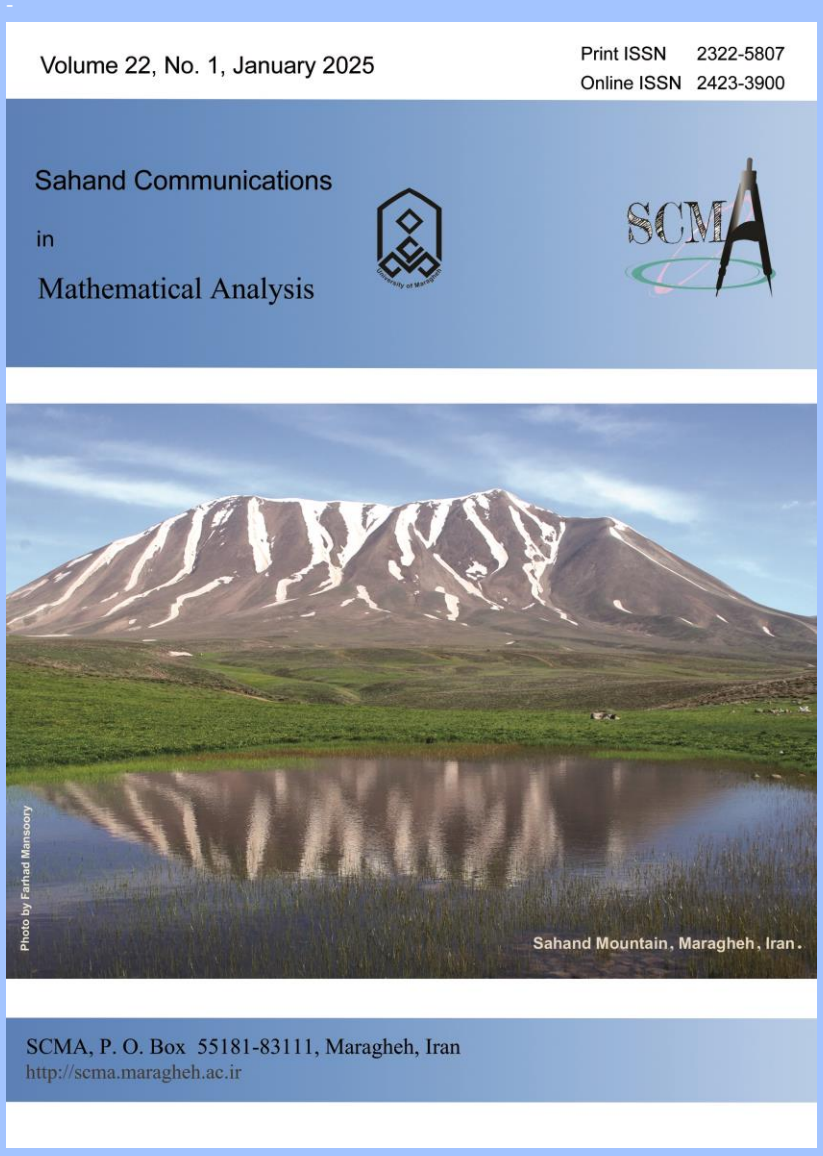
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Remarks on Some Results of Extended Interpolative Hardy-Rogers-Geraghty-Wardowski Contractions and Ćirić-Reich-Rus Type F-Contractions

Dušan Bajović[1*], Slobodanka Mitrović[2], Sandra Buhmiler[3] and Stojan Radenović[4]

Abstract. In this paper, we improve the previous results from fixed point theory related to four types of contractions: interpolative Kannan contraction, extended interpolative and Geraghty type contraction, interpolative Hardy-Rogers contraction and Wardowski F-contraction. The basis for our generalization is the results from "Extended Interpolative Hardy-Rogers-Geraghty-Wardowski Contractions and an Application" as well as "On extended interpolative Ćirić-Reich-Rus type F-contractions and an application".

1. Introduction and Preliminaries

In the paper [4], the authors tried to connect known contractive conditions: Interpolation, Hardy-Rogers, Geraghty and Wardowski contraction. To this end, they first state their Definition 6, in the Main Results section. They denoted by $\mathcal{V}$ the function that maps $[0, +\infty)$ to $[0, 1)$ with the property that if $\mathcal{V}(x_n)$ tends to 1^- then x_n tends to 0^+. We observe that the function $\mathcal{V}$ can be a constant k, where k belongs to $[0, 1)$. If we set $\mathcal{V}(t) = 0 = k$ in their Definition 6, then a contradiction arises from condition (1) ($Q(0^+) = -\infty$). An error was made in the mentioned Definition 6, because $\mathcal{V}(t)$ must be multiplied by t, where in it t is product of four factors. If $\mathcal{V}(t) = k$, where k belongs to $(0, 1)$, we need to check if Theorem 9 from [4] is correct, of course with the

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* Corresponding author.

catch that we fixed condition (1) in Definition 6 by multiplying $\mathcal{V}(t)$ by t, where t is product of four factors. Let us also note that the other two conditions for the constants ρ, ϱ and σ are redundant as they obviously follow from the first condition $\rho + \varrho + \sigma < 1$.

