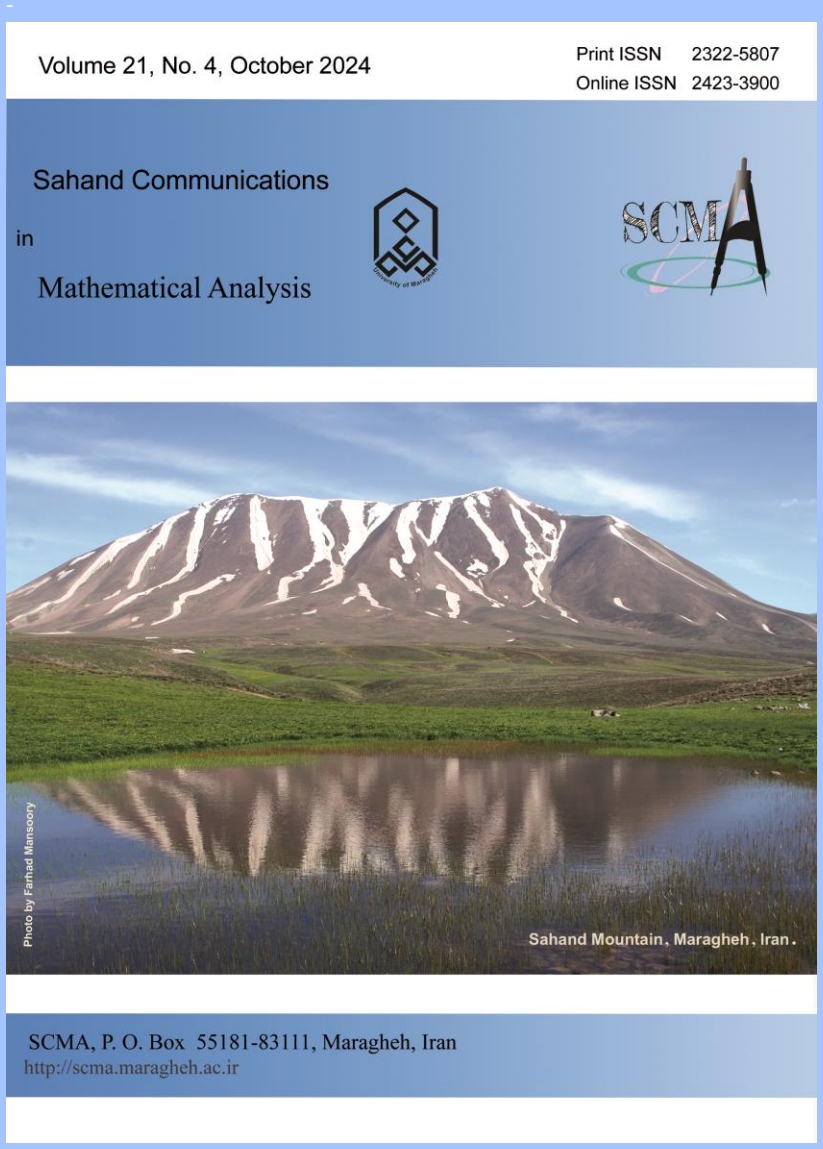
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Mohammad Shahriari and Vladimir Vladicic

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On Recovering Sturm–Liouville-Type Operator with Delay and Jump Conditions

Mohammad Shahriari[1*] and Vladimir Vladicic[2]

Abstract. In this manuscript, the second order differential operators with constant delay and transmission boundary conditions are studied. The asymptotic forms of the characteristic functions and eigenvalues of the operators are obtained. Therefore, an inverse spectral problem of recovering the operators using the spectra of two different Dirichlet-Dirichlet and Dirichlet-Neumann problems, is investigated. For this purpose, to prove the uniqueness of the potential, the integral equations of the potential function are formulated and then utilized. Indirectly, by calculating the coefficients of Fourier series, the potential function is reconstructed.

1. Introduction

Let us consider the delay Sturm–Liouville problem $L_j := L_j(q; d, p)$,

$$\ell y := -y''(x) + q(x)y(x-d) = \mu y(x), \quad x \in (0,p) \cup (p,\pi) \tag{1.1}$$

with boundary and jump conditions

$$y(0) = y^{(j)}(\pi) = 0, \qquad j = 0, 1 \tag{1.2}$$

$$y(p+0) = ay(p-0), \qquad y'(p+0) = by'(p-0), \tag{1.3}$$

such that $q(x) \in L^2(d,\pi)$ is real-valued function and $q(x) = 0$ for $x < d$; $\frac{\pi}{3} < p \leq d < \frac{\pi}{2}$ and $ab > 0$. The coefficients a, b and p are assumed to be known a priori and fixed. Without losing of generality, [17, Lemma 2.3] can be considered $b = a^{-1}$ for the problem (1.1)–(1.3).

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* Corresponding author.

Sturm–Liouville problem is one of the most classical and important problems in mathematics, physics and engineering. This problem arises in the modeling of many systems in vibration theory, quantum mechanics, hydrodynamics and etc., [7, 10, 19]. The inverse Sturm–Liouville problem can be considered from three different aspects: existence, uniqueness and the reconstruction coefficient of the problems. There are many results about inverse spectral problems with discontinuity conditions inside of the interval, [7, 8, 11–13, 17]. In this paper, we study the reconstruction of the operators with two spectra and uniqueness theorem, when the coefficients of the jump conditions are known.

