J. Nonlinear Convex Anal., 23 (2022), pp. 741–753.
30. H.M. Srivastava, B. Khan, N. Khan, A. Hussain, N. Khan and M. Tahir, *Applications of certain basic (or) q-derivatives to subclasses of multivalent Janowski type q-starlike functions involving conic domains*, J. Nonlinear Var. Anal., 5 (2021), pp. 531-547.
31. H.M. Srivastava, G. Murugusundaramoorthy and T. Bulboacă, *The second Hankel determinant for subclasses of bi-univalent functions associated with a nephroid domain*, RACSAM Rev. R. Acad. Cienc.

Exactas Frıs. Nat. Ser. A Mat., 116 (2022), pp. 145.
32. H.M. Srivastava, T.M. Seoudy and M. K. Aouf. *A generalized conic domain and its applications to certain subclasses of multivalent functions associated with the basic (or q) calculus[J]*, AIMS Math., 6 (2021), pp. 6580-6602.
33. L.A. Wani and A. Swaminathan, *Starlike and convex functions associated with a nephroid domain*, Bull. Malays. Math. Sci. Soc., 44 (2021), pp. 79-104.

---

[1]Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-3697, Tehran, Iran.
Email address: v_soltani@pnu.ac.ir

[1]Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-3697, Tehran, Iran.
Email address: najafzadeh1234@yahoo.ie