Let us recall the Geraghty contraction [11]:

$$d(fx, fy) \leq g(d(x, y))d(x, y)$$

where g maps $[0, +\infty)$ in $[0, 1)$ with the property that if $g(x_n)$ tends to 1^- then x_n tends to 0^+. Note: If there is an expression instead of $d(x, y)$, then g of that expression must be multiplied by that expression which is not case in Definition 6, of the paper [4]. We are now considering how to improve the work by connecting the four known types of contractions. First, we take condition (1) from [4] to be in the form $\tau + F(d(Tx, Ty)) \leq F(\mathcal{V}(t) \cdot t), \tau > 0$, where F is strictly increasing on $(0, +\infty)$ and the expression t is, as in the paper [4], an argument of $\mathcal{V}$. For more details on all types of contractive mappings see ([1–3], [5–19],[21–26]).

We now list the definitions of the terms and the results from the papers [4, 20]. Please note that when writing this paper, we will use common, i.e., traditional notations from mathematical literature.

Definition 1.1 ([4, Definition 6]) Let $(\Lambda, \mathcal{N})$ be a metric space and let L be a mapping from Λ to itself. We say that the mapping L is an extended interpolative Hardy-Rogers-type-Geraghty $\mathcal{D}$-contraction if there exist $\rho, \varrho, \sigma \in (0, 1)$ with $\rho + \varrho + \sigma < 1, \varrho + \sigma < 1, \rho + \varrho < 1$ and $\mathcal{D} \in \mathbb{Q}$ such that

$$\begin{aligned}\mathcal{D}\left(\mathcal{N}\left(Lx, Ly\right)\right) \leq \mathcal{D}\Bigg(\mathcal{V}\Bigg(& \left(\mathcal{N}\left(x, y\right)\right)^{\rho} \cdot \left(\mathcal{N}\left(x, Lx\right)\right)^{\varrho} \cdot \left(\mathcal{N}\left(y, Ly\right)\right)^{\sigma} \\ & \cdot \left(\frac{\mathcal{N}\left(x, Ly\right) + \mathcal{N}\left(y, Lx\right)}{2}\right)^{\omega}\Bigg)\Bigg) + \rho\mathcal{D}\left(\mathcal{N}\left(x, y\right)\right) \\ & + \varrho\mathcal{D}\left(\mathcal{N}\left(x, Lx\right)\right) + \sigma\mathcal{D}\left(\mathcal{N}\left(y, Ly\right)\right) \\ & + \left(1 - \rho - \varrho - \sigma\right)\mathcal{D}\left(\frac{\mathcal{N}\left(x, Ly\right) + \mathcal{N}\left(y, Lx\right)}{2}\right)\end{aligned}$$

for all $x, y \in \Lambda \backslash Fix\left(L\right)$ with $\omega = 1 - \rho - \varrho - \sigma$ and $\mathcal{N}\left(Lx, Ly\right) > 0$ and $\mathcal{V} \in \mathcal{G}$ where $\mathcal{G}$ is the class of functions $\mathcal{V} : [0, +\infty) \to [0, 1)$ so that $\mathcal{V}\left(a_n\right) \to 1$ implies $a_n \to 0$ as $n \to +\infty$.

Theorem 1.2 ([4, Theorem 9]) *Any extended interpolative Hardy-Rogers-type Geraghty $\mathcal{D}$-contraction self-mapping L on a complete metric space $(\Lambda, \mathcal{N})$ possesses a fixed point in Λ.*

• Note that the function $\mathcal{D}$ satisfies the conditions ([4], page 266, lines 5+,6+):

1. $\mathcal{D}$ is strictly increasing on $(0,+\infty)$ and continuous,
2. $\lim_{n\to+\infty} a_n = 1$ if and only if $\lim_{n\to+\infty} \mathcal{D}(a_n) = 0$, for any sequence $a_n \in (0,+\infty)$.

The authors introduced in [20] the notion of extended interpolative Ćirić-Reich-Rus type F-contractions.

Definition 1.3 ([20, Definition 2.1]) Let (X,d) be a metric space. The self-mapping T on X is an extended interpolative Ćirić-Reich-Rus type F-contraction if there exist $\alpha,\beta \in (0,1)$ with $\alpha+\beta<1, \tau>0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(Tx,Ty)) \leq \alpha F(d(x,y)) + \beta F(d(x,Tx)) + (1-\alpha-\beta) F(d(y,Ty)) \tag{1.1}$$

for all $x,y \in X\backslash Fix(T)$ with $d(Tx,Ty)>0$.

Also, in [20] the authors formulated and proved the next result:

Theorem 1.4 ([20, Theorem 2.2]) *An extended interpolative Ćirić-Reich-Rus type F-contraction self-mapping on a complete metric space admits a fixed point in X.*

• Note that $\mathcal{F}$ in previously Definition and Theorem represents the collection of all functions $F:(0,+\infty)\to\mathbb{R}$ so that:

(F1) F is strictly increasing.