There are many works and results for direct and inverse problems with one or more constant delays, [1, 6, 8, 14–16, 18, 20–25]. In the case $d \geq \frac{\pi}{2}$ the characteristic functions of (1.1) and (1.2), with or without jump conditions (1.3), linearly depends on the potential function. In the paper [18] authors proved uniqueness of the potential from two spectra without jump conditions, while in [16] authors proved uniqueness of the potential from two spectra with jump conditions. In the case $d < \frac{\pi}{2}$ the characteristic functions of (1.1) and (1.2), with or without jump conditions, nonlinearly depends on the potential function. In this case, the inverse problems is much more complex. For $\frac{2\pi}{5} \leq d < \frac{\pi}{2}$ in [1] and [9] authors independently shows uniqueness of the potential functions from two spectra without jump conditions. For $\frac{\pi}{3} < d < \frac{2\pi}{5}$ in [3] and [4], authors shows that potential is not unique ordered from two spectra without jump conditions. As far as we know there are no result in the case $d < \frac{\pi}{2}$ with jump conditions. Our work is an extension of paper [16] to the nonlinear case. More precisely, we will proved uniqueness of a potential and a delay from the spectra of boundary value problem $L_j := L_j(q; d, p)$ $(j = 0, 1)$, in the case when $\frac{\pi}{3} < p \leq d \leq \frac{\pi}{2}$ and the coefficients a and p are known. For this reason, we obtain the asymptotic form of the solutions, characteristic functions and eigenvalues of the problems. Then, we consider the inverse problem for recovering operators with a constant delay from their two spectra, under Dirichlet-Dirichlet and Dirichlet-Newmann boundary conditions.

2. Asymptotic form of Solutions and Eigenvalues

Let $\mathcal{S}(x, \mu)$ be the solution of Eq. (1.1) under the initial conditions $\mathcal{S}(0, \mu) = 0$, $\mathcal{S}'(0, \mu) = 1$ and the jump conditions (1.3). For each fixed x, the function $\mathcal{S}(x, \mu)$ is an entire in μ of order $1/2$. It is easy to check that the function $\mathcal{S}(x, \mu)$ is the unique solution of the integral equation

$$\mathcal{S}(x, \mu) = \mathcal{S}_0(x, \mu) + \int_d^x \frac{\sin \varrho(x - t)}{\varrho} q(t) \mathcal{S}(t - d, \mu) dt, \tag{2.1}$$

where

$$\mathcal{S}_0(x,\mu)=\begin{cases}\frac{\sin\varrho x}{\varrho}, & x<p,\\ a^+\frac{\sin\varrho x}{\varrho}+a^-\frac{\sin\varrho(2p-x)}{\varrho} & x>p,\end{cases} \tag{2.2}$$

$$\mathcal{S}_0'(x,\mu)=\begin{cases}\cos\varrho x, & x<p,\\ a^+\cos\varrho x-a^-\cos\varrho(2p-x), & x>p.\end{cases} \tag{2.3}$$

such that $\varrho^2=\mu$, $\varrho=\sigma+i\tau$,

$$a^+=\frac{a+1/a}{2},\qquad a^-=\frac{a-1/a}{2}. \tag{2.4}$$

The function $\mathcal{S}_0(x,\mu)$ is the solution of (1.1) with $q(x)=0$ and the initial conditions $\mathcal{S}_0(0,\mu)=0,\mathcal{S}_0'(0,\mu)=1$ and jump conditions (1.3). Solving (2.1) by the method of successive approximations, we get

$$\mathcal{S}(x,\mu)=\mathcal{S}_0(x,\mu)+\mathcal{S}_1(x,\mu)+\mathcal{S}_2(x,\mu). \tag{2.5}$$

Using the formulas (2.1), (2.2) and (2.3) we have

$$\mathcal{S}_1(x,\mu)=\begin{cases}0, & x<d,\\ \int_d^x\frac{\sin\varrho(x-t)}{\varrho}q(t)\mathcal{S}_0(t-d,\mu)dt, & x\geq d,\end{cases} \tag{2.6}$$

$$\mathcal{S}_1'(x,\mu)=\begin{cases}0, & x<d,\\ \int_d^x\cos\varrho(x-t)q(t)\mathcal{S}_0(t-d,\mu)dt, & x\geq d,\end{cases} \tag{2.7}$$

and

$$\mathcal{S}_2(x,\mu)=\begin{cases}0, & x<2d,\\ \int_{2d}^x\frac{\sin\varrho(x-t)}{\varrho}q(t)\mathcal{S}_1(t-d,\mu)dt, & x\geq 2d,\end{cases} \tag{2.8}$$

$$\mathcal{S}_2'(x,\mu)=\begin{cases}0, & x<2d,\\ \int_{2d}^x\cos\varrho(x-t)q(t)\mathcal{S}_1(t-d,\mu)dt, & x\geq 2d.\end{cases} \tag{2.9}$$

Applying the formulas (2.2)–(2.9), we calculate

(2.10)

$$\mathcal{S}_1(x,\mu) = \begin{cases} 0, & x < d, \\ \int_d^x \frac{\sin\varrho(x-t)}{\varrho} q(t) \frac{\sin\varrho(t-d)}{\varrho} dt, & d \le x < d+p, \\ \int_d^x \frac{\sin\varrho(x-t)}{\varrho} q(t) \left(a^+ \frac{\sin\varrho(t-d)}{\varrho} + a^- \frac{\sin\varrho(2d-t+d)}{\varrho} \right) dt, & x > d+p, \end{cases}$$

$$= \begin{cases} 0, & x < d, \\ \frac{1}{2\varrho^2} \Big(-\cos\varrho(x-d) \int_d^x q(t)dt \\ \quad + \int_d^x \cos\varrho(2t-x-d) q(t) dt \Big), & d \le x < d+p, \\ \frac{a^+}{2\varrho^2} \left(-\cos\varrho(x-d) \int_d^x q(t)dt + \int_d^x \cos\varrho(2t-x-d) q(t) dt \right) \\ \quad + \frac{a^-}{2\varrho^2} \Big(\cos\varrho(2p-x+d) \int_d^x q(t)dt \\ \quad - \int_d^x \cos\varrho(x+2p-2t+d) q(t) dt \Big), & x > d+p, \end{cases}$$

and

(2.11)

$$\mathcal{S}_1{}'(x,\mu) = \begin{cases} 0, & x < d, \\ \frac{1}{2\varrho} \left(\sin\varrho(x-d) \int_d^x q(t)dt + \int_d^x \sin\varrho(2t-x-d) q(t) dt \right), & d \le x < d+p, \\ \frac{a^+}{2\varrho} \left(\sin\varrho(x-d) \int_d^x q(t)dt + \int_d^x \sin\varrho(2t-x-d) q(t) dt \right) \\ + \frac{a^-}{2\varrho} \Big(\sin\varrho(2p-x+d) \int_d^x q(t)dt \\ \quad + \int_d^x \sin\varrho(2p-2t+x+d) q(t) dt \Big), & x > d+p, \end{cases}$$