(F2) For each sequence $\{\alpha_n\}$ in $(0,+\infty)$, $\lim\limits_{n\to+\infty} \alpha_n = 0$ if and only if

$$\lim_{n\to+\infty} F(\alpha_n) = -\infty.$$

(F3) There is $k \in (0,1)$ so that $\lim_{\alpha\to0^+} \alpha^k F(\alpha) = 0$.

2. Main Results

Before formulating and proving our first new result in this paper, we state the following two lemmas, which we have used extensively in earlier works on F-contractions. By their very nature, these lemmas are quite simple, yet also useful, especially when it is necessary to determine the Cauchyness of the Picard sequence in the metric space (X,d). For the proofs of both lemmas see for instance ([10], [22]).

Lemma 2.1 *For the Picard sequence x_n in the metric space (X,d) the following holds: $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$. Then x_n is distinct from x_m whenever n is different from m.*

Lemma 2.2 *In the metric space (X,d), let for the Picard sequence $\{x_n\}$ the following condition holds $\{d(x_n,x_{n+1})\}$ is non-increasing sequence and $\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0$. If the sequence $\{x_{2n}\}$ is not a Cauchy in*

(X,d), then for some $\varepsilon>0$ there are two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers, $n_k>m_k>k$, such that the following sequences

$$d\left(x_{2m_k},x_{2n_k}\right),d\left(x_{2m_k},x_{2n_k-1}\right),d\left(x_{2m_k+1},x_{2n_k}\right),$$
$$d\left(x_{2m_k-1},x_{2n_k+1}\right),d\left(x_{2m_k+1},x_{2n_k+1}\right),\ldots$$

tend to ε^+ when $k\to+\infty$.

Remark 2.3 This lemma holds without the hypothesis that the sequence $\{d\left(x_n,x_{n+1}\right)\}$ is non-increasing. Obviously, in that case we have that the following sequences

$$d(x_{m_k},x_{n_k}),d(x_{m_k},x_{n_k-1}),$$
$$d(x_{m_k+1},x_{n_k}),d(x_{m_k-1},x_{n_k+1}),d(+x_{m_k+1},x_{n_k+1}),\ldots$$

tend to ε^+ when $k\to+\infty$.

• Let us recall that every strictly increasing function F from $(0,+\infty)$ in $\mathbb{R}$ is continuous almost everywhere and that at every point a from $(0,+\infty)$ there are left and right limits $F(a-0)$ and $F(a+0)$ respectively and the following applies to them: $F(a-0)\leq F(a)\leq F(a+0)$. In our work, we will assume that $F(0^+)=-\infty$, because there are examples of strictly increasing functions from $(0,+\infty)$ in $\mathbb{R}$ for which $F(0^+)$ is finite.

• Since, based on the above, both Definition 6 and its corresponding Theorem 9, from [4] are at least suspicious, if not outright incorrect, in this paper we will try to either alleviate these inaccuracies or, in some cases, fundamentally correct them.

With the following definition, we improve Definition 6, from [4]. And then, with the theorem that follows, we give a new approach to connecting four types of contractions. Using the above two important lemmas we arrive at what we strongly believe is an essential correction of Theorem 9, from [4].

Definition 2.4 Let (X,d) be the given metric space and let T be the mapping from X to itself. For the mapping T, we will say that it is the interpolative-Hardy-Rogers-Geraghty type-F-contraction if there are numbers a,b,c from $(0,1)$ in $a+b+c<1$, Geraghty function $g:[0,+\infty)\to(0,1)$ with the property: if $g\left(x_n\right)\to1^-$ then $x_n\to0$, a strictly increasing function F from $(0,+\infty)$ in $\mathbb{R}$ with $F(0+)=-\infty$ and $\tau>0$ such that the following holds

$$\tau+F\left(d\left(Tx,Ty\right)\right)\leq F\left(g\left(v\left(x,y,T,a,b,c\right)\right)\cdot v\left(x,y,T,a,b,c\right)\right),\tag{2.1}$$

for all $x,y\in X\backslash Fix\left(T\right)$ with $d(Tx,Ty)>0$, where

$$v\left(x,y,T,a,b,c\right)=\left(d\left(x,y\right)\right)^a\cdot\left(d\left(x,Tx\right)\right)^b\cdot\left(d\left(y,Ty\right)\right)^c$$

$$\cdot \left(\frac{d\left(x, Ty\right) + d\left(y, Tx\right)}{2} \right)^{1-a-b-c}.$$

Theorem 2.5 *Every interpolative-Hardy-Rogers-Geraghty type F-contraction on the complete metric space (X, d) has a fixed point.*

Proof. Since the Geraghty function g is bounded and the function F is strictly increasing, the following new contractive condition follows from (2.1):

$$\tau + F\left(d\left(Tx, Ty\right)\right) < F\left(v\left(x, y, T, a, b, c\right)\right). \tag{2.2}$$

Thus, the Geraghty function g is eliminated. Again, due to the strict increasing of the function F from condition (2.2), a new strict condition follows:

$$d\left(Tx, Ty\right) < v\left(x, y, T, a, b, c\right), \tag{2.3}$$

whenever $d(Tx, Ty) > 0$). We will now try to show that condition (2.3) implies the existence of at least one fixed mapping point T. Let us proceed in the usual way by first taking any point x_0 from X. Then for the corresponding Picard sequence $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$.we have two separate possibilities:

1. There exists n_0 from $\mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$.

2. For each n in $\mathbb{N}$ we have that x_{n+1} is different from x_n, i.e., $d(x_n, x_{n+1}) > 0$ for each n in $\mathbb{N}$. The first case gives us that x_{n_0} is a fixed point of the mapping T and then the proof of the Theorem 2.5, is complete. Starting from the second case, we will prove that the sequence $\{x_n\}$ is Cauchy. Then, due to the completeness of the metric space (X, d), it converges to some point, let us say z from X. For the purpose of the above, let us assume that for some k from $\mathbb{N}$: $d\left(x_{k-1}, x_k\right) \leq d\left(x_k, x_{k+1}\right)$. Then from (2.3) we get

$$\begin{aligned} d\left(x_k, x_{k+1}\right) &< \left(d\left(x_k, x_{k+1}\right)\right)^{a+b} \cdot \left(d\left(x_k, x_{k+1}\right)\right)^{c} \cdot \left(\frac{2d\left(x_k, x_{k+1}\right)}{2} \right)^{1-a-b-c} \\ &= \left(d\left(x_k, x_{k+1}\right)\right)^{a+b+c+1-a-b-c}, \end{aligned}$$

This leads to a contradiction, as we cannot have $d(x_k, x_{k+1}) < d(x_k, x_{k+1})$. So the sequence $\left\{d(x_n, x_{n+1})\right\}$ is strictly decreasing and according to Lemma 2.1, all its members are mutually different. What more, it converges to some non-negative number $\delta \geq 0$. Let us prove that $\delta = 0$. To prove this fact we have to return to condition (2.2) where the function F participates. Substituting $x = x_{n-1}$ and $y = x_n$, we obtain:

$$\begin{aligned} &\tau + F\left(d\left(x_n, x_{n+1}\right)\right) \\ &\quad \leq F\left(\left(d\left(x_{n-1}, x_n\right)\right)^{a+b} \cdot \left(d\left(x_n, x_{n+1}\right)\right)^{c} \cdot \left(\frac{d\left(x_{n-1}, x_{n+1}\right)}{2} \right)^{1-a-b-c} \right) \end{aligned}$$

$$\leq F\left(\left(d\left(x_{n-1},x_n\right)\right)^{a+b}\cdot\left(d\left(x_n,x_{n+1}\right)\right)^{c}\cdot\left(\frac{d\left(x_{n-1},x_n\right)+d\left(x_n,x_{n+1}\right)}{2}\right)^{\omega}\right),$$

where $\omega = 1-a-b-c$. By taking the limit as n tends to $+\infty$ and using the property of the function F that in every point from $(0,+\infty)$ has left and right limit, we obtain

$$\tau + F\left(\delta^{+}\right) \leq F\left(\delta^{+}\right),$$

which is a contradiction to $\tau > 0$. This proves sequence $\{d\left(x_n,x_{n+1}\right)\}$ tends to zero. Let us prove that the sequence $\{x_n\}$ is a Cauchy by positing the opposite. Now we move on to the application of Lemma 2.2. For this purpose, we put in (2.3) $x = x_{n_k}, y = x_{m_k}$. Then we get the following strict inequality:

$$\begin{aligned} d\left(x_{n_k+1},x_{m_k+1}\right) &< \left(d\left(x_{n_k},x_{m_k}\right)\right)^{a}\cdot\left(d\left(x_{n_k},x_{n_k+1}\right)\right)^{b}\cdot\left(d\left(x_{m_k},x_{m_k+1}\right)\right)^{c} \\ &\quad\cdot\left(\frac{d\left(x_{n_k},x_{m_k+1}\right)+d\left(x_{n_k+1},x_{m_k}\right)}{2}\right)^{1-a-b-c}. \end{aligned}$$

If in the resulting inequality we take limit when k tends to $+\infty$, we get that

$$\begin{aligned} \varepsilon &\leq (\varepsilon)^{a}\cdot(0)^{b}\cdot(0)^{c}\cdot\left(\frac{\varepsilon+\varepsilon}{2}\right)^{1-a-b-c} \\ &= 0, \end{aligned}$$

which is a contradiction with $\varepsilon > 0$.