(2.12)

$$\mathcal{S}_2(x,\mu) = \begin{cases} 0, & x < 2d, \\ \int_{2d}^{x} \frac{\sin \varrho(x-t)}{\varrho} q(t) \mathcal{S}_1(t-d,\mu)dt, & x \geq 2d, \end{cases}$$

$$= \begin{cases} 0, & x < 2d, \\ \frac{1}{2\varrho^3} \int_{2d}^{x} \sin \varrho(x-t) q(t) \Big(-\cos \varrho(t-2d) \int_{d}^{t-d} q(u)du \\ \quad + \int_{d}^{t-d} \cos \varrho(2u-t) q(u)du \Big) dt, & x \geq 2d \end{cases}$$

$$= \begin{cases} 0, & x < 2d, \\ \frac{1}{4\varrho^3} \Big(-\sin \varrho(x-2d) \int_{2d}^{x} q(t) \int_{d}^{t-d} q(u)dudt \\ \quad - \int_{2d}^{x} \sin \varrho(x-2t+2d) q(t) \int_{d}^{t-d} q(u)dudt \\ \quad + \int_{2d}^{x} \int_{d}^{t-d} \sin \varrho(x+2u-2t) q(t)q(u)dudt \\ \quad + \int_{2d}^{x} \int_{d}^{t-d} \sin \varrho(x-2u) q(t)q(u)dudt \Big), & x \geq 2d, \end{cases}$$

and

(2.13)

$$\mathcal{S}_2{}'(x,\mu) = \begin{cases} 0, & x < 2d, \\ \frac{1}{4\varrho^2} \Big[-\cos \varrho(x-2d) \int_{2d}^{x} q(t) \int_{d}^{t-d} q(u)dudt \\ \quad - \int_{2a}^{x} \cos \varrho(x-2t+2a) q(t) \int_{a}^{t-a} q(u)du\, dt \\ \quad + \int_{2d}^{x} \int_{d}^{t-d} \cos \varrho(x+2u-2t) q(t)q(u)dudt \\ \quad + \int_{2d}^{x} \int_{d}^{t-d} \cos \varrho(x-2u) q(t)q(u)dudt \Big], & 2d \leq x. \end{cases}$$

We use $\Delta_j(\mu) := \mathcal{S}^{(j)}(\pi,\mu)$, $j = 0,1$ for the characteristic function of (1.1)–(1.3). Note that these functions are entire in μ of order $1/2$ and

the set of zeroes of these functions are coincide with the spectra μ_{nj} of L_j. From Eqs. (2.5)–(2.13) as $|\varrho| \to \infty$, we get

$$
\begin{aligned}
\Delta_0(\mu) &= \frac{1}{\varrho}\left(a^+\sin\varrho\pi + a^-\sin\varrho(2p-\pi)\right) \\
&\quad + \frac{1}{2\varrho^2}\Bigg[w(-a^+\cos\varrho(\pi-d) + a^-\cos\varrho(2p-\pi+d)) \\
&\quad + a^+\int_d^\pi \cos\varrho(2t-\pi-d)q(t)dt \\
&\quad - a^-\int_a^\pi \cos\varrho(2t-\pi-2p-d)q(t)dt\Bigg] \\
&\quad + \frac{1}{4\varrho^3}\Bigg[-\sin\varrho(\pi-2d)\int_{2a}^\pi q(t)\int_d^{t-d} q(u)dudt \\
&\quad - \int_{2d}^\pi \sin\varrho(\pi-2t+2d)q(t)\int_d^{t-d} q(u)dudt \\
&\quad + \int_{2d}^\pi\int_d^{t-d} \sin\varrho(\pi+2u-2t)q(t)q(u)dudt \\
&\quad + \int_{2d}^\pi\int_d^{t-d} \sin\varrho(\pi-2u)q(t)q(u)dudt\Bigg] \\
&= a^+\frac{\sin\varrho\pi}{\varrho} + a^-\frac{\sin\varrho(2p-\pi)}{\varrho} \\
&\quad + \frac{w}{2\varrho^2}\left(-a^+\cos\varrho(\pi-d) + a^-\cos\varrho(2p-\pi+d)\right) \\
&\quad + o\left(\frac{\exp(|\tau|(\pi-d))}{\varrho^2}\right),
\end{aligned}
\tag{2.14}
$$

and

$$
\begin{aligned}
\Delta_1(\mu) &= \mathcal{S}'(\pi,\mu) \\
&= a^+\cos\varrho\pi - a^-\cos\varrho(2p-\pi) \\
&\quad + \frac{1}{2\varrho}\Bigg[w\left(a^+\sin\varrho(\pi-d) + a^-\sin\varrho(2p-\pi+d)\right) \\
&\quad + a^+\int_a^\pi \sin\varrho(2t-\pi-d)q(t)dt \\
&\quad - a^-\int_d^\pi \sin\varrho(2t-\pi-2p-d)q(t)dt\Bigg] \\
&\quad + \frac{1}{4\varrho^2}\Bigg[-\cos\varrho(\pi-2d)\int_{2d}^\pi q(t)\int_d^{t-d} q(u)dudt
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
&- \int_{2d}^{\pi} \cos\varrho(\pi - 2t + 2d) q(t) \int_{d}^{t-d} q(u) du dt \\
&+ \int_{2d}^{\pi} \int_{d}^{t-d} \cos\varrho(\pi + 2u - 2t) q(t) q(u) du dt \\
&+ \int_{2d}^{x} \int_{d}^{t-d} \cos\varrho(\pi - 2u) q(t) q(u) du dt \Big] \\
&= a^{+}\cos\varrho\pi - a^{-}\cos\varrho(2p - \pi) \\
&+ \frac{w}{2\varrho}\left(a^{+}\sin\varrho(\pi - d) + a^{-}\sin\varrho(2p - \pi + d)\right) \\
&+ o\left(\frac{\exp(|\tau|(\pi - d))}{\varrho}\right),
\end{aligned}
$$

where $w = \int_d^{\pi} q(t)dt$. Applying [15, Lemma 2.1] with the similar method, we have:

Lemma 2.1. *The eigenvalues of* (1.1)–(1.3) *i.e.* $\mu_{n0} = \varrho_{n0}^2$ *and* $\mu_{n1} = \varrho_{n1}^2, n \geq 1$ *as* $n \to \infty$, *have the following asymptotic forms:*

$$\varrho_{n0} = \varrho_{n0}^{\circ} + \frac{\eta_n}{\varrho_{n0}^{\circ}} + \frac{\kappa_n}{\varrho_{n0}^{\circ}}, \tag{2.16}$$

$$\varrho_{n1} = \varrho_{n1}^{\circ} + \frac{\nu_n}{\varrho_{n1}^{\circ}} + \frac{\kappa_n}{\varrho_{n1}^{\circ}}, \tag{2.17}$$

such that ϱ_{n0}° *and* ϱ_{n1}° *are the roots of* $\frac{1}{\varrho}(a^{+}\sin\varrho\pi + a^{-}\sin\varrho(2d - \pi))$ *and* $a^{+}\cos\varrho\pi - a^{-}\cos\varrho(2p - \pi)$, *respectively. Also, we have*

$$\eta_n = \frac{w}{2}\frac{a^{+}\cos\varrho_{n0}^{\circ}(\pi - d) - a^{-}\cos\varrho_{n0}^{\circ}(2p - \pi + d)}{a^{+}\pi\cos\varrho_{n0}^{\circ}\pi + a^{-}(2p - \pi)\cos\varrho_{n0}^{\circ}(2p - \pi)},$$

$$\nu_n = -\frac{w}{2}\frac{a^{+}\sin\varrho_{n1}^{\circ}(\pi - d) - a^{-}\sin\varrho_{n1}^{\circ}(2p - \pi + d)}{a^{+}\pi\sin\varrho_{n1}^{\circ}\pi - a^{-}(2p - \pi)\sin\varrho_{n1}^{\circ}(2p - \pi)},$$

and $\kappa_n = o(1)$.

Proof. Suppose that $\mu_{n0}^{\circ} := (\varrho_{n0}^{\circ})^2$ are the zeros of the function

$$\Delta_0^{\circ}(\mu) = \frac{1}{\varrho}(a^{+}\sin\varrho\pi + a^{-}\sin\varrho(2d - \pi)) \tag{2.18}$$

then from (2.14) we obtain

$$\varrho_{n0} = \varrho_{n0}^{\circ} + o(1), \quad n \to \infty. \tag{2.19}$$

The roots of $\Delta_0^{\circ}(\mu)$ are

$$\varrho_{n0}^{\circ} = n + \vartheta_n,$$

where $\sup_n \ \vartheta_n < M$ for all $n \in \mathbb{N}$. For sufficiently large n, $p = \frac{p_1}{q_1}\pi$ and $p_1, q_1 \in \mathbb{N}$, one can check that $0 < \vartheta_n < 1$ [**?** , Lemma 2.6]. By a simple calculation and using (2.15), (2.18) and (2.19), we get the asymptotic form of (2.16). Using the similar calculation, we obtain the asymptotic form of ϱ_{n1} in (2.17). □

Lemma 2.2. *When the coefficients a and p are known, then specification of the spectra $\{\mu_{nj}\}$, $n \geq 1$ and $j = 0, 1$, uniquely determines the characteristic functions $\Delta_j(\mu), j = 0, 1$ of the boundary value problems $L_j, j = 0, 1$.*

Proof. Using Hadamard's theorem [2, P. 289], the functions $\Delta_j(\mu),\ j = 0, 1$ are uniquely determined by a constant multiplicative of their zeros:

$$\Delta_0(\mu) = C_0(a^+\pi + a^-(2p - \pi)) \prod_{n=1}^{\infty} \frac{\mu_{n0} - \mu}{{\varrho_{n0}^{\circ}}^2}, \tag{2.20}$$

$$\Delta_1(\mu) = C_1 a \prod_{n=1}^{\infty} \frac{\mu_{n1} - \mu}{{\varrho_{n1}^{\circ}}^2}.$$

From Eq. (2.18) the function $\Delta_0^{\circ}(\mu)$ is an entire function of order $1/2$ then we have

$$\begin{aligned}\Delta_0^{\circ}(\mu) &= \frac{1}{\varrho}(a^+ \sin \varrho\pi + a^- \sin \varrho(2p - \pi)) \\ &= (a^+\pi + a^-(2p - \pi)) \prod_{n=1}^{\infty} \left(1 - \frac{\mu}{{\varrho_n^{\circ}}^2}\right).\end{aligned}$$

So, we get

$$\frac{\Delta_0(\mu)}{\Delta_0^{\circ}(\mu)} = C_0 \prod_{n=1}^{\infty} \frac{{\varrho_{n0}^{\circ}}^2}{\mu_{n0}} \prod_{n=1}^{\infty} \left(1 + \frac{\mu_{n0} - {\varrho_{n0}^{\circ}}^2}{{\varrho_{n0}^{\circ}}^2 - \mu}\right).$$

Using (2.14) and (2.16), we get

$$\lim_{\mu \to -\infty} \frac{\Delta_0(\mu)}{\Delta_0^{\circ}(\mu)} = 1, \qquad \lim_{\mu \to -\infty} \left(1 + \frac{\mu_{n0} - {\varrho_{n0}^{\circ}}^2}{{\varrho_{n0}^{\circ}}^2 - \mu}\right) = 1$$

and finally

$$C_0 = \prod_{n=1}^{\infty} \frac{\mu_{n0}}{{\varrho_{n0}^{\circ}}^2}.$$

Substituting this into (2.20), we proved uniqueness of the $\Delta_0(\mu)$. We prove $\Delta_1(\mu)$ by the same of $\Delta_0(\mu)$. □

Lemma 2.3. *If coefficients a and p are known and $w \neq 0$, then delay point d is unique ordered by spectra $\{\mu_{nj}\}$, $n \geq 1$, of the problems L_j, for $j = 0, 1$.*