Hence, we obtain that $d\left(x_n,x_m\right)\to 0$ as $n,m\to+\infty$, i.e., since (X,d) is a complete metric space there exists some $z\in X$ such that $x_n\to z$ as $n\to+\infty$. Further, including in (2.3) $x = x_n, y = z$ where z is the limit of the sequence x_n, we get

$$\begin{aligned} d\left(x_{n+1},Tz\right) &< \left(d\left(x_n,z\right)\right)^{a}\cdot\left(d\left(x_n,x_{n+1}\right)\right)^{b}\cdot\left(d\left(z,Tz\right)\right)^{c} \\ &\quad\cdot\left(\frac{d\left(x_n,Tz\right)+d\left(x_{n+1},z\right)}{2}\right)^{1-a-b-c} \\ &\leq \left(d\left(x_n,z\right)\right)^{a}\cdot\left(d\left(x_n,x_{n+1}\right)\right)^{b}\cdot\left(d\left(z,Tz\right)\right)^{c} \\ &\quad\cdot\left(\frac{d\left(x_n,z\right)+d\left(z,Tz\right)+d\left(x_{n+1},z\right)}{2}\right)^{1-a-b-c}. \end{aligned} \tag{2.4}$$

By passing to limit in (2.4) when n tends to $+\infty$, we get that $d(z,Tz) = 0$, i.e., z is a fixed point of the mapping T. The proof of Theorem 2.5 is complete. □

Corollary 2.6 *In [4] (pages 271, 272) the authors obtained two corollaries. The first corollary is obtained for case $\mathcal{D}(t) = -\frac{1}{t}+1$ and second one is continuation of the first for the case $\mathcal{V}(t) = k, k\in(0,1)$. As previously mentioned, problem arise in accordance with Definition 1.1.*

For strictly increasing and continuous function $\mathcal{D}(t) = -\frac{1}{t} + 1$ *on* $(0, +\infty)$ *holds* $\lim_{t\to 0^+} \mathcal{D}(t) = -\infty$. *Note that based on Definition 1.1, we now obtain for the function* $\mathcal{V}(t) = 0, t \in [0, +\infty)$ *for arbitrary* $j, j' \in \Lambda \setminus Fix(L)$

$$\mathcal{D}(\mathcal{N}Lj, Lj')) \leq -\infty.$$

Since $\mathcal{D}(\mathcal{N}(Lj, Lj'))$ *is finite, we have an error. Second corollary initially excludes* 0 *from image of Geraghty function so it is correct in accordance with Definition 2.4 in this paper.*

3. An Important Note

It often happens that a function is quickly eliminated from the some definition of a contractive condition, such as Wardowski's function F or Geraghty's function g. But the limitations that were at the beginning for those functions remain in the newly obtained relationship, which is more general than the initial one. When it comes to the connection of several types of contractions, then the usual ones can be redefined to make the contractive condition meaningful, i.e., to be well defined. Thus, the authors in [4] did not redefine the Geraghty function $\mathcal{V}$ and because of this, the situation arose that due to the definition of the function $\mathcal{D}$, the stated contractive condition does not make sense. It had to be taken, as we already mentioned, that the Geraghty function takes values in $(0, 1)$ and not, as usual, in its definition $[0, 1)$.

We have seen from the above that (2.1) implies (2.2) and that (2.2) implies the more general condition (2.3). From (2.3) and only once from (2.1) we conclude that the mapping T has a fixed point. If we assume that x and y are two different fixed points of the mapping T, using the condition (2.3) we get:

$$\begin{aligned} d(x,y) = d(Tx,Ty) &< (d(x,y))^a \cdot (d(x,Tx))^b \cdot (d(y,Ty))^c \\ &\cdot \left(\frac{d(x,Ty) + d(y,Tx)}{2}\right)^{1-a-b-c} \\ &= (d(x,y))^a \cdot (0)^b (0)^c \cdot \left(\frac{d(x,y) + d(y,x)}{2}\right)^{1-a-b-c} \\ &= (d(x,y))^{1-b-c} \cdot 0 \cdot 0 \\ &= 0. \end{aligned}$$

We obtain that $d(x, y) = 0$, i.e., $x = y$ thus we proved that the mapping T, if it satisfies the conditions (2.3) has a unique fixed point.

With the previous Theorem 2.5, we have in some way improved the main considerations from the paper [4] Definition 6 and Theorem 9. We used two well-known lemmas and properties of strictly increasing

mapping $F : (0, +\infty) \to \mathbb{R}$. Now employing similar arguments—namely, Lemmas 2.1 and 2.2 as well as the properties **(F1)** and**(F2)** of function F, we will significantly improve the main result from ([20], Theorem 2.2). This part of the work is based on the already introduced Definition 1.3, as well as Theorem 1.4.

Proof of Theorem 1.4. In the usual way, for a given arbitrary point $x_0 \in X$, the corresponding Picard sequence of the form $x_{n+1} = Tx_n$ has the following two distinct properties. There exists k in $\mathbb{N}$ such that $x_k = x_{k-1}$ or for all members of the sequence x_n holds x_{n-1} is different from x_n. From the first case it follows that x_{n-1} is a fixed point of the mapping T. For the second case, we first prove that $d(x_n, x_{n+1}) < d\left(x_{n-1}, x_n\right)$. Assume the opposite, i.e., there exists m from $\mathbb{N}$ such that $d(x_{m-1}, x_m) \leq d\left(x_m, x_{m+1}\right)$. Then by using the contractive condition in Definition 1.3., we get