Proof. First, we put $\varrho = n$, $n \in \mathbb{N}$ into (2.14) and (2.15), we have asymptotic formulas for $n \to \infty$

(2.21)
$$\Delta_0(n^2) = a^- \frac{(-1)^n \sin 2nd}{n} + \frac{w(-1)^n}{2n^2}\left(-a^+ \cos nd + a^- \cos n(2p+d)\right) + o\left(\frac{1}{n^2}\right),$$

(2.22)
$$\Delta_1(n^2) = a^-(-1)^n - a^-(-1)^n \cos 2np + \frac{w(-1)^n}{2n}\left(a^+ \sin nd + a^- \sin n(2p+d)\right) + o\left(\frac{1}{n}\right).$$

Now, we introduce function

$$D(n) := \left(2n^2\left(\Delta_0(n^2) - a^- \frac{(-1)^n \sin 2np}{n}\right)\right)^2 + \left(2n\left(\Delta_1(n^2) - a^-(-1)^n + a^-(-1)^n \cos 2nd\right)\right)^2.$$

Since, the characteristic functions $\Delta_j(\mu)$ are ordered by spectra $\{\mu_{nj}\}$, Lemma 2.2 and coefficients b and p are known. We conclude that function $D(n)$ is ordered by spectra $\{\mu_{nj}\}$. Using (2.21) and (2.22), we have

$$D(n) = w^2\left(a^{+2} + a^{-2} - 2a^+a^- \cos 2n(a+d)\right) + o(1), \quad n \to \infty.$$

Using this asymptotic formula, we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{D(n-2) - D(n+2)}{D(n-1) - D(n+1)} \\
&\quad = \lim_{n\to\infty} \frac{\cos(2(n+2)(d+p)) - \cos(2(n-2)(d+p))}{\cos(2(n+1)(d+p)) - \cos(2(n-1)(d+p))} \\
&\quad = \lim_{n\to\infty} \frac{2\sin(2n(d+p))\sin(4(d+p))}{2\sin(2n(d+p))\sin(2(d+p))} \\
&\quad = \lim_{n\to\infty} \frac{2\sin(2(d+p))\cos(2(d+p))}{\sin(2(d+p))} \\
&\quad = 2\cos 2(d+p).
\end{aligned}$$

Finally,

$$d = -p + \frac{1}{2}\arccos\left(\frac{1}{2}\lim_{n\to\infty} \frac{D(n-2) - D(n+2)}{D(n-1) - D(n+1)}\right).$$

Since $d+p \in \left(\frac{2\pi}{3}, \pi\right)$ are infinitely many $n \in \mathbb{N}$ with property $\sin 2n(a+d) \neq 0$. □

Lemma 2.4. *If the coefficients a and p are known and $w \neq 0$, then $w = \int_d^\pi q(t)dt$ is unique ordered by spectra $\{\mu_{nj}\}$, $n \geq 1$, of the boundary value problems L_j, for $j = 0, 1$.*

Proof. If we put $\varrho = im,\ m \in \mathbb{N}$ in (2.14), we have

(2.23)

$$\begin{aligned}\Delta_0(-m^2) = a^+ \frac{\sin im\pi}{im} + a^- \frac{\sin im(2p-\pi)}{im} \\ + \frac{w}{-2m^2}\left(-a^+ \cos im(\pi - d) + a^- \cos im(2p - \pi + d)\right) \\ + o\left(\frac{\exp(m(\pi - d))}{-im^3}\right), \qquad m \to \infty.\end{aligned}$$

From (2.23), we have

$$w = \frac{2}{a^+} \lim_{m\to\infty} \frac{\Delta_0(-m^2) - a^+ \frac{\sin im\pi}{im} - a^- \frac{\sin im(2d-\pi)}{im}}{\frac{\exp(m(\pi-a))}{m^2}}.$$

Since, the function $\Delta_0(\mu)$ is ordered by spectrum $\{\mu_{n0}\}$, Lemma 2.2, delay point d is ordered by spectra $\{\mu_{nj}\}$, $n \geq 1$, Lemma 2.3 and coefficients a and p are known, then we proved that integral $w = \int_d^\pi q(t)dt$ is unique ordered by spectra $\{\mu_{nj}\}$, $n \geq 1$. □

3. Potential Function Construction

In this section, we will construct an integral equation on the potential q using the Fourier series. Also, we will solve this integral equations and indirectly proved uniqueness of the solution of this integral equation. From the previous part, we conclude that functions $\Delta_j(\mu) = F_j(\varrho)$ are ordered by spectra $\{\mu_{nj}\}$, $j = 0, 1,\ n \geq 1$.

We introduce notation $W := \int_{2d}^\pi q(t) \int_d^{t-d} q(t_1)\, dt_1 dt$ and we also define the functions $K : [0, \pi] \to \mathbb{R}$, $q_1 : [-\frac{d}{2}, \pi - \frac{d}{2}] \to \mathbb{R}$ and $q_2 : \left[-p - \frac{d}{2}, \pi - p - \frac{d}{2}\right] \to \mathbb{R}$ of the following forms

$$K(t) = \begin{cases} q(t+a) \int_d^t q(u)du - q(t) \int_{t+d}^\pi q(u)du \\ \quad - \int_{t+d}^\pi q(u-t)q(u)du, & t \in [d, \pi - d], \\ \\ 0, & \text{else}, \end{cases}$$

$$q_1(t) = \begin{cases} q\left(t + \frac{d}{2}\right), & t \in [\frac{d}{2}, \pi - \frac{d}{2}], \\ 0, & \text{else}, \end{cases}$$

and

$$q_2(t) = \begin{cases} q\left(t+p+\frac{d}{2}\right), & t \in [\frac{d}{2}-p, \pi-p-\frac{d}{2}], \\ 0, & \text{else.} \end{cases}$$

It is obvious that $K(t) \in L^2[0,\pi]$, $q_1(t) \in L^2[0,\pi]$ and $q_2(t) \in L^2[-p-\frac{d}{2}, \pi-p-\frac{d}{2}]$. Using the following formulas and similar formulas of the cosine function

$$\int_{2d}^{\pi}\int_{d}^{t-d} q(t)q(u)\sin\varrho(\pi-2u)dudt = \int_{d}^{\pi-d}\sin\varrho(\pi-2t)q(t)\int_{t+d}^{\pi} q(u)dudt,$$

$$\int_{2d}^{\pi}\int_{d}^{t-d} q(t)q(u)\sin\varrho(\pi-2t+2d)dudt = \int_{d}^{\pi-d}\sin\varrho(\pi-2t)q(t+d)\int_{d}^{t} q(u)dudt,$$

$$\int_{2d}^{\pi}\int_{d}^{t-d} q(t)q(u)\sin\varrho(\pi-2t+2u)dudt = \int_{d}^{\pi-d}\sin\varrho(\pi-2t)\int_{t+d}^{\pi} q(u)q(u-t)dudt,$$

and notations

$$a_{c_1}(\varrho) = \int_{\frac{d}{2}}^{\pi-\frac{d}{2}} q_1(t)\cos\varrho(2t-\pi)dt,$$

$$a_{s_1}(\varrho) = \int_{\frac{d}{2}}^{\pi-\frac{d}{2}} q_1(t)\sin\varrho(2t-\pi)dt,$$

$$a_{c_2}(\varrho) = \int_{\frac{d}{2}-d}^{\pi-p-\frac{d}{2}} q_2(t)\cos\varrho(2t-\pi)dt,$$

$$a_{s_2}(\varrho) = \int_{\frac{d}{2}-p}^{\pi-p-\frac{d}{2}} q_2(t)\sin\varrho(2t-\pi)dt,$$

$$k_c(\varrho) = \int_{d}^{\pi-d} K(t)\cos\varrho(\pi-2t)dt,$$

$$k_s(\varrho) = \int_{d}^{\pi-d} K(t)\sin\varrho(\pi-2t)dt.$$

Using (2.14) and (2.15), we get

$$\begin{aligned}(3.1)\quad F_0(\varrho) &= a^+\frac{\sin\varrho\pi}{\varrho} + a^-\frac{\sin\varrho(2p-\pi)}{\varrho}\\ &\quad + \frac{1}{2\varrho^2}\left[w(-a^+\cos\varrho(\pi-d) + a^-\cos\varrho(2p-\pi+d))\right]\\ &\quad + \frac{1}{2\varrho^2}(a^+a_{c_1} - a^-a_{c_2}) - \frac{1}{4\varrho^3}(\sin\varrho(\pi-2d)W + k_s(\varrho)),\end{aligned}$$

and

$$\begin{aligned}(3.2)\quad F_1(\varrho) &= a^+\cos\varrho\pi - a^-\cos\varrho(2p-\pi)\\ &\quad + \frac{1}{2\varrho}\left[w\left(a^+\sin\varrho(\pi-d) + a^-\sin\varrho(2p-\pi+d)\right)\right]\\ &\quad + \frac{1}{2\varrho}(a^+a_{s_1} - a^-a_{s_2}) - \frac{1}{4\varrho^2}(W\cos\varrho(\pi-2d) + k_c(\varrho)).\end{aligned}$$

Using $\int_d^{\pi-d} K(t)dt = -W$, we have

$$\begin{aligned}k_s(\varrho) &= \int_d^{\pi-d} K(t)\sin\varrho(\pi-2t)dt\\ &= \int_d^{\pi-d}\left(\int_d^t K(s)ds\right)'\sin\varrho(\pi-2t)dt\\ &= -\int_d^{\pi-d} K(t)dt\sin\varrho(\pi-2d)\\ &\quad + 2\varrho\int_d^{\pi-d}\left(\int_d^t K(s)ds\right)\cos\varrho(\pi-2t)dt\\ &= W\sin\varrho(\pi-2d) + 2\varrho K_c^\circ(t),\end{aligned}$$

and by a similar calculation, we get

$$k_c(\varrho) = -W\cos\varrho(\pi-2d) - 2\varrho K_s^\circ(\varrho),$$

such that

$$\begin{aligned}K_c^\circ(\varrho) &= \int_d^{\pi-d}\int_d^t K(s)ds\cos\varrho(\pi-2t)dt,\\ K_s^\circ(\varrho) &= \int_d^{\pi-d}\int_d^t K(s)ds\sin\varrho(\pi-2t)dt.\end{aligned}$$

From (3.1) and (3.2), we get

$$\begin{aligned}F_0(\varrho) &= a^+\frac{\sin\varrho\pi}{\varrho} + a^-\frac{\sin\varrho(2p-\pi)}{\varrho}\\ &\quad + \frac{1}{2\varrho^2}\left[w(-a^+\cos\varrho(\pi-d) + a^-\cos\varrho(2p-\pi+d))\right]\end{aligned}$$

$$+\frac{1}{2\varrho^2}(a^+a_{c_1}-a^-a_{c_2})$$
$$-\frac{1}{4\varrho^3}(\sin\varrho(\pi-2d)W+W\sin\varrho(\pi-2d)+2\varrho K_c^\circ(t)),$$

$$F_1(\varrho)=a^+\cos\varrho\pi-a^-\cos\varrho(2p-\pi)$$
$$+\frac{1}{2\varrho}\left[w\left(a^+\sin\varrho(\pi-d)+a^-\sin\varrho(2p-\pi+d)\right)\right]$$
$$+\frac{1}{2\varrho}(a^+a_{s_1}-a^-a_{s_2})$$
$$-\frac{1}{4\varrho^2}(W\cos\varrho(\pi-2d)-W\cos\varrho(\pi-2d)-2\varrho K_s^\circ(\varrho)).$$

Let us define the functions

$$K_s^*(\varrho)=-\int_d^{\pi-d}\int_t^{\pi-d}K(s)ds\sin\varrho(\pi-2t)dt,$$
$$K_c^*(\varrho)=-\int_d^{\pi-d}\int_t^{\pi-d}K(s)ds\cos\varrho(\pi-2t)dt.$$