$$\begin{aligned}
&\tau + F\left(d\left(x_m, x_{m+1}\right)\right) \\
&\leq aF\left(d\left(x_{m-1}, x_m\right)\right) + bF\left(d\left(x_{m-1}, x_m\right)\right) + (1-a-b)\, F\left(d\left(x_m, x_{m+1}\right)\right) \\
&\leq aF\left(d\left(x_m, x_{m+1}\right)\right) + bF\left(d\left(x_m, x_{m+1}\right)\right) + (1-a-b)\, F\left(d\left(x_m, x_{m+1}\right)\right) \\
&= (a+b+1-a-b)F\left(d\left(x_m, x_{m+1}\right)\right) \\
&= F\left(d\left(x_m, x_{m+1}\right)\right),
\end{aligned}$$

which is a contradiction with $\tau > 0$. This proves that the sequence $\{d\left(x_n, x_{n+1}\right)\}$ is strongly decreasing and tends to some $\delta \geq 0$. By the standard way on can get that $\delta = 0$. Now the famous Lemmas 2.1 and 2.2 enter the scene. From the first it follows that all members of the sequence $\{x_n\}$ are different from each other and from the second putting $x = x_{n_k}, y = x_{m_k}$ in (1.1) which according to Lemma 2.1 is valid $(d(Tx, Ty) > 0)$.

So we have

$$\begin{aligned}
\tau + F\left(d\left(x_{n_k+1}, x_{m_k+1}\right)\right) \leq\ & aF\left(d\left(x_{n_k}, x_{m_k}\right)\right) + bF\left(d\left(x_{n_k}, x_{n_k+1}\right)\right) \\
&+ (1-a-b)\, F\left(d\left(x_{m_k}, x_{m_k+1}\right)\right).
\end{aligned}$$

By switching to limit when k tends to $+\infty$ we get

$$\tau + F\left(\varepsilon^+\right) \leq aF\left(\varepsilon^+\right) + (1-a)\, F\left(0^+\right)$$

i.e,

$$\tau + (1-a)\, F\left(\varepsilon^+\right) \leq (1-a)\, F\left(0^+\right) = -\infty,$$

which is obviously a contradiction. Hence, we obtain that $d\left(x_n, x_m\right) \to 0$ as $n, m \to +\infty$, that is, since (X, d) is a complete metric space there exists some $z \in X$ such that $x_n \to z$ as $n \to +\infty$. Now, suppose that $z \neq Tz$, i.e., $d\left(z, Tz\right) > 0$. It is clear that there exists natural number k

such that $z, Tz \neq x_n$ for all $n \geq k$. In this case, substituting $x = x_n, y = z$, into (1.1) we have

$$\tau + F\left(d\left(x_{n+1}, Tz\right)\right) \leq \alpha F\left(d\left(x_n, z\right)\right) + \beta F\left(d\left(x_n, x_{n+1}\right)\right) + \left(1 - \alpha - \beta\right) F\left(d\left(z, Tz\right)\right). \tag{3.1}$$

Letting $n \to +\infty$ in the inequality (3.1), we obtain that

$$F\left(d\left(x_{n+1}, Tz\right)\right) \to -\infty.$$

According to the property **(F2)** of the mapping F it follows that $x_{n+1} \to Tz$ as $n \to +\infty$. This implies that $Tz = z$, i.e., z is a fixed point of the mapping T. The proof of Theorem 1.4 is now complete. □

Example 3.1 Let $X = \{1, 2, 3\}$ be endowed with the metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \frac{5}{3} & \text{if } (x, y) \in \{(2, 3), (3, 2)\}, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, (X, d) is complete. Take $T1 = T3 = 1$ and $T2 = 3$. For $x = 1$ and $y = 2$, we have

$$F(d(T1, T2)) = F(d(1, 3)) = F(1), \qquad F(d(x, y)) = F(d(1, 2)) = F(1).$$

Therefore, we cannot find $\tau > 0$ such that $\tau + F(d(Tx, Ty)) \leq F(d(x, y))$, i.e., theorem for fixed point of $F-$contraction in [26], is not applicable. On the other hand, let $x, y \in X \setminus \mathrm{Fix}(T)$ with $d(Tx, Ty) > 0$. Hence $(x, y) \in \{(2, 3), (3, 2)\}$. Without loss of generality, take $(x, y) = (2, 3)$. Choose $a = \frac{1}{2}$, $b = \frac{1}{3}$, $c = \frac{1}{8}$, $F(t) = \ln(t)$, $g(t) = \frac{9}{10}$ and $\tau = \ln\left(\frac{10}{9}\right)$. We have

$$\begin{aligned} \tau + F\left(d(Tx, Ty)\right) &= \tau + \ln(d(T2, T3)) \\ &= \tau + \ln(d(3, 1)) \\ &= \tau + \ln(1) \\ &= \ln\left(\frac{10}{9}\right). \end{aligned}$$

Since $28 \ln 5 + 49 \ln 2 < 76 \ln 3$, after several transformations, we obtain

$$\ln\left(\frac{10}{9}\right) \leq \ln\left(\frac{9}{10} \cdot \left(\frac{5}{3}\right)^{\frac{5}{6}} \cdot \left(\frac{1}{2}\right)^{\frac{1}{24}}\right).$$