Using $\int_d^{\pi-d}K(t)dt=-W$, we get

$$\int_d^t K(s)ds=-W-\int_t^{\pi-d}K(s)ds,$$

and

$$K_c^\circ(\varrho)=\int_d^{\pi-d}\int_d^t K(s)ds\cos\varrho(\pi-2t)dt$$
$$=\int_d^{\pi-d}\left(-W-\int_t^{\pi-d}K(s)\right)ds\cos\varrho(\pi-2t)dt$$
$$=-W\int_d^{\pi-d}\cos\varrho(\pi-2t)dt-\int_d^{\pi-d}\int_t^{\pi-a}K(s)ds\cos\varrho(\pi-2t)dt$$
$$=-\frac{W\sin\varrho(\pi-2d)}{\varrho}+K_c^*(\varrho),$$
$$K_s^\circ(\varrho)=K_s^*(\varrho).$$

So, we have

$$F_0(\varrho)=a^+\frac{\sin\varrho\pi}{\varrho}+a^-\frac{\sin\varrho(2p-\pi)}{\varrho}$$
$$+\frac{1}{2\varrho^2}\left[w(-a^+\cos\varrho(\pi-d)+a^-\cos\varrho(2p-\pi+d))\right]$$
$$+\frac{1}{2\varrho^2}(a^+a_{c_1}-a^-a_{c_2})-\frac{1}{2\varrho^2}K_c^*(\varrho)),\tag{3.3}$$

and

$$
\begin{aligned}
F_1(\varrho) = {} & a^+ \cos \varrho\pi - a^- \cos \varrho(2p-\pi) \\
& + \frac{1}{2\varrho}\left[w\left(a^+ \sin \varrho(\pi-d) + a^- \sin \varrho(2p-\pi+d)\right)\right] \\
& + \frac{1}{2\varrho}(a^+ a_{s_1} - a^- a_{s_2}) + \frac{1}{2\varrho}K_s^*(\varrho)).
\end{aligned} \tag{3.4}
$$

Now, we define the functions

$$
\begin{aligned}
A(\varrho) = {} & 2\varrho^2(F_0(\varrho) - 2\varrho a^+ \sin \varrho\pi - 2\varrho a^- \sin \varrho(2p-\pi) \\
& + w(-a^+ \cos \varrho(\pi-d) - a^- \cos \varrho(2p-\pi+d)),
\end{aligned}
$$

$$
\begin{aligned}
B(\varrho) = {} & -2\varrho F_1(\varrho) + 2\varrho a^+ \cos \varrho\pi - 2\varrho a^- \cos \varrho(2p-\pi) \\
& - w\left(a^+ \sin \varrho(\pi-d) + a^- \sin \varrho(2p-\pi+d)\right).
\end{aligned}
$$

From (3.3) and (3.4), we have

$$
A(\varrho) = a^+ a_{c_1} - a^- a_{c_2} - K_c^*(\varrho), \qquad B(\varrho) = -a^+ a_{s_1} + a^- a_{s_2} - K_s^*(\varrho). \tag{3.5}
$$

Using Lemmas 2.2 and 2.3, it is obviously that the functions $A(\varrho)$ and $B(\varrho)$ are ordered from spectra. Now, we will put $\varrho = m$ for $m \in \mathbb{N}$ into (3.5) and we introduce function $Q_2 : [0,\pi] \to \mathbb{R}$ with $Q_2(t) = q_2(t), t \in \left[0, \pi - p - \frac{d}{2}\right]$ and on the rest of interval $Q_2(t) = q_2(t-\pi), t \in \left(\pi - p - \frac{d}{2}, \pi\right]$. It is easy to check that for $m \in \mathbb{N}$ it is fulfilled

$$
\int_{\frac{d}{2}-p}^{\pi-p-\frac{d}{2}} q_2(t) \sin 2mt dt = \int_{-p-\frac{d}{2}}^{\pi-p-\frac{d}{2}} q_2(t) \sin 2mt dt = \int_0^{\pi} Q_2(t) \sin 2mt dt.
$$

$$
\int_{\frac{d}{2}-p}^{\pi-p-\frac{d}{2}} q_2(t) \cos 2mt dt = \int_{-p-\frac{d}{2}}^{\pi-p-\frac{d}{2}} q_2(t) \cos 2mt dt = \int_0^{\pi} Q_2(t) \cos 2mt dt,
$$

and

$$
\int_{\frac{d}{2}-p}^{\pi-p-\frac{d}{2}} q_2(t) dt = \int_{-p-\frac{d}{2}}^{\pi-p-\frac{d}{2}} q_2(t) dt = \int_0^{\pi} Q_2(t) dt.
$$

We get

$$
\begin{aligned}
(-1)^m A(m) = {} & a^+ \int_{\frac{d}{2}}^{\pi-\frac{d}{2}} q_1(t) \cos 2mt\, dt - a^- \int_0^{\pi} Q_2(t) \cos 2mt\, dt \\
& - \int_d^{\pi-d} \int_t^{\pi-d} K^*(s) ds \cos 2mt\, dt,
\end{aligned}
$$

$$(-1)^{m+1}B(m) = a^+ \int_{\frac{d}{2}}^{\pi-\frac{d}{2}} q_1(t)\sin 2mt\,dt - a^- \int_0^{\pi} Q_2(t)\sin 2mt\,dt - \int_d^{\pi-d}\int_t^{\pi-d} K^*(s)\,ds\sin 2mt\,dt.$$

Also from (3.5), we have

$$\lim_{\varrho\to 0} A(\varrho) = a^+ \int_{\frac{d}{2}}^{\pi-\frac{d}{2}} q_1(t)dt - a^- \int_0^{\pi} Q_2(t) - \int_d^{\pi-d}\int_t^{\pi-d} K^*(s)dsdt.$$

It is easy to prove that $A(m)$ and $B(m)$ belong to the space l^2. Therefore, according to the Riesz-Fischer theorem, there is a function f from $L^2[0,\pi]$ such that

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{+\infty}(-1)^m A(m)\cos 2mt + (-1)^{m+1}B(m)\sin 2mt.$$