Hence, we have

$$\begin{aligned} \tau + F\left(d(Tx, Ty)\right) &= \ln\left(\frac{10}{9}\right) \\ &\leq \ln\left(\frac{9}{10} \cdot \left(\frac{5}{3}\right)^{\frac{5}{6}} \cdot \left(\frac{1}{2}\right)^{\frac{1}{24}}\right) \end{aligned}$$

$$
\begin{aligned}
&= \ln\left(\frac{9}{10}\cdot\left(\frac{5}{3}\right)^{a+b}\cdot\left(\frac{1}{2}\right)^{1-a-b-c}\right) = \\
&= \ln\left(\frac{9}{10}\cdot d(2,3)^a\cdot d(2,3)^b\cdot d(3,1)^c\cdot\left(\frac{d(2,1)}{2}\right)^{1-a-b-c}\right) \\
&= F\Bigg(g\Bigg(\left(d\left(x,y\right)\right)^a\cdot\left(d\left(x,Tx\right)\right)^b\cdot\left(d\left(y,Ty\right)\right)^c \\
&\cdot\left(\frac{d\left(x,Ty\right)+d\left(y,Tx\right)}{2}\right)^{1-a-b-c}\Bigg)\cdot\left(d\left(x,y\right)\right)^a\cdot\left(d\left(x,Tx\right)\right)^b \\
&\cdot\left(d\left(y,Ty\right)\right)^c\cdot\left(\frac{d\left(x,Ty\right)+d\left(y,Tx\right)}{2}\right)^{1-a-b-c}\Bigg)
\end{aligned}
$$

that is, Theorem 2.5 holds for all $x, y \in X \setminus \mathrm{Fix}(T)$ with $d(Tx, Ty) > 0$. Therefore, from Theorem 2.5 T has a unique fixed point ($u = 1$).

Example 3.2 If we consider $F(t) = \ln(t)$ for $t > 0$ in Theorem 2.5 contraction becomes

$$d(Tx, Ty) \leq e^{-\tau}\cdot g\left(v(x, y, T, a, b, c)\right)\cdot v(x, y, T, a, b, c).$$

As we can observe, Theorem 2.5 extends the interpolative Geraghty contraction.

4. A New Approach

Now we introduce the extended interpolative Ćirić-Reich-Rus-Hardy-Rogers type F-contraction and prove the corresponding result.

Definition 4.1 Let (X, d) be a metric spaces. The self-mapping T on X is an extended interpolative Ćirić-Reich-Rus-Hardy-Rogers type F-contraction if there exist $\alpha, \beta, \gamma \in (0, 1), \tau > 0$ and $F \in \mathcal{F}$ such that

(4.1)

$$
\begin{aligned}
\tau + F\left(d\left(Tx, Ty\right)\right) \leq{}& \alpha F\left(d\left(x, y\right)\right) + \beta F\left(d\left(x, Tx\right)\right) + \gamma F\left(d\left(y, Ty\right)\right) \\
&+ \left(1 - \alpha - \beta - \gamma\right) F\left(\frac{d\left(x, Ty\right) + d\left(y, Tx\right)}{2}\right)
\end{aligned}
$$

for all $x, y \in X \backslash Fix\left(T\right)$ with $d\left(Tx, Ty\right) > 0$.

Remark 4.2 At this point F is strongly increasing on $(0, +\infty)$ and $F(0^+) = -\infty$.

Theorem 4.3 *An extended interpolative Ćirić-Reich-Rus-Hardy-Rogers type F-contraction self-mapping on a complete metric space admits a fixed point in X if at least one of the following conditions*

(i) *T is continuous;*

(ii) *F is continuous,*

holds.

Proof. Let $x_0 \in X$ be any element. Further for the corresponding Picard sequence $x_n = Tx_{n-1}, n = 1, 2, \ldots$ two possibilities are:

Either $x_k = x_{k-1}$ for some $k \in \mathbb{N}$ or $x_n \neq x_{n-1}$ for each $n \in \mathbb{N}$. In the first case it is obvious that x_{n-1} is a fixed point of T. In the second case, we will prove that the sequence $\{x_n\}$ is a Cauchy. Firstly, we consider the sequence $d(x_n, x_{n+1})$. For this, we get the next: putting $x = x_{n-1}, y = x_n$ in (4.1):

(4.2)

$$\tau + F(d(x_n, x_{n+1})) \leq (\alpha + \beta) F(d(x_{n-1}, x_n)) + \gamma F(d(x_n, x_{n+1})) + (1 - \alpha - \beta - \gamma) F\left(\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right).$$

Suppose that for some $m \in \mathbb{N}$ we have $d(x_{m-1}, x_m) \leq d(x_m, x_{m+1})$. Then from (4.2), it follows

$$\begin{aligned}\tau + F(d(x_m, x_{m+1})) &\leq (\alpha + \beta + \gamma) F(d(x_m, x_{m+1})) \\ &\quad + (1 - \alpha - \beta - \gamma) F\left(\frac{d(x_m, x_{m+1}) + d(x_m, x_{m+1})}{2}\right) \\ &= ((\alpha + \beta + \gamma) + (1 - \alpha - \beta - \gamma)) F(d(x_m, x_{m+1}))\end{aligned}$$

that is,

$$\tau + F(d(x_m, x_{m+1})) \leq F(d(x_m, x_{m+1})). \tag{4.3}$$

From (4.3) it follows a contradiction with $\tau > 0$.

Hence, the sequence $\{d(x_n, x_{n+1})\}$ is a strongly decreasing. Therefore, there exists a $\delta \geq 0$ such that $d(x_{n+1}, x_n) \to \delta^+$. If $\delta > 0$ we get according to the (4.3):

$$\tau + F(\delta^+) \leq F(\delta^+),$$

what is a new contradiction with $\tau > 0$. Hence, $d(x_{n+1}, x_n) \to 0^+$ as $n \to +\infty$.

Now, according to the Lemma 2.2, by putting $x = x_{n_k}, y = x_{m_k}$ (it is possible since it is clear that $x \neq y$) we get

$$\begin{aligned}&\tau + F(d(x_{n_k+1}, x_{m_k+1})) \\ &\quad \leq \alpha F(d(x_{n_k}, x_{m_k})) + \beta F(d(x_{n_k}, x_{n_k+1})) + \gamma F(d(x_{m_k}, x_{m_k+1})) \\ &\qquad + (1 - \alpha - \beta - \gamma) F\left(\frac{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})}{2}\right),\end{aligned}$$

from which if $k \to +\infty$, it follows:

$$\tau + F(\varepsilon^+) \leq \alpha F(\varepsilon^+) + \beta F(0^+) + \gamma F(0^+)$$

$$\begin{aligned}&+(1-\alpha-\beta-\gamma)F\left(\frac{\varepsilon^{+}+\varepsilon^{+}}{2}\right)\\&=\alpha F\left(\varepsilon^{+}\right)+(\beta+\gamma)F\left(0^{+}\right)+(1-\alpha-\beta-\gamma)F\left(\varepsilon^{+}\right)\\&=(1-\beta-\gamma)F\left(\varepsilon^{+}\right)+(\beta+\gamma)F\left(0^{+}\right)\end{aligned}$$

or,

$$\tau+(\beta+\gamma)F\left(\varepsilon^{+}\right)\leq-\infty,$$

what is obviously a contradiction. This means that the sequence $\{x_n\}$ is a Cauchy. Since the metric space (X,d) is a complete, there exists $z\in X$ such that $x_n\to z$ as $n\to+\infty$. We will prove that $Tz=z$ if either (i) or (ii) holds. Suppose that $Tz\neq z$. In the case (i) it is impossible. If the condition (ii) holds then there is natural n_0 such that for $n\geq n_0$ we have that $z,Tz\notin\{x_n:n\geq n_0\}$. Now, by putting $x=x_n,n\geq n_0,y=z$ in (4.2) we get:

(4.4)

$$\begin{aligned}\tau+F\left(d\left(x_{n+1},Tz\right)\right)\leq{}&\alpha F\left(d\left(x_n,z\right)\right)+\beta F\left(d\left(x_{n-1},x_n\right)\right)+\gamma F\left(d\left(z,Tz\right)\right)\\&+(1-\alpha-\beta-\gamma)F\left(\frac{d\left(x_n,Tz\right)+d\left(z,x_{n+1}\right)}{2}\right).\end{aligned}$$

Putting the limit in (4.4) as $n\to+\infty$, since F is continuous, we obtain,

$$\begin{aligned}\tau+F\left(d\left(z,Tz\right)\right)&\leq\alpha F\left(0^{+}\right)+\beta F\left(0^{+}\right)+\gamma F\left(d\left(z,Tz\right)\right)\\&\quad+(1-\alpha-\beta-\gamma)F\left(\frac{d\left(z,Tz\right)}{2}\right)\\&\leq(\alpha+\beta)F\left(0^{+}\right)+\gamma F\left(d\left(z,Tz\right)\right)\\&\quad+(1-\alpha-\beta-\gamma)F\left(d\left(z,Tz\right)\right)\end{aligned}$$

that is,

$$\tau+(\alpha+\beta)F\left(d\left(z,Tz\right)\right)\leq-\infty,$$

what is a contradiction. Hence, the assumption that $z\neq Tz$ is impossible. □

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[1] Department of Mathematics and Physics, Faculty of Electrical Engineering, University of Banja Luka, Patre 5, 78000 Banja Luka, Bosnia and Herzegovina.
Email address: dusan.bajovic@etf.unibl.org

[2] Department of Applied Mathematics, Faculty of Forestry, University of Beograd, Kneza Višeslava 1, 11000 Beograd, Serbia.
Email address: slobodanka.mitrovic@sfb.bg.ac.rs

[3] Department of Mathematics, Faculty of Technical Sciences, University of Novi Sad, Trg Dositeja Obradovića 6, 21000 Novi Sad, Serbia.
Email address: sandrabu@uns.ac.rs

[4] Department of Mathematics, Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia.
Email address: radens@beotel.rs