Function $f(t)$ is obviously ordered by spectra $\mu_{nj}, j = 0,1$. From this equations we have integral equation on the potential q. We multiply the first equation with $\frac{2}{\pi}\cos 2mt,\ t \in (0,\pi)$, second equation with $\frac{2}{\pi}\sin 2mt, t \in (0,\pi)$ and then sum them up. Using definition of function q_1 and f, we get the integral equation on the interval $(0,\pi)$

$$f(t) = a^+ q_1(t) - a^- Q_2(t) - K^*(t),$$

where $K^* : [0,\pi] \to \mathbb{R}$ is defined with

$$K^*(t) = \begin{cases} \int_t^{\pi-d} K(s)ds, & t \in (d, \pi-d), \\ 0, & \text{else.} \end{cases}$$

Using definition of the function Q_2 we get two integral equation. Firstly, on the interval $(0, \pi - p - \frac{d}{2})$

$$f(t) = a^+ q_1(t) - a^- q_2(t) - K^*(t), \tag{3.6}$$

and second on the interval $(\pi - p - \frac{d}{2}, \pi)$

$$f(t) = a^+ q_1(t) - a^- q_2(t-\pi) - K^*(t). \tag{3.7}$$

Using this integral equations we will prove the uniqueness of the potential.

Theorem 3.1. *If coefficients a and p are known, then spectra $\{\mu_{nj}\},\ n \geq 1$ of boundary spectral problems $L_j,\ j = 0,1$, uniquely determine potential q on the interval $(0,\pi)$ when $\frac{\pi}{3} < p \leq d < \frac{\pi}{2}$.*

Proof. From integral equations (3.6) and (3.7) we construct the potential q on the interval $[0,\pi]$. Since $p > \frac{\pi}{3}$ and $d < \frac{\pi}{2}$, we can conclude that it is $p > \frac{d}{2}$. We will study two cases, first when $\frac{3d}{2} + p \geq \pi$ and second when $\frac{3d}{2} + p < \pi$.

1. $\frac{3d}{2}+p\geq\pi$
   1.1. When $t\in\big(0,\frac{d}{2}\big)\subset\big(0,\pi-p-\frac{d}{2}\big)$, integral equation (3.6) has the form
   $$f(t)=-a^{-}q_2(t).$$
   Since $f(t)$ is ordered by spectra, we conclude that potential q is known on the $\big(\frac{d}{2}+p,d+p\big)$.
   1.2. When $t\in\big(\pi-p+\frac{d}{2},\pi\big)\subset\big(\pi-p-\frac{d}{2},\pi\big)$, integral equation (3.7) has the form
   $$f(t)=-a^{-}q_2(t-\pi).$$
   Since $f(t)$ is ordered by spectra, we conclude that potential function q is known on the $\big(d,\frac{d}{2}+p\big)$.
   1.3. When $t\in\big(\frac{d}{2},\pi-p-\frac{d}{2}\big)\subset\big(0,\pi-p-\frac{d}{2}\big)$, since $\pi-p-\frac{d}{2}<d$ integral equation (3.6) has the form
   $$f(t)=a^{+}q_1(t)-a^{-}q_2(t).$$
   Since $(d,\pi-p)\subset(d,d+p)$ from [1.1.] and [1.2.] we have $q_1(t)$ is known on the interval $(d,\pi-p)$ and we conclude that q is known on the $(d+p,\pi)$.
   Since $q\in L^2(0,\pi)$, we proved that potential q is unique on the interval $(0,\pi)$, when $\frac{3d}{2}+p\geq\pi$.
2. $\frac{3d}{2}+p<\pi$
   2.1. When $t\in\big(0,\frac{d}{2}\big)\subset\big(0,\pi-p-\frac{d}{2}\big)$, integral equation (3.6) has the form
   $$f(t)=-a^{-}q_2(t).$$
   Since function $f(t)$ is ordered by spectra, we conclude that potential q is known on the $\big(\frac{d}{2}+p,d+p\big)$.
   2.2. When $t\in\big(\pi-p+\frac{d}{2},\pi\big)\subset\big(\pi-p-\frac{d}{2},\pi\big)$,integral equation (3.7) has the form
   $$f(t)=-a^{-}q_2(t-\pi).$$
   Since $f(t)$ is ordered by spectra, we conclude that potential q is known on the $\big(d,\frac{d}{2}+p\big)$.
   2.3. When $t\in\big(\frac{d}{2},d\big)$, since $d<\pi-p-\frac{d}{2}$ we have $\big(\frac{d}{2},d\big)\subset\big(0,\pi-p-\frac{d}{2}\big)$ and integral equation (3.6) has the form
   $$f(t)=a^{+}q_1(t)-a^{-}q_2(t).$$
   Since $\big(d,\frac{3d}{2}\big)\subset(d,d+p)$ from [2.1.] and [2.2.] we have $q_1(t)$ is known on the interval $\big(d,\frac{3d}{2}\big)$ and we conclude that potential q is known on the $\big(d+p,\frac{3d}{2}+p\big)$.

2.4. When $t \in \left(\pi - d, \pi - \frac{d}{2}\right) \subset \left(\pi - p - \frac{d}{2}, \pi\right)$, integral equation (3.7) has the form

$$f(t) = a^{+} q_1(t).$$

Since $f(t)$ is ordered by spectra, we conclude that potential q is known on the $\left(\pi - \frac{d}{2}, \pi\right)$.

Since $q \in L^2(0, \pi)$ and $p + \frac{3d}{2} > \pi - \frac{d}{2}$ we proved that potential q is unique on the interval $(0, \pi)$, when $\frac{3d}{2} + p < \pi$. □

4. Conclusion

In this manuscript, the second order differential operators with constant delay and transmission boundary conditions are studied. By obtaining the asymptotic forms of eigenvalues and eigenfunctions, we find an algorithm to the recovering the operators by using the spectrums of two different Dirichlet-Dirichlet and Dirichlet-Neumann problems. Indirectly, by calculating the coefficients of Fourier series, the potential is reconstructed.

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[1] Department of Mathematics, Faculty of Science, University of Maragheh, P.O. Box 55136-553, Maragheh, Iran.
Email address: shahriari@maragheh.ac.ir

[2] Department of Mathematics, University of East Sarajevo, 71123, East Sarajevo, Bosnia and Herzegovina.
Email address: vladimir.vladicic@ff.ues.rs.ba